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ON Fπ-REGULAR RINGS

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Abstract. This paper aims to examine Fπ-regular rings and identify some of their properties. We introduce the characterization of Fπ-regular rings under duo rings and FGP-injective rings. Additionally, we offer the motivating set for any element $\alpha \in \mathscr{A}$ as $M(\alpha) = \{0 \neq c \in \mathscr{A} : \alpha c = \alpha c \mathfrak{b} \alpha c$ for some $\mathfrak{b} \in \mathscr{A}\}$ and demonstrate its connection to Fπ-regular rings. Moreover, we define $F^*\pi$-regular rings which leads to Fπ-regular rings. We also investigate the classification of 2-primal $F^*\pi$-regular rings and its relationship to strongly π-regular and π-regular rings.

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1. Introduction

The pioneering work of Emmy Noether on commutative algebra during the 1900s [17,28] laid the foundation for what became the current notion of regularity. Noether defined many of the properties of commutative rings that enabled their classification into types, although none of those types were in exact correspondence with the modern definition of regular rings. Towards the middle of the 20th century, the properties of algebraic structures and the generalizations that could be made based on their commutative or non-commutative properties, began to be explored in great depth. One of the foremost mathematicians of the period was Irving Kaplansky, who made seminal contributions to our understanding of associative rings and particularly to regular rings under conditions of non-commutativity.

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Near the mid of the 1900s, one of the greatest contributors to algebra and the increasing interest in regular rings was the outstanding mathematician and polymath, Von Neuman. Neuman was highly influential, with his work on regular rings resulting in a class of rings named after him 'Regular Rings'. Regular rings were receiving considerable attention at this time and their relations to other mathematical structures gradually being uncovered. This enabled the generalizations and extensions of regular rings to be discovered, including the characterization of rings into different levels of regularity. For example, extensions of regularity that impose further conditions to the essential properties, led to classes of rings being identified as having strong or weak regularity. Such generalizations were furthered by McCoy's 1939 [23] generalization of regular rings to π-RR. π-regularity has been extensively explored by mathematicians [5,27] since McCoy made the generalization. Interest in associative rings continued to increase during the 1950's, typified by Kaplansky's work [4] on regular rings that continues to be influential in Ring theory.

Kaplansky's work led to the definition of a Sπ-RR. Formal proof that all Sπ-RR are also π-RR was first given by Azumaya [5], while El-Astal [1] showed the equality of all Sπ-RR and π-RR that form a duo. Examples of Sπ-RR are also given in [15]. Further extensions of π-RR [2,12,21,26] and Sπ-RR have been investigated by many mathematicians [9,10,14,18]. Interest subsequently focused on the similarities between different classes of π-RR and regular rings and the further properties that could be derived from the fact that a ring was π-RR or regular [6,7]. The relations between properties of idempotents in π-RR and regular rings, and their implications for module theory [20], received much attention. Mathematicians then began exploring the connection between these rings and group rings, FGP-injective rings, as well as other algebraic structures [3,4,11,19,24].

The work presented in this paper characterises a new type of regular ring that represents an extension of both π-RR and regular rings and is called Fπ-RR. The properties of these rings are investigated under both duo rings and FGP-injective rings. $F^*\pi$-RR are defined here and how these relate to Fπ-RR is also presented.

2. Preliminaries

Thought this article, $\mathscr{A}$ indicates an associated ring with identity and $\mathscr{A}^* = \mathscr{A} \setminus \{0\}$. For a subset $\mathcal{X} \subset \mathscr{A}$, its right and left annihilators are denoted by $\mathbf{r}(\mathcal{X})$ and $\mathbf{l}(\mathcal{X})$ respectively. The abbreviation $\mathbf{r}(\alpha)$ (or $\mathbf{l}(\alpha)$) is utilized if $\mathcal{X} = \{\alpha\}$. $\mathbb{P}(\mathscr{A})$ denotes the prime radical of $\mathscr{A}$ and $\mathbb{N}(\mathscr{A})$ denotes the nilpotent elements of $\mathscr{A}$. In addition, $\mathbb{U}(\mathscr{A})$ denotes the set of all unit elements of $\mathscr{A}$ and $\mathbb{Z}(\mathscr{A})$ denotes the set of all zero divisors in $\mathscr{A}$. The greatest common divisor of $0 \neq x \in \mathbb{Z}_n$ is denoted by $gcd(x, n)$. The idempotent element in $\mathscr{A}$ is denoted by $\mathfrak{e}$.

Definition 2.1. *Let $\mathscr{A}$ be a ring then:*

(*i*) *$\mathscr{A}$ is RG-R if for every $\alpha \in \mathscr{A}$, an $\mathfrak{b} \in \mathscr{A}$ with $\alpha = \alpha\mathfrak{b}\alpha$ exists* [25].

(*ii*) *$\mathscr{A}$ is π-RR if for every element $\alpha \in \mathscr{A}$, there is a positive integer n where α^n is regular* [23].

(*iii*) *$\mathscr{A}$ is Sπ-RR in which every $\alpha \in \mathscr{A}$, there are $\mathfrak{b} \in \mathscr{A}$ and an m where $\alpha^m = \alpha^{m+1}\mathfrak{b}$* [5].

(*iv*) *$\mathscr{A}$ is called reduced, if $\mathscr{A}$ contains no non-zero nilpotent elements* [2].

(*v*) *$\mathscr{A}$ is called a right (left) duo if and only if every right (left) ideal of $\mathscr{A}$ is a two sided ideal* [22].

(*vi*) *$\mathscr{A}$ is called a duo if and only if it is a right and left duo ring* [22].

(*vii*) *$\mathscr{A}$ is 2-PR if $\mathbb{P}(\mathscr{A}) = \mathbb{N}(\mathscr{A})$, or equivalently, if $\mathscr{A}/\mathbb{P}(\mathscr{A})$ is a reduced ring* [21].

(*viii*) *$\mathscr{A}$ is NI-ring if $\mathbb{N}(\mathscr{A})$ forms a two sided ideal. Notice that a right (or left) duo ring is 2-PR. Moreover, every 2-PR is an NI-ring* [21].

Definition 2.2. *Let $\mathscr{A}$ be a ring, and $\mathbf{K} \subset \mathscr{A}$ is an ideal in $\mathscr{A}$, then:*

(*i*) *$\mathbf{K}$ is regular if for each $\alpha \in \mathbf{K}$, there is $\mathfrak{b} \in \mathbf{K}$ where $\alpha = \alpha\mathfrak{b}\alpha$* [13].

(*ii*) *$\mathbf{K}$ is called pure if for all $\alpha \in \mathbf{K}$, there is $\mathfrak{b} \in \mathbf{K}$ where $\alpha = \alpha\mathfrak{b}$* [2].

(*iii*) *$\mathbf{K}$ is completely prime if $\alpha\mathfrak{b} \in \mathbf{K}$ implies either $\alpha \in \mathbf{K}$ or $\mathfrak{b} \in \mathbf{K}$ where $\alpha, \mathfrak{b} \in \mathscr{A}$* [8].

Definition 2.3. [3] *A ring $\mathscr{A}$ is FGP-injective ring if for any $0 \neq \alpha \in \mathscr{A}$, there is $0 \neq c \in \mathscr{A}$ where $0 \neq \alpha c = c\alpha$ and any right $\mathscr{A}$-homomorphism from $\alpha c\mathscr{A}$ to $\mathscr{A}$ extends to an endomorphism of $\mathscr{A}$.*

Theorem 2.4. [16] *If $\mathbf{I}$ is an ideal in a ring $\mathscr{A}$, then*

$$\{\textit{All ideals of } \mathscr{A} \textit{ containing } \mathbf{I}\} \overset{\textit{1-1}}{\longleftrightarrow} \{\textit{All ideals of } \mathscr{A}/\mathbf{I},\ \textit{given by } \mathbf{J} \to \mathbf{J}/\mathbf{I}\}.$$

In view of the above assertion, every ideal in $\mathscr{A}/\mathbf{I}$ can be written as $\mathbf{J}/\mathbf{I}$, where $\mathbf{J}$ is an ideal of $\mathscr{A}$ containing $\mathbf{I}$.

Lemma 2.5. [22] *Let $\mathscr{A}$ be a reduced ring. Then $\mathbf{r}(\alpha) = \mathbf{l}(\alpha)$ for all non-zero element $\alpha \in \mathscr{A}$.*

Lemma 2.6. [8] *A ring $\mathscr{A}$ is 2-PR if and only if every minimal prime ideal is completely prime.*

3. Fπ-regular rings

In this section, we define Fπ-RR as well as $F^*\pi$-RR and outline some of their basic properties.

Definition 3.1. *An element α in a ring $\mathscr{A}$ is said to be Fπ-regular if there exist $0 \neq c \in \mathscr{A}$ and $\mathfrak{b} \in \mathscr{A}$ where $\alpha c = \alpha c\mathfrak{b}\alpha c$. A ring $\mathscr{A}$ is said to be an Fπ-RR if and only if every element of $\mathscr{A}$ is an Fπ-regular element.*

Example 3.2. *The following rings are Fπ-RR:*

(1) *Boolean rings.*

(2) *$\mathbb{Z}_6$, $\mathbb{Z}_{10}$, and $\mathbb{Z}_{14}$.*

(3) *$\mathscr{A}_{2\times 2}(\mathbb{Z})$, the ring of 2×2 matrices over $\mathbb{Z}_2$.*

Clearly, RG-R and π-RR are Fπ-RR; however, the opposite is untrue. For example, $\mathbb{Z}_4$ is an Fπ-RR that is not RG-R. In addition, if $\mathscr{A} = \mathbb{Z} \times \mathbb{Z}_6$, then $\mathscr{A}$ is an Fπ-RR which is not π-RR.

Theorem 3.3. *A ring $\mathscr{A}$ is Fπ-RR if and only if for every $\alpha \in \mathscr{A}$, there is $0 \neq c \in \mathscr{A}$ such that the principle right (left) ideal $\alpha c\mathscr{A}$ ($\mathscr{A}\alpha c$) of $\mathscr{A}$ is generated by an idempotent.*

Proof. Let $\mathscr{A}$ be Fπ-RR and $\alpha \in \mathscr{A}$. Then, there exist $0 \neq c \in \mathscr{A}$ and $\mathfrak{b} \in \mathscr{A}$ where $\alpha c = \alpha c\mathfrak{b}\alpha c$. Set $\mathfrak{e} = \alpha c\mathfrak{b}$. Let $x \in \alpha c\mathscr{A}$, then $x = \alpha cr$ for some $r \in \mathscr{A}$. Thus, $x = \alpha c\mathfrak{b}\alpha cr = \mathfrak{e}\alpha cr \in \mathfrak{e}\mathscr{A}$, and therefore, $\alpha c\mathscr{A} \subseteq \mathfrak{e}\mathscr{A}$. Conversely, if $d \in \mathfrak{e}\mathscr{A}$, then $d = \mathfrak{e}s$ for some $s \in \mathscr{A}$. Thus, $d = \alpha c\mathfrak{b}s \in \alpha c\mathscr{A}$. Hence, $\mathfrak{e}\mathscr{A} \subseteq \alpha c\mathscr{A}$, and therefore, $\alpha c\mathscr{A} = \mathfrak{e}\mathscr{A}$. To prove that $\mathscr{A}\alpha c = \mathscr{A}\mathfrak{e}$, put $\mathfrak{e} = \mathfrak{b}\alpha c$ in $\mathscr{A}$. Conversely, let $\alpha \in \mathscr{A}$. Choose $\mathfrak{e} \in \mathscr{A}$ where $\alpha c\mathscr{A} = \mathfrak{e}\mathscr{A}$ for some $0 \neq c \in \mathscr{A}$. Then, $\mathfrak{e} = \alpha c\mathfrak{b}$ for some $\mathfrak{b} \in \mathscr{A}$ and $\alpha c = \mathfrak{e}m$ for some $m \in \mathscr{A}$. Hence, $\mathfrak{e}\alpha c = \alpha c\mathfrak{b}\alpha c$ and $\mathfrak{e}\alpha c = \mathfrak{e}\mathfrak{e}m = \mathfrak{e}m = \alpha c$. Therefore, $\alpha c = \alpha c\mathfrak{b}\alpha c$. Thus, $\mathscr{A}$ is Fπ-RR. □

Theorem 3.4. *Let $\mathscr{A}$ be an Fπ-RR. Then for every $\alpha \in \mathscr{A}$, there is $0 \neq c \in \mathscr{A}$ such that the principle right (left) ideal $\alpha c\mathscr{A}$ ($\mathscr{A}\alpha c$) of $\mathscr{A}$ is a right (left) annihilator of an element of $\mathscr{A}$.*

Proof. Suppose $\mathscr{A}$ be an Fπ-RR, with $\alpha \in \mathscr{A}$. Then there exist $0 \neq c \in \mathscr{A}$ and $\mathfrak{b} \in \mathscr{A}$ where $\alpha c = \alpha c\mathfrak{b}\alpha c$. Set $\mathfrak{e} = \alpha c\mathfrak{b}$. Now, $1-\mathfrak{e}$ is also an element in $\mathscr{A}$, and hence $\alpha c\mathscr{A} = \mathbf{r}(1-\mathfrak{e})$. To prove that $\alpha c\mathscr{A} \subseteq \mathbf{r}(1-\mathfrak{e})$, let $y \in \alpha c\mathscr{A}$ and then $y = \alpha cr$ for some $r \in \mathscr{A}$. Hence $y = \alpha c\mathfrak{b}\alpha cr = \mathfrak{e}\alpha cr = \mathfrak{e}y$ and this means that $y - \mathfrak{e}y = (1-\mathfrak{e})y = 0$. Therefore, $y \in \mathbf{r}(1-\mathfrak{e})$. Conversely, let $x \in \mathbf{r}(1-\mathfrak{e})$. Then $(1-\mathfrak{e})x = 0$ and this implies $x = \mathfrak{e}x = \alpha c\mathfrak{b}x$. Therefore, $x \in \alpha c\mathscr{A}$. If $\mathfrak{e} = \mathfrak{b}\alpha c$, then we can easily prove that $\mathscr{A}\alpha c = \mathbf{l}(1-\mathfrak{e})$. □

Corollary 3.5. *If $\mathscr{A}$ is a reduced ring where every maximal ideal of $\mathscr{A}$ is a right annihilator, then $\mathscr{A}$ is an Fπ-RR.*

Proof. Straightforward. □

Theorem 3.6. *If $\mathscr{A}$ is an Fπ-RR without zero divisors, then $\mathscr{A}$ is a division ring.*

Proof. Let $0 \neq \alpha \in \mathscr{A}$. Since $\mathscr{A}$ is Fπ-RR, there exist $0 \neq c \in \mathscr{A}$ and $\mathfrak{b} \in \mathscr{A}$ where $\alpha c = \alpha c\mathfrak{b}\alpha c$. Then, $(1-\alpha c\mathfrak{b})\alpha c = 0$ implies $1 - \alpha m = 0$, where $m = c\mathfrak{b} \in \mathscr{A}$. Hence, $1 = \alpha m$. In addition, $\alpha(1 - c\mathfrak{b}\alpha)c = 0$ implies $1 = c\mathfrak{b}\alpha = m\alpha$, and hence α is invertible. Therefore, $\mathscr{A}$ is a division ring. □

Corollary 3.7. *Let $\mathscr{A}$ be an integral domain. If $\mathscr{A}$ is an Fπ-RR, then $\mathscr{A}$ is a field.*

Theorem 3.8. *The direct product of rings with identities is an Fπ-RR if at least one of them is an Fπ-RR.*

Proof. Let $\mathscr{A}$ and $\mathfrak{S}$ be rings where $\mathscr{A}$ is an Fπ-RR. It is enough to show that $\mathfrak{T} = \mathscr{A} \times \mathfrak{S}$ is Fπ-RR. Let $(\alpha_1, \alpha_2) \in \mathfrak{T}$ where $\alpha_1 \in \mathscr{A}$ and $\alpha_2 \in \mathfrak{S}$. Take $(0,0) \neq c = (c_1, 0) \in \mathfrak{T}$ with $c_1 \neq 0$ and $\mathfrak{b} = (\mathfrak{b}_1, \mathfrak{b}_2) \in \mathfrak{T}$.

Now, $(\alpha_1, \alpha_2)(c_1, 0)(\mathfrak{b}_1, \mathfrak{b}_2)(\alpha_1, \alpha_2)(c_1, 0) = (\alpha_1 c_1 \mathfrak{b}_1 \alpha_1 c_1, 0) = (\alpha_1 c_1, 0) = (\alpha_1, \alpha_2)(c_1, 0)$. Therefore, $\mathfrak{T}$ is an Fπ-RR.

□

If both rings are not Fπ-RR, then the direct product is not an Fπ-RR. For example, $\mathbb{Z}\times\mathbb{Z}$ is not an Fπ-RR.

The homomorphic image of an Fπ-RR is not Fπ-RR. For example, if $\mathscr{A} = \mathbb{Z} \times \mathbb{Z}_6$ and $\mathbf{I} = \{(0, x) : x \in \mathbb{Z}_6\}$ is an ideal of $\mathscr{A}$. Then, $\mathscr{A}$ is an Fπ-RR. Now, $\mathscr{A}/\mathbf{I} \cong \mathbb{Z}$ which is not an Fπ-RR.

Theorem 3.9. *Let* $\mathbf{I}$ *be an ideal of a ring* $\mathscr{A}$*. If* $\mathscr{A}/\mathbf{I}$ *is F*π*-RR and* $\mathbf{I}$ *is regular, then* $\mathscr{A}$ *is F*π*-RR.*

Proof. Let $\mathbf{I}$ be a regular ideal of $\mathscr{A}$ and $\alpha \in \mathscr{A}$. Then, $\bar{\alpha} \in \mathscr{A}/\mathbf{I}$. If $\bar{\alpha} = \mathbf{I}$, then $\alpha \in \mathbf{I}$ and $\mathbf{I}$ is regular. Thus, there is $d \in \mathbf{I}$ where $\alpha = \alpha d \alpha$; hence, $\mathscr{A}$ is Fπ-RR. If $\bar{\alpha} \neq \mathbf{I}$, then $\alpha \notin \mathbf{I}$, and this implies that $\bar{\alpha} \in \mathscr{A}/\mathbf{I}$. As $\mathscr{A}/\mathbf{I}$ is Fπ-RR, there exist $\bar{0} \neq \bar{c} \in \mathscr{A}/\mathbf{I}$ and $\bar{\mathfrak{b}} \in \mathscr{A}/\mathbf{I}$ with $\bar{\alpha}\bar{c} = \bar{\alpha}\bar{c}\bar{\mathfrak{b}}\bar{\alpha}\bar{c}$. Thus, $\alpha c - \alpha c \mathfrak{b} \alpha c \in \mathbf{I}$ and since $\mathbf{I}$ is regular, there is $z \in \mathbf{I}$ satisfying $(\alpha c - \alpha c \mathfrak{b} \alpha c) = (\alpha c - \alpha c \mathfrak{b} \alpha c) z (\alpha c - \alpha c \mathfrak{b} \alpha c)$. Hence, $\alpha c = \alpha c(\mathfrak{b} + z - z\alpha c \mathfrak{b} - \mathfrak{b}\alpha c z + \mathfrak{b}\alpha c z \alpha c \mathfrak{b})\alpha c$. Set $h = \mathfrak{b} + z - z\alpha c\mathfrak{b} - \mathfrak{b}\alpha c z + \mathfrak{b}\alpha c z \alpha c \mathfrak{b}$, and since $\bar{c} \neq \bar{0}$, then $c \neq 0$. Consequently, for all $\alpha \in \mathscr{A}$, there exist $0 \neq c \in \mathscr{A}$ and $h \in \mathscr{A}$ with $\alpha c = \alpha c h \alpha c$. Hence, $\mathscr{A}$ is Fπ-RR. □

Definition 3.10. *A ring* $\mathscr{A}$ *is* $F^*\pi$*-RR if for every* $\alpha \in \mathscr{A}$ *and* $\alpha \notin \mathbf{I}$ *for some ideal* $\mathbf{I}$ *of* $\mathscr{A}$*, there exist* $c \notin \mathbf{I}$ *and* $\mathfrak{b} \in \mathscr{A}$ *where* $\alpha c = \alpha c \mathfrak{b} \alpha c$*.*

Clearly, $F^*\pi$-RR is Fπ-RR, but the converse is not true. For example, $\mathbb{Z}_{12}$ is an $F^*\pi$-RR, while $\mathbb{Z}_{16}$ is an $F\pi$-RR which is not $F^*\pi$-RR.

Theorem 3.11. *Let* $\mathbf{I}$ *be an ideal of a ring* $\mathscr{A}$*. If* $\mathscr{A}$ *is* $F^*\pi$*-RR, then* $\mathscr{A}/\mathbf{I}$ *is* $F^*\pi$*-RR.*

Proof. Let $\mathscr{A}$ be $F^*\pi$-RR. Suppose $\bar{\alpha} \in \mathscr{A}/\mathbf{I}$ and $\bar{\alpha} \notin \bar{\mathbf{J}}$, where $\bar{\mathbf{J}}$ is an ideal of $\mathscr{A}/\mathbf{I}$. Then, $\bar{\mathbf{J}} = \mathbf{J}/\mathbf{I}$ with $\mathbf{I} \subset \mathbf{J}$. Thus, $\alpha \in \mathscr{A}$, and $\alpha \notin \mathbf{J}$ as $\mathbf{J}$ is an ideal of $\mathscr{A}$. Since $\mathscr{A}$ is $F^*\pi$-RR, there exist $c \notin \mathbf{J}$ and $\mathfrak{b} \in \mathscr{A}$ where $\alpha c = \alpha c \mathfrak{b} \alpha c$. Hence, $\bar{\alpha}\bar{c} = \bar{\alpha}\bar{c}\bar{\mathfrak{b}}\bar{\alpha}\bar{c}$. Since $c \notin \mathbf{J}$, we have $\bar{c} \notin \bar{\mathbf{J}}$ and $\bar{\mathfrak{b}} \in \mathscr{A}/\mathbf{I}$. Therefore, $\mathscr{A}/\mathbf{I}$ is $F^*\pi$-RR. □

Theorem 3.12. $\prod \mathscr{A}_i$ *is* $F^*\pi$*-RR if and only if* $\mathscr{A}_i$ *is* $F^*\pi$*-RR for all* i*.*

Proof. Let $\prod \mathscr{A}_i$ is $F^*\pi$-RR. Let $\alpha \in \mathscr{A}_i$ and $\alpha \notin \mathbf{J}$ where $\mathbf{J}$ is ideal in $\mathscr{A}_i$. Then, $\bar{\mathbf{J}} = \{(\alpha_1, \alpha_2, \ldots, \alpha_i, \ldots) : \alpha_j \in \mathscr{A}_j, \forall j \neq i, \alpha_i \in \mathbf{J}\}$ is an ideal of $\prod \mathscr{A}_i$ and $\bar{\alpha} = (0, \ldots, \alpha, 0, \ldots) \notin \bar{\mathbf{J}}$. Since $\prod \mathscr{A}_i$ is $F^*\pi$-RR, then there exist $\bar{c} \notin \bar{\mathbf{J}}$ with $\bar{c} = (c_1, c_2, \ldots, c_i, \ldots)$, and $\bar{\mathfrak{b}} \in \prod \mathscr{A}_i$ where $\bar{\alpha}\bar{c} = \bar{\alpha}\bar{c}\bar{\mathfrak{b}}\bar{\alpha}\bar{c}$. As $\bar{c} \notin \bar{\mathbf{J}}$, we have $c_i \notin \mathbf{J}$; otherwise a contradiction. Thus, $\alpha c_i = \alpha c_i \mathfrak{b}_i \alpha c_i$ in $\mathscr{A}_i$. Hence $\mathscr{A}_i$ is $F^*\pi$-RR. Conversely, let $\mathscr{A}_i$ is $F^*\pi$-RR, for all i. Suppose $\alpha = (\alpha_1, \alpha_2, \ldots) \in \prod \mathscr{A}_i$ and $\alpha \notin \bar{\mathbf{J}}$ where $\bar{\mathbf{J}}$ is an ideal in $\prod \mathscr{A}_i$. Since $\alpha \notin \bar{\mathbf{J}}$, there is at least $\alpha_i \notin \mathbf{J}_i = \{j_i : (j_1, j_2, \ldots, j_i, \ldots) \in \bar{\mathbf{J}}\}$. Since $\mathscr{A}_i$ is $F^*\pi$-RR, there is $c_i \notin \mathbf{J}_i$ and

$\mathfrak{b}_i \in \mathscr{A}_i$ where $\alpha_i c_i = \alpha_i c_i \mathfrak{b}_i \alpha_i c_i$. Hence, $\alpha c = \alpha c \mathfrak{b} \alpha c$ where $c = (0, \ldots, c_i, 0, \ldots) \notin \bar{\mathbf{J}}$; otherwise, $c_i \in \mathbf{J}_i$ a contradiction. □

4. The motivating set and Fπ-regular rings

In this section, we introduce the motivating set for any element $\alpha \in \mathscr{A}$ as $M(\alpha) = \{0 \neq c \in \mathscr{A} : \alpha c = \alpha c \mathfrak{b} \alpha c$ for some $\mathfrak{b} \in \mathscr{A}\}$ and demonstrate its connection to Fπ-RR. Let us pave the way to explore such a connection by presenting the following two assertions.

proposition 4.1. *Let $\mathscr{A}$ be a ring, then the following holds:*

(1) *If $a \in \mathbb{U}(\mathscr{A})$, then a is regular [13].*
(2) *If $a \in \mathbb{N}(\mathscr{A}) \cup \mathbb{U}(\mathscr{A})$, then a is π-regular [5].*
(3) *If $a \in \mathbb{Z}(\mathscr{A})$, then a is Fπ-regular.*
(4) *If $a \in \mathbb{U}(\mathscr{A})$, then $\mathbb{U}(\mathscr{A}) \subset M(a)$.*

Corollary 4.2. *$\mathbb{Z}_n$ is Fπ-RR for all positive integer $n \geq 2$.*

Proof. Let $x \in \mathbb{Z}_n$. If $0 \neq x \in \mathbb{U}(\mathbb{Z}_n)$, then x is regular by Proposition 4.1, and then $\mathbb{Z}_n$ Fπ-RR. Otherwise, if $0 \neq x \notin \mathbb{U}(\mathbb{Z}_n)$, then $gcd(x, n) = d \neq 1$. Thus, $x = sd$ and $n = td$ where s, t are positive integers and $t < n$. Hence, $tx = std = sn$ and then x is a zero divisor which proves that $\mathbb{Z}_n$ is Fπ-RR by Proposition 4.1. □

proposition 4.3. *$\mathscr{A}$ is a RG-R if and only if $M(\alpha) = \mathscr{A}^*$, for all $\alpha \in \mathscr{A}$.*

Proof. Let $\mathscr{A}$ be a RG-R and $0 \neq \alpha \in \mathscr{A}$. Clearly, $M(\alpha) \subset \mathscr{A}^*$. To prove that $\mathscr{A}^* \subset M(\alpha)$, let $y \in \mathscr{A}^*$. Then $\alpha y \in \mathscr{A}$. Since $\mathscr{A}$ is RG-R, there is $\mathfrak{b} \in \mathscr{A}$ where $\alpha y = \alpha y \mathfrak{b} \alpha y$. Therefore, $y \in M(\alpha)$. Hence, $M(\alpha) = \mathscr{A}^*$. Conversely, let $x \in \mathscr{A}$ where $M(x) = \mathscr{A}^*$. Since $\mathscr{A}$ is a ring with identity, we have $1 \in M(x)$. Therefore, $x.1 = x.1.\mathfrak{b}.x.1$ for some $\mathfrak{b} \in \mathscr{A}$. Hence, $x = x\mathfrak{b}x$ which proves that $\mathscr{A}$ is RG-R. □

proposition 4.4. *A ring $\mathscr{A}$ is π-RR if and only if for all $\alpha \notin \mathbb{N}(\mathscr{A}) \cup \mathbb{U}(\mathscr{A})$, there is a positive integer n where $\alpha^n \in M(\alpha)$.*

Proof. Let $\mathscr{A}$ be π-RR and $\alpha \notin \mathbb{N}(\mathscr{A}) \cup \mathbb{U}(\mathscr{A})$. Then $\alpha^n = \alpha^n \mathfrak{b} \alpha^n$ with $n \in \mathbb{Z}^+$ and $\mathfrak{b} \in \mathscr{A}$. Thus, $\alpha^{n-1} \in M(\alpha)$. Conversely, let $\alpha \in \mathscr{A}$. If $\alpha \in \mathbb{N}(\mathscr{A}) \cup \mathbb{U}(\mathscr{A})$, then by Proposition 4.1, α is π-regular. If $\alpha \notin \mathbb{N}(\mathscr{A}) \cup \mathbb{U}(\mathscr{A})$ and $\alpha^n \in M(\alpha)$ for some positive integer n, then $\alpha^{n+1} = \alpha^{n+1} \mathfrak{b} \alpha^{n+1}$ for some $\mathfrak{b} \in \mathscr{A}$ which proves that $\mathscr{A}$ is π-RR. □

proposition 4.5. *A ring $\mathscr{A}$ is Fπ-RR if and only if for all $\alpha \notin \mathbb{Z}(\mathscr{A}) \cup \mathbb{U}(\mathscr{A})$, $M(\alpha) \neq \phi$.*

Proof. Let $\mathscr{A}$ be Fπ-RR and $x \notin \mathbb{Z}(\mathscr{A}) \cup \mathbb{U}(\mathscr{A})$. Then for every $x \in \mathscr{A}$, there is $0 \neq c \in \mathscr{A}$ and $\mathfrak{b} \in \mathscr{A}$ where $xc = xc\mathfrak{b}xc$. Thus, $c \in M(x)$ and hence $M(x) \neq \phi$. Conversely, let $x \in \mathscr{A}$. If $x \in \mathbb{Z}(\mathscr{A}) \cup \mathbb{U}(\mathscr{A})$, then x is Fπ-regular by Proposition 4.1. If $x \notin \mathbb{Z}(\mathscr{A}) \cup \mathbb{U}(\mathscr{A})$, and since $M(x) \neq \phi$, there is $0 \neq c \in \mathscr{A}$ and $\mathfrak{b} \in \mathscr{A}$ where $xc = xc\mathfrak{b}xc$. Thus, $\mathscr{A}$ is Fπ-RR. □

Corollary 4.6. *If $\mathscr{A}$ is Fπ-RR, and $\alpha \notin \mathbb{Z}(\mathscr{A})$, then $M(\alpha)$ is a regular set.*

Proof. Let $\mathscr{A}$ be Fπ-RR and $\alpha \notin \mathbb{Z}(\mathscr{A})$. By Proposition 4.5, $M(\alpha) \neq \phi$. Let $x \in M(\alpha)$. Then $0 \neq x \in \mathscr{A}$ and $\alpha x = \alpha x\mathfrak{b}\alpha x$ for some $\mathfrak{b} \in \mathscr{A}$. Hence, $\alpha(x - x\mathfrak{b}\alpha x) = 0$. Since $\alpha \notin \mathbb{Z}(\mathscr{A})$, $x = x\mathfrak{b}\alpha x = xrx$ where $r = \mathfrak{b}\alpha \in \mathscr{A}$. Therefore, $M(\alpha)$ is regular.

□

Corollary 4.7. *A ring $\mathscr{A}$ is Fπ-RR if and only if for all $\alpha \notin \mathbb{Z}(\mathscr{A})$, there exist $\mathfrak{b} \in \mathscr{A}$ and $0 \neq c \in \mathscr{A}$ where $c = c\mathfrak{b}\alpha c$.*

Theorem 4.8. *Let $\mathscr{A}$ be a ring such that for every $\alpha \in \mathscr{A}$, there is $c \in M(\alpha)$ where $\mathbf{l}(\alpha c) = \mathbf{l}(\alpha)$. Then the following statements are equivalent:*

(1) *$\mathscr{A}$ is RG-R.*
(2) *$\mathscr{A}$ is a π-RR.*
(3) *$\mathscr{A}$ is an Fπ-RR.*

Proof. It is clear that $(1) \Rightarrow (2) \Rightarrow (3)$. We only prove $(3) \Rightarrow (1)$. Let $\mathscr{A}$ be an Fπ-RR where $\alpha \in \mathscr{A}$ and $c \in M(\alpha)$ with $\mathbf{l}(\alpha c) = \mathbf{l}(\alpha)$. Then $\alpha c = \alpha c\mathfrak{b}\alpha c$. Therefore, $(1 - \alpha c\mathfrak{b})\alpha c = 0$ implies $(1 - \alpha c\mathfrak{b}) \in \mathbf{l}(\alpha c)$. Hence $(1 - \alpha c\mathfrak{b}) \in \mathbf{l}(\alpha)$, and $\alpha - \alpha c\mathfrak{b}\alpha = 0$. Set $m = c\mathfrak{b} \in \mathscr{A}$. Thus, $\alpha = \alpha m\alpha$ and then $\mathscr{A}$ is a RG-R. □

Corollary 4.9. *Let $\mathscr{A}$ be a ring and for every $\alpha \in \mathscr{A}$, there exist $d \in M(\alpha)$ and $d \notin \mathbb{Z}(\mathscr{A})$. Then $\mathscr{A}$ is RG-R.*

5. Duo rings and Fπ-regular rings

This section provides some properties of Fπ-RR under duo rings.

Theorem 5.1. *Let $\mathscr{A}$ be a duo ring. Then $\mathscr{A}$ is Fπ-RR if and only if for every $\alpha \in \mathscr{A}$, there is $0 \neq c \in \mathscr{A}$ such that the principle ideal $\alpha c\mathscr{A}$ is an idempotent.*

Proof. Suppose $\mathscr{A}$ be an Fπ-RR. Let us show that $(\alpha c\mathscr{A})^2 = (\alpha c)^2\mathscr{A}$ for all $\alpha, c \in \mathscr{A}$. Since $\mathscr{A}$ is a unital ring, we have $(\alpha c)^2\mathscr{A} \subset \alpha c\mathscr{A}\alpha c\mathscr{A}$. On the other hand, let $y \in (\alpha c\mathscr{A})^2$. So, $y = \alpha cm\alpha cn$ for some $m, n \in \mathscr{A}$. Since $\mathscr{A}$ is duo, then $y = (\alpha c)^2\acute{m}n \in (\alpha c)^2\mathscr{A}$. Since $\mathscr{A}$ is Fπ-RR, then for every $\alpha \in \mathscr{A}$ there exist $0 \neq c \in \mathscr{A}$ and $\mathfrak{b} \in \mathscr{A}$ where $\alpha c = \alpha c\mathfrak{b}\alpha c$. To prove that $\alpha c\mathscr{A}$ is an idempotent its enough to show that $\alpha c\mathscr{A} = (\alpha c)^2\mathscr{A}$. Clearly, $(\alpha c)^2\mathscr{A} \subseteq \alpha c\mathscr{A}$. To prove that $\alpha c\mathscr{A} \subseteq (\alpha c)^2\mathscr{A}$, let $x \in \alpha c\mathscr{A}$. So, $x = \alpha cr$ for some $r \in \mathscr{A}$. Hence, $x = \alpha c\mathfrak{b}\alpha cr = (\alpha c)^2\bar{\mathfrak{b}}r = (\alpha c)^2\bar{r} \in (\alpha c)^2\mathscr{A}$. Therefore, $\alpha c\mathscr{A}$ is an idempotent ideal. Conversely, since for every $\alpha \in \mathscr{A}$, there is $0 \neq c \in \mathscr{A}$ where $\alpha c\mathscr{A}$ is an idempotent, implies $\alpha c\mathscr{A} = (\alpha c\mathscr{A})^2$. Hence, $\alpha c = (\alpha c)r(\alpha c)$ for some $r \in \mathscr{A}$. Therefore, $\mathscr{A}$ is Fπ-RR.

□

Theorem 5.2. *Let $\mathscr{A}$ be a duo ring. Then, $\mathscr{A}$ is an Fπ-RR if and only if for every $\alpha \in \mathscr{A}$, there is $0 \neq c \in \mathscr{A}$ such that $\alpha c = u\mathfrak{e}$ for some unit $u \in \mathscr{A}$ and for some idempotent $\mathfrak{e} \in \mathscr{A}$.*

Proof. Let $\mathscr{A}$ be Fπ-RR and $\alpha \in \mathscr{A}$. Then, there exist $0 \neq c \in \mathscr{A}$ and $\mathfrak{b} \in \mathscr{A}$ where $\alpha c = \alpha c\mathfrak{b}\alpha c$. Hence, $\mathfrak{e} = \alpha c\mathfrak{b}$. Since $\mathscr{A}$ is duo, then $\alpha c = \mathfrak{e}\alpha c = \alpha c\mathfrak{e}$. Now, $\alpha c = \mathfrak{e} - \mathfrak{e}^2 + \alpha c = (1 - \mathfrak{e} + \alpha c)\mathfrak{e} = u\mathfrak{e}$ where $u = 1 - \mathfrak{e} + \alpha c$. Notice that u is a unit in $\mathscr{A}$ as $(1-\mathfrak{e}+\alpha c)(1-\mathfrak{e}+\mathfrak{e}\mathfrak{b}) = 1$. Therefore, $u\mathfrak{e} = \alpha c$. Conversely, if $\alpha \in \mathscr{A}$, there is $0 \neq c \in \mathscr{A}$ where $\alpha c = u\mathfrak{e}$ for some unit $u \in \mathscr{A}$ and $\mathfrak{e} \in \mathscr{A}$. Hence $\mathfrak{e} = r\alpha c$; where r is the inverse of u in $\mathscr{A}$. Thus, $\alpha c = \alpha c\mathfrak{e} = \alpha cr\alpha c$, for some $r \in \mathscr{A}$. Therefore, $\mathscr{A}$ is an Fπ-RR. □

Theorem 5.3. *If $\mathscr{A}$ be a duo ring, then $\mathscr{A}$ is an Fπ-RR if and only if for every $\alpha \in \mathscr{A}$, there is $0 \neq c \in \mathscr{A}$ where $\alpha c\mathscr{A}$ is a pure ideal.*

Proof. Let $\mathscr{A}$ be an Fπ-RR. Then for every $\alpha \in \mathscr{A}$, there exist $0 \neq c \in \mathscr{A}$ and $\mathfrak{b} \in \mathscr{A}$ where $\alpha c = \alpha c\mathfrak{b}\alpha c$. Now $\alpha cr \in \alpha c\mathscr{A}$ for some $r \in \mathscr{A}$ and as $\mathscr{A}$ is duo, we obtain $\alpha cr = \bar{r}\alpha c = \bar{r}\alpha c\mathfrak{b}\alpha c = \alpha cr.\alpha c\bar{\mathfrak{b}} = \alpha crd$ where $d = \alpha c\bar{\mathfrak{b}} \in \alpha c\mathscr{A}$ and both $\bar{r}$, $\bar{\mathfrak{b}}$ are in $\mathscr{A}$. Thus, $\alpha c\mathscr{A}$ is a pure ideal. Conversely, let $\alpha \in \mathscr{A}$, there is $0 \neq c \in \mathscr{A}$ where $\alpha c\mathscr{A}$ is a pure ideal. Then $\alpha c = \alpha cy$ for some $y \in \alpha c\mathscr{A}$. Since $\mathscr{A}$ is duo, we have $y \in \mathscr{A}\alpha c$. Therefore, $y = \mathfrak{b}\alpha c$ for some $\mathfrak{b} \in \mathscr{A}$. Hence, $\alpha c = \alpha c\mathfrak{b}\alpha c$ so $\mathscr{A}$ is Fπ-RR. □

6. FGP-INJECTIVE RINGS AND Fπ-REGULAR RINGS

This section presents the connection between FGP-injective rings and Fπ-RR.

Theorem 6.1. *If for all $\alpha \in \mathscr{A}$, there exists $0 \neq c \in \mathscr{A}$ where $\alpha c\mathscr{A}$ is FGP-injective, then $\mathscr{A}$ is Fπ-RR.*

Proof. Assume $ac\mathscr{A}$ is FGP-injective and $\alpha \in \mathscr{A}$. Consider the identity mapping $\mathbf{i} : \alpha c\mathscr{A} \to \alpha c\mathscr{A}$ for some $0 \neq c \in \mathscr{A}$. Since $\alpha c\mathscr{A}$ is FGP-injective, there is $z \in \alpha c\mathscr{A}$ where $\mathbf{i}(\alpha cb) = z\alpha c\mathfrak{b}$ for all $\mathfrak{b} \in \mathscr{A}$. Now, $\alpha c = \mathbf{i}(\alpha c) = z\alpha c$ and since $z \in \alpha c\mathscr{A}$ implies $z = \alpha cr$ for some $r \in \mathscr{A}$. Hence, $\alpha c = \alpha cr\alpha c$. □

Theorem 6.2. *Let $\mathscr{A}$ be reduced ring with every maximal right ideal is FGP-injective. Then, $\mathscr{A}$ is an Fπ-RR.*

Proof. Let $\alpha \in \mathscr{A}$. We claim that $\alpha c\mathscr{A} + \mathbf{r}(\alpha c) = \mathscr{A}$. If not, there is a maximal right ideal $\mathcal{M} \subset \mathscr{A}$ where $\alpha c\mathscr{A} + \mathbf{r}(\alpha c) \subseteq \mathcal{M}$. Define the canonical injective $\mathbf{v} : \alpha c\mathscr{A} \to \mathcal{M}$ by $\mathbf{v}(\alpha c\mathfrak{b}) = \alpha c\mathfrak{b}$ for every $\mathfrak{b} \in \mathscr{A}$. Since $\mathcal{M}$ is an FGP-injective, there is $y \in \mathcal{M}$ where $\mathbf{v}(\alpha c\mathfrak{b}) = y\alpha c\mathfrak{b}$. Hence, $\alpha c = y\alpha c$ which is $\alpha c - y\alpha c = 0$, and then $(1 - y) \in \mathbf{l}(\alpha c)$. Since $\mathscr{A}$ is reduced, $(1 - y) \in \mathbf{r}(\alpha c)$. So, $1 - y \in \mathcal{M}$ and as a result $1 \in \mathcal{M}$ which is a contradiction. Now, $1 = \alpha cd + n$ for some $d \in \mathscr{A}$ and $n \in \mathbf{r}(\alpha c)$ and this implies that $\alpha cd\alpha c + n\alpha c = \alpha c$. Since $\mathscr{A}$ is reduced, $n \in \mathbf{l}(\alpha c)$. Thus, $\alpha c = \alpha cd\alpha c$. Hence, $\mathscr{A}$ is an Fπ-RR. □

Theorem 6.3. *Let $\mathscr{A}$ be a ring where for every $\alpha \in \mathscr{A}$, $\mathscr{A}/\mathbf{r}(\alpha c)$ is an FGP-injective for some $0 \neq c \in \mathscr{A}$. Then $\mathscr{A}$ is an $F\pi$-RR.*

Proof. Fix $0 \neq \alpha \in \mathscr{A}$ and let $\mathbf{v} : \alpha c\mathscr{A} \to \mathscr{A}/\mathbf{r}(\alpha c)$ be a function defined by $\mathbf{v}(\alpha cx) = x + \mathbf{r}(\alpha c)$ for every $x \in \mathscr{A}$. Clearly, $\mathbf{v}$ is a well-defined right homomorphism. Since $\mathscr{A}/\mathbf{r}(\alpha c)$ is FGP-injective, there is $\bar{y} \in \mathscr{A}/\mathbf{r}(\alpha c)$ where $\mathbf{v}(\alpha cx) = (y + \mathbf{r}(\alpha c))\alpha cx = y\alpha cx + \mathbf{r}(\alpha c)$. In particular, $\mathbf{v}(\alpha c) = 1 + \mathbf{r}(\alpha c)$, and this implies that $1 + \mathbf{r}(\alpha c) = y\alpha c + \mathbf{r}(\alpha c)$. Hence, $1 - y\alpha c \in \mathbf{r}(\alpha c)$ and then $\alpha c(1 - y\alpha c) = 0 = \alpha c - \alpha cy\alpha c$. Therefore, $\alpha c = \alpha cy\alpha c$ and $\mathscr{A}$ is Fπ-RR. □

Theorem 6.4. *Let $\mathscr{A}$ be a duo ring. If for every $\alpha \in \mathscr{A}$, $\mathscr{A}/(\alpha c\mathscr{A})$ is FGP-injective with $\mathbf{r}(\alpha c) = 0$ for some $0 \neq c \in \mathscr{A}$, then $\mathscr{A}$ is a division ring.*

Proof. Let $0 \neq \alpha \in \mathscr{A}$. Define $\mathbf{v} : \alpha c\mathscr{A} \to \mathscr{A}/(\alpha c\mathscr{A})$ by $\mathbf{v}(\alpha cx) = x + \alpha c\mathscr{A}$, for every $x \in \mathscr{A}$. Since $\mathbf{r}(\alpha c) = 0$, then $\mathbf{v}$ is well defined. Since $\mathscr{A}/(\alpha c\mathscr{A})$ is FGP-injective, there is $\bar{z} \in \mathscr{A}/\alpha c\mathscr{A}$ where $\mathbf{v}(\alpha cx) = (z + \alpha c\mathscr{A})\alpha cx = z\alpha cx + \alpha c\mathscr{A}$. So, $\mathbf{v}(\alpha c) = z\alpha c + \alpha c\mathscr{A}$. In particular, $\mathbf{v}(\alpha c) = 1 + \alpha c\mathscr{A}$ hence $1 - z\alpha c \in \alpha c\mathscr{A}$. But $z\alpha c \in \mathscr{A}\alpha c = \alpha c\mathscr{A}$, implies $1 \in \alpha c\mathscr{A}$. Hence $\alpha c\mathscr{A} = \mathscr{A}$. Thus $\alpha c\mathfrak{b} = 1$ for some $\mathfrak{b} \in \mathscr{A}$. Let $c\mathfrak{b} = m$, thus $\mathscr{A}$ is a division ring. □

Theorem 6.5. *Let $\mathscr{A}$ be a ring and for every $\alpha \in \mathscr{A}$, $\alpha c\mathscr{A}$ is a right annihilator generated by the same element for some $0 \neq c \in \mathscr{A}$. If $\mathscr{A}/\alpha c\mathscr{A}$ is FGP-injective, then $\mathscr{A}$ is Fπ-RR.*

Proof. Let $\alpha \in \mathscr{A}$. Define $\mathbf{v} : \alpha c\mathscr{A} \to \mathscr{A}/(\alpha c\mathscr{A})$ by $\mathbf{v}(\alpha cx) = x + \alpha c\mathscr{A}$, for every $x \in \mathscr{A}$. Clearly, $\mathbf{v}$ is a well defined right homomorphism. Since $\mathscr{A}/(\alpha c\mathscr{A})$ is FGP-injective, there is $\bar{z} \in \mathscr{A}/\alpha c\mathscr{A}$ where $\mathbf{v}(\alpha cx) = (z + \alpha c\mathscr{A})\alpha cx = z\alpha cx + \alpha c\mathscr{A}$. So, $\mathbf{v}(\alpha c) = z\alpha c + \alpha c\mathscr{A}$. In particular, $\mathbf{v}(\alpha c) = 1 + \alpha c\mathscr{A}$ hence $1 - z\alpha c \in \alpha c\mathscr{A} = r(\alpha c)$. Thus $\alpha c = \alpha cz\alpha c$. Therefore, $\mathscr{A}$ is Fπ-RR. □

7. 2-PRIMAL RINGS AND $F^*\pi$-REGULAR RINGS

This section provides the relation between 2-PR $F^*\pi$-RR and RG-R, Sπ-RR and π-RR.

Lemma 7.1. *If $\mathscr{A}$ is a 2-PR, and $\mathscr{A}/\mathbb{P}(\mathscr{A})$ is Fπ-RR. Then every prime ideal of $\mathscr{A}$ is maximal.*

Proof. Let $P \subset \mathscr{A}$ be a prime ideal. Since $\mathscr{A}$ is 2-PR, by the above lemma, there is a minimal prime ideal M of $\mathscr{A}$ which is completely prime. Since, $\mathscr{A}/\mathsf{M}$ is Fπ-RR, then for every $0 \neq \alpha + \mathsf{M} \in \mathscr{A}/\mathsf{M}$, there exist $0 \neq c + \mathsf{M} \in \mathscr{A}/\mathsf{M}$ and $\mathfrak{b} + \mathsf{M} \in \mathscr{A}/\mathsf{M}$ where $(\alpha + \mathsf{M})(c + \mathsf{M}) = (\alpha + \mathsf{M})(c + \mathsf{M})(\mathfrak{b} + \mathsf{M})(\alpha + \mathsf{M})(c + \mathsf{M})$. Hence $\alpha(c - c\mathfrak{b}\alpha c) \in \mathsf{M}$. As $\alpha \notin \mathsf{M}$ we have $(c - c\mathfrak{b}\alpha c) \in \mathsf{M}$ and then $(1 - c\mathfrak{b}\alpha)c \in \mathsf{M}$. However, since $c \notin \mathsf{M}$, we have $1 - c\mathfrak{b}\alpha \in \mathsf{M}$. Thus $1 + \mathsf{M} = (c\mathfrak{b} + \mathsf{M})(\alpha + \mathsf{M})$ and therefore $\mathscr{A}/\mathsf{M}$ is a division ring. Hence, M is maximal and so is P. □

Corollary 7.2. *Let $\mathscr{A}$ be 2-PR. If $\mathscr{A}$ is $F^*\pi$-RR, then $\mathscr{A}$ is a π-RR/ Sπ-RR.*

Corollary 7.3. *Let $\mathscr{A}$ be 2-PR. If $\mathscr{A}$ is $F^*\pi$-RR, then every prime factor ring of $\mathscr{A}$ forms a division ring.*

Proof. Suppose $\mathscr{A}$ is $F^*\pi$-RR, and notice that using Theorem 3.11, $\mathscr{A}/P$ is $F^*\pi$-RR. As a result, $\mathscr{A}/P$ is an Fπ-RR for every prime ideal P. Since $\mathscr{A}$ is 2-PR, then by Lemma 7.1, $\mathscr{A}/P$ is a division ring. □

Lemma 7.4. *Let $\mathscr{A}$ be reduced. If $\mathscr{A}$ is $F^*\pi$-RR, then $\mathscr{A}$ is a RG-R/ π-RR / Sπ-RR.*

Lemma 7.5. *Let $\mathscr{A}$ be reduced. If $\mathscr{A}$ is $F^*\pi$-RR, then every prime factor ring of $\mathscr{A}$ is a division ring.*

Proof. Suppose $\mathscr{A}$ is a reduced $F^*\pi$-RR. Then it is 2-PR $F^*\pi$-RR and by Corollary 7.3, every prime factor ring is a division ring.

□

Lemma 7.6. *Let $\mathscr{A}$ be a 2-PR. If $\mathscr{A}$ is $F^*\pi$-RR, then $\mathscr{A}/\mathbb{P}(\mathscr{A})$ is RG-R.*

Proof. Let $\mathscr{A}$ be $F^*\pi$-RR. Since $\mathscr{A}$ is 2-PR, then $\mathscr{A}/\mathbb{P}(\mathscr{A})$ is reduced and the result holds by Lemma 7.4. □

Lemma 7.7. *Let $\mathscr{A}$ be NI-ring. If $\mathscr{A}$ is $F^*\pi$-RR, then $\mathscr{A}/\mathbb{N}(\mathscr{A})$ is RG-R.*

Proof. Suppose $\mathscr{A}$ is an NI-ring and $F^*\pi$-RR, then $\mathscr{A}/\mathbb{N}(\mathscr{A})$ is reduced $F^*\pi$-RR and hence by Lemma 7.4, $\mathscr{A}/\mathbb{N}(\mathscr{A})$ is RG-R. □

8. Conclusion

In this paper, we discovered Fπ-RR which are extension of both π-RR and RG-R and clarify some important properties. The properties of these rings are also investigated under both duo rings and FGP-injective rings. In addition, we explored $F^*\pi$-RR and showed its relations to Fπ-RR. Such an extension plays a vital role to reformulate the applications of regular rings on algebraic K-theory and algebraic geometry (specifically for the smoothness of varieties).

Authors' Contributions

The final draft of the work has been read and approved by all writers. Each author made an equal contribution to this work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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