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METRIZABILITY OF ASYMMETRIC QUASI NORMED SPACE

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Abstract. Our present work is divided into three parts. Firstly, we present two modified theorems after giving a definition of the asymmetric normal cone. In this way it is confirmed that a non-asymmetric normal cone can be converted to an asymmetric normal with constant one. In the next part, as the main purpose of this paper, after recalling the asymmetric quasi 2-norm function it is showed that an asymmetric 2-normed space is metrizable. At the end of the paper we present a way to build a topological space by creating its basis, using the asymmetric quasi 2-norm function and than prove that it is a normal topological space.

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1. Introduction

The major motivation of many mathematicians is to generalize a certain structure and search for new results. Sometimes there are generalizations which contribute a lot to the subject. In 1906, Fréchet did generalized the notion of the distance function and introduced the concept of metric space. Now we have a huge amount of articles that generalizes the metric structure.

In 2007, Huan and Zhang [9] introduced the cone metric space. A lot of research work is done after that with the aim to generalize the cone metric space and to prove the metrizability of these new con metric spaces. In 1993, Czewiwik [7] did a modification to the triangle inequality of the metric function, giving the definition of the b-metric as a generalization of the metric function. In 1998 [8], he expanded his generalization by replacing the 2 coefficient in the right hand of the inequality, by an arbitrary coefficient $K \geq 1$. After that, other researches proved metrizability of such spaces and gave more results about b-metric spaces [1], [5], [12]. Khamsi and Hussain [11] are two other authors that

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considered the concept of b-metric with the change of the coefficient to be $K > 0$.
Some other articles dealt with fixed point theorems in those spaces or with the structure of the spaces themselves [2, 10, 13–16]. Considering this, in [3] and [4] it is given the notion of the asymmetric quasi norm function and the asymmetric quasi 2-normed function with the aim to obtain useful results, and in this paper we present some results using these definitions.

Definition 1.1. The function $p : X \to \mathbb{R}^+$ is called *the asymmetric quasi norm function* if:

a) $p(x) \geq 0$ for every $x \in X$,
b) $p(\lambda x) = \lambda p(x)$ for every $x \in X$ and $\lambda > 0$,
c) $p(x + y) \leq k[p(x) + p(y)]$ for every $x, y \in X$ and $k \geq 1$.

Definition 1.2. The function $q(x) : X \times X \to [0, \infty)$ is called *the asymmetric quasi 2-normed function* if it completes the following conditions:

a) $q(x, y) \geq 0$, $q(x, y) = 0 \Leftrightarrow x = y$,
b) $q(x, y) = q(y, x)$,
c) $q(\lambda x, y) = \lambda q(x, y)$ for $\lambda > 0$,
d) $q(x, y) \leq q(x, z) + q(z, y)$ for all $x, y, z \in X$ and $\lambda > 0$.

There have been a lot of research work after the publication of the paper of Huang and Zhang [9], dealing with cone metric spaces. Cone metric spaces are similar to metric spaces with a positive cone in a Banach space but replacing positive real numbers. We begin with a short introduction to cone metric spaces.

Definition 1.3. A subset P of a real Banach space E is called a *cone* if and only if:

P1) P is closed, nonempty and $P \neq \{0\}$,
P2) If $a, b \in \mathbb{R}$, $a, b \geq 0$ and $x, y \in P$, then $ax + by \in P$,
P3) If both $x \in P$ and $-x \in P$ then $x = 0$.

The space X can be partially ordered by the cone $P \subset X$ in this way $x \ll y$ if and only if $y - x \in P$. We also can write $x \ll y$ if $y - x \in int(P)$.

The cone P is called *normal* if there exists a constant $K > 0$ such that for all $a, b \in E$, $0 \leq a \leq b \Rightarrow \|a\| \leq K\|b\|$.

Example. The cone $[0, \infty)$ in $(\mathbb{R}, |.|)$ and the cone $P = \{(x, y) : x \geq 0, y \geq 0\}$ in $\mathbb{R}^2$ are normal cones with constant $K = 1$.

Definition 1.4. A *con metric space* is an ordered pair (X, d), where X is any set and $d : X \times X \to E$ is a mapping satisfying:

1) $0 < d(x, y)$ for all $x, y \in X$,
2) $d(x, y) = 0$ if and only if $x = y$,

3) $d(x, y) = d(y, x)$ for all $x, y \in X$,
4) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

2. Materials and Methods

A metrizable space is a topological space that is homeomorphic to a metric space. In other words, a topological space (X, T) is called a *metrizable* space if there is a metric $d : X \times X \to [0, \infty)$ such that the topology induced by the metric d is T.

Before proceeding further, we might need some lemmas.

Lemma 2.1. Let $y \in Int(P)$. Then $\forall x(x \geq y \Rightarrow x \in Int(P))$.

Lemma 2.2. Let (X, P) be a cone space with $x \in P$ and $y \in Int(P)$. Then one can find $n \in \mathbb{N}$ such that $x \ll y$.

Lemma 2.3. In a cone space (X, P) we have

$$x \leq y \ll z \Rightarrow x \ll z.$$

Now let us present a new definition of a normal cone in the space (X, p) created by the asymmetric quasi norm function presented in Definition 1.1, and called *the asymmetric quasi norm space*.

Definition 2.4. A cone P is called *asymmetric normal cone* if there exists a constant $C > 0$ such that $0 \leq x \leq y \Rightarrow p(x) \leq Cp(y)$, where $p(x)$ and $p(y)$ are asymmetric quasi norms.

Suppose that (X, p) is an asymmetric quasi norm space, P is an asymmetric normal cone in X with $int(P) \neq \Phi$ and $\leq$ is the partial ordering with respect to P. The following theorem states that we can convert every asymmetric normal cone with constant $C > 1$ to an asymmetric normal cone with $C = 1$.

Theorem 2.5. Let P be an asymmetric normal cone with $C > 1$ in the asymmetric quasi normed space (X, p). There exists an asymmetric quasi norm on X such that P is an asymmetric normal cone with constant $C = 1$.

Proof. Firstly we define $p'(x) : X \times X \to [0, \infty)$ a function such that

$$p'(x) = c \cdot \inf \{p(s) : x \leq s\} + c \cdot \inf \{p(t) : t \leq x\}$$

for all $x \in X$, where $p(s)$, $p(t)$ are asymmetric quasi norms on X. After presenting the function we shall show that $p'(x)$ is an asymmetric quasi norm on X. Let us try to prove every condition.

a) $p'(x) \geq 0$, for every $x \in X$, it is evident that $p'(x) > 0$ because $p(s), p(t) > 0$ since they are asymmetric quasi norms on X. Also if $x = 0 \Rightarrow p'(x) = 0$. If $p'(x) = 0$ then there exists $s_n, t_n \in X$ with the property $t_n \leq x \leq s_n$, where $s_n \to 0$, and $t_n \to 0$ as $n \to \infty$. Since P is an asymmetric normal cone then $x = 0$.

b) $p'(\lambda x) = \lambda p'(x)$ for $\lambda > 0$,

$$\begin{aligned} p'(x) &= c \inf \{p(s) : \lambda x \leq s\} + c \inf \{p(t) : t \leq \lambda x\} \\ &= c \inf \{\lambda p(\frac{1}{\lambda}s) : x \leq \frac{1}{\lambda}s\} \\ &+ c \inf \{\lambda p(\frac{1}{\lambda}t) : \frac{1}{\lambda}t \leq x\} \\ &= \lambda[c \inf \{p(s') : x \leq s'\} \\ &+ c \inf \{p(t') : t' \leq x\}] \\ &= \lambda p'(x) \end{aligned} \tag{1}$$

c) Now let us prove the triangle inequality

$$p'(x+y) \leq kp'(x) + kp'(y).$$

For every $\epsilon > 0$, $\exists s_1, t_1 > 0$ such that $t_1 \leq x \leq s_1$,

$$\begin{aligned} &p'(x) \leq kp(s_1) + kp(t_1) \\ &\qquad \Rightarrow kp(s_1) + kp(t_1) - \epsilon < p'(x). \end{aligned} \tag{2}$$

For every $\epsilon > 0$, $\exists s_2, t_2 > 0$ such that $t_2 \leq x \leq s_2$

$$\begin{aligned} &p'(y) \leq kp(s_2) + kp(t_2) \\ &\qquad \Rightarrow kp(s_2) + kp(t_2) - \epsilon < p'(y). \end{aligned} \tag{3}$$

Now for $t_1 + t_2 \leq x + y \leq s_1 + s_2$, we have

$$\begin{aligned} p'(x+y) &\leq kp(t_1+t_2) + kp(s_1+s_2) \\ &\leq k(p'(x) + p'(y)) + 2\epsilon. \end{aligned} \tag{4}$$

Choosing an arbitrary $\epsilon > 0$, we obtain

$$p'(x+y) \leq kp'(x) + kp'(y),$$

therefore $p'(x)$ is an asymmetric quasi norm on X.

It is left to show that P is an asymmetric normal cone with constant $C = 1$ with the respect to $p'(x)$. Let $0 \leq x \leq y$, we have:

$$\inf \{p(s) : s \leq x\} = \inf \{p(s) : s \leq y\} = 0.$$

Therefore $p'(x) \leq p'(y)$. □

Theorem 2.6. Let P be a positive cone on an asymmetric quasi normed space (X, p). There exists an asymmetric quasi norm on X such that P is an asymmetric normal cone with constant $C = 1$ with respect to this asymmetric quasi norm.

Proof. Let us define $h(x) : X \to [0, \infty)$ such a function fulfilling the following conditions:

a) $x = 0 \Rightarrow h(x) = 0$,
   $h(x) = p(x)$ or $h(x) = 0 \Rightarrow x = 0$,

b) $h(\lambda x) = \lambda h(x)$ for $\lambda > 0$,

c) $p(x) - p(y) \leq h(x) - h(y)$

$$p(x + y) - kp(x) - kp(y) \leq h(x + y) - kh(x) - kh(y).$$

By the first part of the third condition, if we choose $x = 0$ then $h(x) \leq p(x)$, $\forall x \in P$. Now let us define $s(x) = p(x) - h(x)$, $\forall x \in X$ and show that this function is an asymmetric quasi norm.

a) $s(x) \geq 0$ because $s(x) = p(x) - h(x)$ and
   $h(x) \leq p(x)$, for $x = 0 \Rightarrow s(x) = 0$.

b) $s(\lambda x) = p(\lambda x) - h(\lambda x) = \lambda p(x) - \lambda h(x) = \lambda s(x)$.

c)

$$\begin{aligned} s(x + y) &= p(x + y) - h(x + y) \\ &\leq k(p(x) + p(y)) - k(h(x) - h(y)) \\ &= k(p(x) - h(x) + p(y) - h(y)) \\ &= k(s(x) + s(y)). \end{aligned} \tag{5}$$

For every $x, y \in X$ and $0 \leq x \leq y$ we obtain

$$s(x) \leq s(y).$$

□

In the year 1917, the theorem called Chittenden's metrization theorem gave a lot of conditions to prove that a topological space is metrizable [6].

Definition 2.7. Let the distance function $F : X \times X \to [0, \infty)$ a topological space X if it satisfies the following conditions:

a) $F(x, y) = 0 \Leftrightarrow x = y$ for all $x, y \in X$,

b) $F(x, y) = F(y, x)$ for all $x, y \in X$,

c) $\forall \epsilon > 0$, and $x, y, z \in X$, there exists $\phi(\epsilon) > 0$ such that if

$$F(x, z) < \phi(\epsilon),$$

and

$$F(z,y) < \phi(\epsilon) \Rightarrow F(x,y) < \epsilon$$

(this is called the uniformly regular condition)

then the topological space X is metrizable.

Theorem 2.8. If (X,q) is an asymmetric quasi 2-norm space then X is metrizable.

Proof. Let (X,q) be an asymmetric quasi 2-normed space. The asymmetric quasi 2-norm function $q : X \times X \to [0,\infty)$ on X satisfies the two conditions of Definition 2.7 because of its own definition. Us such

a) $q(x,y) = 0 \Leftrightarrow x = y$ for all $x, y \in X$,

b) $q(x,y) = q(y,x)$ for all $x, y \in X$.

For the third condition (the uniformly regular condition) we get $\epsilon > 0$, and $x, y, z \in X$. It is obvious that if $x = y$, then $q(x,y) = 0$. For our purpose, we can have $\phi(\epsilon) = s$ (where s a positive real number). If $x \neq y \Rightarrow q(x,y) > 0$, by using the definition of an asymmetric quasi 2-normed space, we have

$$q(x,y) \leq q(x,z) + q(z,y).$$

If We choose $\phi(\epsilon) = \frac{\epsilon}{2}$, then $q(x,z) < \frac{\epsilon}{2}$, $q(z,y) < \frac{\epsilon}{2}$ and

$$q(x,z) + q(z,y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

In conclusion we have $q(x,y) < \epsilon$, which means that the asymmetric quasi 2-normed space X is metrizable. □

After introducing the asymmetric quasi 2-norm function in this section, we present *the asymmetric quasi 2-normed topological space*. As such,we start by defining the *quasi-ϵ-ball* centered at x.

Definition 2.9. Let $q(x,y)$ be an asymmetric quasi 2-normed space. The *quasi-ϵ-ball* centered at x is defined as

$$B_q(x,\epsilon) = \{y \in X | q(x,y) < \epsilon\}.$$

Now we have to show that the collection of all *quasi-ϵ-balls* centered at x, for all $x \in X$, is a basis for a topology on X. This topology will be called *the asymmetric quasi 2-normed topology* induced by q.

1) Clearly for all $\epsilon > 0$, and $a \in X$, we have $a \in B_q(a,\epsilon)$.

2) To prove the second condition of the basis for a topological space, let us take

$$u \in B_q(a,\epsilon_1) \cap B_q(b,\epsilon_2), \forall a, b, u \in X.$$

Now we consider ϵ to as follows

$$\epsilon = \min\{\epsilon_1 - q(a,u), \epsilon_2 - q(b,u)\}.$$

As such, the subsequent step involves demonstrating that

$$B_q(u,\epsilon) \subset B_q(a,\epsilon_1) \cap B_q(b,\epsilon_2).$$

If, $s \in B_q(u,\epsilon)$ then $q(u,s) < \epsilon$. Additionally, we also have

$$u \in B_q(a,\epsilon_1) \Rightarrow q(a,u) < \epsilon_1.$$

With q representing an asymmetric quasi 2-norm, we can express the following

$$\begin{aligned} q(a,s) &\leq q(a,u) + q(u,s) \\ &\leq q(a,u) + \epsilon. \end{aligned} \tag{6}$$

Since

$$\epsilon + q(a,u) < \epsilon_1 - q(a,u) + q(a,u) = \epsilon_1,$$

then

$$q(a,s) < \epsilon_1 \Rightarrow s \in B_q(a,\epsilon_1).$$

Similarly, we can establish the proof's completeness by proving that $s \in B_q(b,\epsilon_2)$.

Proposition 2.10. Each asymmetric quasi 2-norm topological space is a normal topological space.

Proof. Let U, V be two closed subsets of an asymmetric quasi 2-normed topological space (X,q) with $S = \bigcup_{u \in U} B_q(u, \frac{\epsilon_1}{2})$, and $T = \bigcup_{v \in V} B_q(v, \frac{\epsilon_2}{2})$, where $\epsilon_1, \epsilon_2 > 0$, and

$$B_q(u,\epsilon_1) \cap B_q(v,\epsilon_2) = \Phi.$$

Assuming that S, T are disjoint open subsets containing U, V respectively, we have $z \in S \cap T$. As such there are $u, v \in X$, such that $z \in B_q(u, \frac{\epsilon_1}{2})$, and $z \in B_q(v, \frac{\epsilon_2}{2})$. Therefore we have

$$\begin{aligned} q(u,v) &\leq q(v,z) + q(z,u) \\ &\leq \frac{\epsilon_1}{2} + \frac{\epsilon_2}{2}. \end{aligned} \tag{7}$$

Assuming that $\epsilon_1 < \epsilon_2$ (without loos of generality) we have

$$q(v,u) \leq \epsilon_2 \Rightarrow u \in B_q(v,\epsilon_2).$$

Given the fact that $B_q(u,\epsilon_1) \cap B_q(v,\epsilon_2) = \Phi$, it is evident that this is untenathereby completing the proof. □

3. Conclusions

The paper explores properties of the asymmetric quasi norm function and 2-norm function. Our focus is on examining the concept of asymmetric quasi norm (2-norm) with a particular focus on metrizability of the spaces created by these concepts. Specifically it is showed that an asymmetric quasi 2-normed space is metrizable. This research work establishes a new topology space after the definition of asymmetric quasi 2-norm function. After that we have shown that this new topology space is a normal topology space and we hope leading to further research options for novel properties pertainig this space.

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Authors' Contributions

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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