APPROXIMATING SOLUTION OF SYSTEM OF VARIATIONAL INEQUALITY CONSTRAINED SPLIT COMMON FIXED POINT PROBLEM IN HILBERT SPACES

F. M. NKWUDA[1,*], J. A. OGUNTUASE[1], H. A. ABASS[2], M. APHANE[2]

[1]Department of Mathematics, Federal University of Agriculture, Abeokuta, Nigeria

[2]Department of Mathematics and Applied Mathematics, Sefako Makgato Health Science University, P.O. Box 94, Pretoria 0204, South Africa

[*]Corresponding author: nkwudafm@funaab.edu.ng

Received Mar. 4, 2024

Abstract. We present a new modified viscosity technique in this article for solving the solution of the system of variational inequality problem, as well as the fixed point of single-valued ρ-strictly pseudocontractive mappings and finite families of multi-valued quasi-nonexpansive mappings within real Hilbert spaces. Using our iterative method, we establish a strong convergence result for approximating the solution of the aforementioned problems without any strict condition being imposed. We present some consequences and applications to validate our main result. Our result complements and generalizes some related results in literature.

2020 Mathematics Subject Classification. 47H06; 47H09; 47J05; 47J25.

Key words and phrases. Split variational inequality problem; fixed point problem; strictly pseudocontractive mapping; multi-valued mapping, Hilbert spaces.

1. Introduction

Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. The Split Feasibility Problem (SFP) introduced in 1994 by Censor and Elfving [8] is to find a point

$$x \in C \text{ such that } Ax \in Q, \tag{1}$$

where A is an $m \times n$ real matrix. The SFP is widely used in various domains, including signal processing, radiation therapy treatment planning, medical image reconstruction and phase retrieval, (for example see [1,3,5,8,12] and the references therein).

Based on SFP (1), Censor et al. [11] introduced the following Split Variational Inequality Problem

DOI: 10.28924/APJM/11-57

(SVIP) which is to find $x^* \in C$ such that

$$\langle f(x^*), x - x^* \rangle \geq 0, \ \forall\, x \in C, \tag{2}$$

and such that

$$y^* = Ax^* \in Q \text{ solves } \langle g(y^*), y - y^* \rangle \geq 0, \ \forall\, y \in Q; \tag{3}$$

where f and g are given mappings. Several authors working in this approach have taken a look at the split common fixed point problem (SCFPP), which is an extension of the SFP (1), (see [2,13,15–17,20] and the references contained in). Let H_1 and H_2 be real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Let $S : H_1 \to H_1$ and $T : H_2 \to H_2$ be two mappings with nonempty fixed point sets $F(S)$ and $F(T)$, respectively. The SCFPP is to find a point

$$x \in F(S) \text{ such that } Ax \in F(T). \tag{4}$$

Observe that when we set $S := P_C$ and $T := P_Q$ to be metric projections where P_C and P_Q, SCFPP (4) reduces to SFP (1).

For solving (2)-(3), Censor *et al.* [7] introduced the following iterative algorithm:

$$x_{n+1} = P_C(I - \lambda f)(x_n + \gamma_n B^*(P_Q(I - g)Bx_n)),$$

for each $n \in \mathbb{N}$. They demonstrated that there is a high convergence of the sequence produced by their technique to (2)-(3).

Recently, Tian and Jiang [21] introduced the following iterative method for approximating solution of (2)-(3):

$$\begin{cases} y_n = P_C(x_n - \gamma_n B^*(I - T)Bx_n), \\ t_n = P_C(y_n - \lambda_n f(y_n)), \\ x_{n+1} = P_C(y_n - \lambda_n f(t_n)), \end{cases}$$

for each $n \in \mathbb{N}$, where $\{\gamma_n\} \subset [c, d]$ for some $c, d \in (0, \frac{1}{k})$, B is a bounded linear operator and T is a nonexpansive mapping. They also established a weak convergence result.

In this article, we consider the following problem:

$$\begin{cases} \text{find } p \in C := \bigcap_{i=1}^{m} F(V_i) \text{ such that } \langle S_i(p), q - p \rangle \geq 0 \\ \forall\, i = 1, 2, \cdots, m, \ q \in C, \\ \text{and } y^* = Bx^* := F(T), \end{cases} \tag{5}$$

where $B : H_1 \to H_2$ is a bounded linear operator, $V_i : H_1 \to CB(H_1)$ is a finite family of quasi-nonexpansive mappings and $S_i : H_1 \to H_1$ is a finite family of σ_i-ism mappings.

Motivated by the results of [7], [21] and other related results in the literature. We introduced the split common fixed point problem of finite family of multi-valued quasi-nonexpansive and strictly pseudocontractive mapping together with a system of variational inequality problem in the framework of real Hilbert spaces. We establish a strong convergence result using a modified viscosity iterative method without imposing any compactness condition. In addition, we state some corollary and application to validate our main result. The result presented in this paper extends many related results in the literature.

2. Preliminaries

We present a few well-known and helpful results that are required for the main theorem's proof. We indicate strong and weak convergence in the sequel by "$\to$" and "$\rightharpoonup$", respectively.

Assume H to be a real Hilbert space, let C be a closed nonempty and convex subset of H. Let $T : C \to C$ be a single-valued mapping, then a point $x \in C$ is called a fixed point of T if $Tx = x$. However, if T is a multi-valued mapping, then a point $x \in C$ is called a fixed point of T, if $x \in Tx$. We denote by $F(T)$, the set of all fixed points of T.

A single-valued mapping $T : H \to H$ is called

(i) nonexpansive, if

$$||Tx - Ty|| \leq ||x - y||, \ \forall \, x, y \in H; \tag{6}$$

(ii) strongly nonexpansive, if T satisfies (i) and

$$\lim_{n\to\infty} ||(x_n - y_n) - (Tx_n - y_n)|| = 0,$$

whenever $\{x_n\}$ and $\{y_n\}$ are bounded sequences in H and

$$\lim_{n\to\infty} (||x_n - y_n|| - ||Tx_n - Ty_n||) = 0;$$

(iii) averaged nonexpansive, if it can be written as

$$T = (1 - \alpha)I + \alpha S,$$

Here, I is the identity operator on H, $\alpha \in (0, 1)$, and $S : H \to H$ is a nonexpansive mapping,

(iv) firmly nonexpansive , if

$$||Tx - Ty||^2 \leq \langle x - y, Tx - Ty\rangle, \ \forall \, x, y \in H;$$

(v) k-strictly pseudocontractive, if for $0 \leq k < 1$,

$$||Tx - Ty||^2 \leq ||x - y||^2 + k||(I - T)x - (I - T)y||^2, \ \forall \, x, y \in H;$$

(vi) monotone, if

$$\langle Tx - Ty, x - y\rangle \geq 0, \ \forall \, x, y \in H;$$

(vii) α-inverse strongly monotone (α-ism) if a constant $\alpha > 0$ exists and

$$\langle Tx - Ty, x - y\rangle \geq \alpha ||Tx - Ty||^2, \; \forall\, x, y \in H.$$

For H being a real Hilbert space , one can easily see that (v) is equivalent to

$$\langle Tx - Ty, x - y\rangle \leq ||x - y||^2 - \frac{1-\lambda}{2}||(I - T)x - (I - T)y||^2.$$

The intimate relationship between the class of pseudocontractive mappings and the well-known class of monotone mappings makes them particularly noteworthy. The zeroes of the monotone mapping $A := I - T$ are fixed points of the pseudocontractive mapping T.
Numerious writers have examined the category of strictly pseudocontractive mappings and the class of nonexpansive mappings together with certain optimization problems, (see [4,9,10] and the references contained in). Let C be a closed, nonempty, convex subset of H, the real Hilbert space. There is a single nearest point $P_C x$ in C such that

$$||x - P_C x|| \leq ||x - y||, \; \forall\, y \in C$$

exists for each point $x \in H$.
P_C is called the metric projection of H onto C and it is well known that P_C is a nonexpansive mapping of H onto C that satisfies the inequality:

$$||P_C x - P_C y|| \leq \langle x - y, P_C x - P_C y\rangle.$$

Moreover, $P_C x$ is characterized by the following properties:

$$\langle x - P_C x, y - P_C x\rangle \leq 0,$$

and

$$||x - y||^2 \geq ||x - P_C x||^2 + ||y - P_C x||^2, \; \forall\, x \in H, \; y \in C.$$

Let $CB(C)$ denote the family of nonempty closed bounded subset of C, the Hausdorff metric on $CB(C)$ is defined by

$$\mathcal{H}(A, B) = \max\Big\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\Big\} \text{ for } A, B \in CB(C),$$

where $d(x, C) = \inf\{||x - y|| : y \in C\}$.
A multi-valued mapping T is said to be *L-Lipschitzian* if there exists $L > 0$ such that

$$\mathcal{H}(Tx, Ty) \leq L||x - y||, \; x, y \in C. \tag{7}$$

In (7), if $L \in (0, 1)$, then T is called a contraction while T is called nonexpansive if $L = 1$.
Also, $T : C \to CB(C)$ is said to be *quasi-nonexpansive*, if $F(T) \neq \emptyset$ and

$$\mathcal{H}(Tx, Ty) \leq ||x - y||, \; \forall\, x \in C, y \in F(T).$$

We now state some of the results needed to establish our strong convergence result.

Lemma 2.1 [26] Let H be a real Hilbert space, then for all $x, y \in H$ and $\alpha \in (0,1)$, the following inequalities holds:

$$||\alpha x + (1-\alpha)y||^2 = \alpha||x||^2 + (1-\alpha)||y||^2 - \alpha(1-\alpha)||x-y||^2.$$

$$2\langle x, y\rangle = ||x||^2 + ||y||^2 - ||x-y||^2 = ||x+y||^2 - ||x||^2 - ||y||^2.$$

Lemma 2.2 [22] Let H be a real Hilbert space and $T : H \to H$ be a nonlinear mapping, then the following hold.

(i) f is nonexpansive if and only if the complement $I - f$ is $\frac{1}{2}$-ism.

(ii) f is ν-ism and $\gamma > 0$, then γf is $\frac{\nu}{\gamma}$-ism.

(iii) f is averaged if and only if the complement $I - f$ is ν-ism for some $\nu > \frac{1}{2}$. Indeed, for $\beta \in (0,1)$, f is β-averaged if and only if $I - f$ is $\frac{1}{2\beta}$-ism.

(iv) If f_1 is β_1-averaged and f_2 is β_2-averaged, where $\beta_1, \beta_2 \in (0,1)$, then the composite $f_1 f_2$ is β-averaged, where $\beta = \beta_1 + \beta_2 - \beta_1\beta_2$.

(v) If f_1 and f_2 are averaged and have a common fixed point, then $F(f_1 f_2) = F(f_1) \cap F(f_2)$.

Lemma 2.3 [21] Let H_1 and H_2 be real Hilbert spaces. Let C be a nonempty, closed and convex subset of H_1. Let $S : H_2 \to H_2$ be a nonexpansive mapping and let $B : H_1 \to H_2$ be a bounded linear operator. Suppose that $C \cap B^{-1}F(S) \neq \emptyset$. Let $\gamma > 0$ and $x^* \in H_1$. Then the following are equivalent.

(i) $x^* = P_C(I - \gamma B^*(I - S)B)x^*$;

(ii) $0 \in B^*(I - S)Bx^* + N_C x^*$;

(iii) $x^* \in C \cap B^{-1}F(S)$.

. **Lemma 2.4** [18] Let E be a uniformly convex Banach space, $B_r(0) := \{x \in E : ||x|| \leq r\}$ be a closed ball with center 0 and radius $r > 0$. Then there exists a continuous strictly increasing and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that

$$\left\|\sum_{n=1}^{m} \lambda_n x_n\right\|^2 \leq \sum_{n=1}^{m} \lambda_n ||x_n||^2 - \lambda_i\lambda_j g(||x_i - x_j||)$$

for any $i, j \in \mathbb{N}$, $i < j$, where $\{x_1, x_2, \cdots, x_m\} \subset B_r(0)$ and $\lambda_n \geq 0$, $\sum\limits_{n=1}^{m} \lambda_n = 1$.

Lemma 2.5 [23] Let H_1 and H_2 be real Hilbert spaces. Let $B : H_1 \to H_2$ be a bounded linear operator with $B \neq 0$, and $S : H_2 \to H_2$ be a nonexpansive mapping. Then $B^*(I - S)B$ is $\frac{1}{2||B||^2}$-ism.

Lemma 2.6 [24] Let C be a nonempty, closed and convex subset of a real Hilbert space H and $T : C \to C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at 0, i.e., if $\{x_n\}$ converges weakly to $x \in C$ and $\{x_n - Tx_n\}$ converges strongly to 0, then $x = Tx$.

Lemma 2.7 [25] Let H be a real Hilbert space and $S : H \to H$ be k-strictly pseudocontractive mapping with $k \in [0,1)$. Let $T_\mu := \mu I + (1-\mu)S$, where $\mu \in [\beta, 1)$. Then,

(i) $F(S) = F(T_\mu)$,

(ii) T_μ is a nonexpansive mapping.

Lemma 2.8 [6] Let C be a nonempty, closed and convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if the following inequality holds.

$$\langle x - z, z - y \rangle \geq 0, \ \forall\, y \in C.$$

Lemma 2.9 [19] Let $\{\alpha_n\}$ be sequence of nonnegative real numbers, $\{a_n\}$ be sequence of real numbers in $(0, 1)$ such that $\sum_{n=1}^{\infty} a_n = \infty$ and $\{b_n\}$ be a sequence of real numbers. Assume that

$$a_{n+1} \leq (1 - \alpha_n) a_n + \alpha_n b_n, \ \forall\, n \geq 1.$$

If $\limsup_{k \in \infty} b_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition

$$\liminf_{k \in \infty} (a_{n_{k+1}} - a_{n_k}) \geq 0,$$

then $\lim_{k \to \infty} a_k = 0$.

3. Main Results

In this article, we modified viscosity technique for solving the solution of the system of variational inequality problem, as well as the fixed point of single-valued ρ-strictly pseudocontractive mappings and finite families of multi-valued quasi-nonexpansive mappings within real Hilbert spaces. We also establish a strong convergence result, present some consequences and application to validate our result. Our main result is stated as follows:

Theorem 3.1 Let H_1 and H_2 be a real Hilbert spaces and C and Q be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $T : H_2 \to H_2$ be ρ-strictly pseudocontractive mapping and $B : H_1 \to H_2$ be a bounded linear operator with B^* being the adjoint of B such that $B \neq 0$. Let $V_i : H_1 \to CB(H_1)$ be a finite family of multi-valued quasi-nonexpansive mappings and for each $i \geq 1$, V_i is demiclosed at 0 and $\bigcap_{i=1}^{m} F(V_i) \neq \emptyset$. For $i = 1, 2, \cdots, m$, $S_i : H_1 \to H_1$ be finite σ_i-inverse strongly monotone mappings. Assume that $\Delta := \{q \in \bigcap_{j=1}^{m} (F(V_i) \cap S_i^{-1}(0)) : Bq \in F(T)\}$ is nonempty and $g : H_1 \to H_1$ is a contraction mapping with coefficient $\mu \in (0, 1)$, then the sequence $\{x_n\}$ generated for arbitrary $x_1, u \in H_1$ is defined by:

$$\begin{cases} u_n = P_C(x_n - \gamma_n B^*(I - U_\gamma)Bx_n), \\ y_n = (I - \alpha^{(m)}\lambda_n^{(m)} S_m) \circ \cdots \circ (I - \alpha^{(2)}\lambda_n^{(2)} S_2) \circ (I - \alpha^{(1)}\lambda_n^{(1)} S_1) u_n \\ x_{n+1} = \eta_n g(y_n) + (\beta_{n,0} - \eta_n) y_n + \sum_{i=1}^{m} \beta_{n,i} z_n^i, \end{cases} \tag{8}$$

where $U_\gamma := \gamma I + (I - \gamma)T$ with $\gamma \in [\rho, 1)$, $z_n^i \in V_i y_n$, $\{\gamma_n\} \subset [a, b]$ for some $a, b \in \left(0, \frac{1}{\|B\|^2}\right)$ and $\{\eta_n\}$, $\{\beta_{n,i}\}$ are sequences in (0,1) that satisfy the following conditions:

(i) $\lim_{n\to\infty} \eta_n = 0$ and $\sum_{n=1}^{\infty} \eta_n = \infty$,

(ii) for each $i \geq 1$, $\liminf_{n\to\infty} \beta_{n,0}\beta_{n,i} > 0$ and $\eta_n < \beta_{n,0}$ for each $i \geq 1$,

(iii) for each $p \in \bigcap_{i=1}^{m} F(S_j)$, $S_j p = \{p\}$, for each $i \geq 1$,

(iv) $\{\alpha^{(i)}\lambda^{(i)}\} \subset [a_i, b_i] \subset (0, 2\sigma_i)$, for each $i \geq 1$.

Then the sequence $\{x_n\}$ converges strongly to $q = P_\triangle g(q)$.

Proof. From Lemma 2.5, Lemma 2.2 (ii), (iii) and (iv), we obtain that $P_C(I - \gamma_n B^*(I - U_\gamma)B)$ is $\frac{1+\gamma_n||B||^2}{2}$-averaged. This implies that $P_C(I - \gamma_n B^*(I - U_\gamma)B) = (1 - \delta_n)I + \delta_n U_n$, where $\delta_n = \frac{1+\gamma_n||B||^2}{2}$ and U_n is nonexpansive for each $n \geq 1$. Thus, u_n can be re-written as

$$u_n = (1 - \delta_n)x_n + \delta_n U_n x_n. \tag{9}$$

Also, let

$$\begin{cases} \psi_n^{(1)} = (I - \alpha^{(1)}\lambda_n^{(1)} S_1)u_n \\ \psi_n^{(2)} = (I - \alpha^{(2)}\lambda_n^{(2)} S_2)\psi_n^{(1)} \\ \vdots \\ \psi_n^{(m)} = y_n = (I - \alpha^{(m)}\lambda_n^{(m)} S_m)\psi_n^{(m-1)}. \end{cases} \tag{10}$$

Now, by applying Lemma 2.5 and the fact that S_1 is σ_1-inverse strongly monotone, then we can rewrite $\psi_n^{(1)}$ as

$$\psi_n^{(1)} = (1 - \theta_n^{(1)})u_n + \theta_n^{(1)} D_n^{(1)} u_n, \tag{11}$$

where $D_n^{(1)}$ is a nonexpansive mapping and $\theta_n^{(1)} = \frac{\alpha^{(1)}\lambda_n^{(1)}}{2\sigma}$ $\forall\, n \in \mathbb{N}$. From (11) and Lemma 2.1, it is obvious that

$$\begin{aligned} \|\psi_n^{(1)} - p\|^2 &= \|(1 - \theta_n^{(1)})u_n + \theta_n^{(1)} D_n^{(1)} u_n - p\|^2 \\ &\leq (1 - \theta_n^{(1)})\|u_n - p\|^2 + \theta_n^{(1)}\|D_n^{(1)}u_n - p\|^2 - \theta_n^{(1)}(1 - \theta_n^{(1)})\|D_n^{(1)}u_n - u_n\|^2 \\ &\leq \|u_n - p\|^2 - \theta_n^{(1)}(1 - \theta_n^{(1)})\|D_n^{(1)}u_n - u_n\|^2. \end{aligned} \tag{12}$$

Following the same argument as in (12) for $i \in \{2, 3, 4, \cdots, m\}$, we obtain

$$\|\psi_n^{(i)} - p\|^2 \leq \|\psi_n^{(i-1)} - p\|^2 - \theta_n^{(i)}(1 - \theta_n^{(i)})\|D_n^i \psi_n^{(i-1)} - \psi_n^{(i-1)}\|^2. \tag{13}$$

From (11) and Lemma 2.1, we obtain that

$$\begin{aligned} \|u_n - p\|^2 &= \|(1 - \delta_n)x_n + \delta_n U_n x_n - p\|^2 \\ &\leq (1 - \delta_n)\|x_n - p\|^2 + \delta_n\|U_n x_n - p\|^2 - \delta_n(1 - \delta_n)\|U_n x_n - x_n\|^2 \\ &\leq \|x_n - p\|^2 - \delta_n(1 - \delta_n)\|U_n x_n - x_n\|^2. \end{aligned} \tag{14}$$

Using (8), the convexity of $\|.\|^2$ and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\eta_n g(y_n) + (\beta_{n,0}-\eta_n)y_n + \sum_{i=1}^{m}\beta_{n,i}z_n^i - p\|^2 \\
&= \|(\beta_{n,0}-\eta_n)(y_n-p) + \sum_{i=1}^{m}\beta_{n,i}(z_n^i-p) + \eta_n(g(y_n)-p)\|^2 \\
&\le (\beta_{n,0}-\eta_n)\|y_n-p\|^2 + \sum_{i=1}^{m}\beta_{n,i}\|z_n^i-p\|^2 + \eta_n\|g(y_n)-p\|^2 \\
&\le (\beta_{n,0}-\eta_n)\|y_n-p\|^2 + \sum_{i=1}^{m}\beta_{n,i}(d(z_{n,i},V_i(p)))^2 + \eta_n\|g(y_n)-g(p)+g(p)-p\|^2 \\
&\le (\beta_{n,0}-\eta_n)\|y_n-p\|^2 + \sum_{i=1}^{m}\mathcal{H}^2(V_iy_n,V_ip) + \eta_n\big[\|g(y_n)-g(p)\|^2 \\
&+ \|g(p)-p\|^2 + 2\langle g(y_n)-g(p), g(p)-p\rangle\big] \\
&\le (\beta_{n,0}-\eta_n)\|y_n-p\|^2 + \sum_{i=1}^{m}\beta_{n,i}\|y_n-p\|^2 + \eta_n\big[\|g(y_n)-g(p)\|^2 \\
&+ \|g(p)-p\|^2 + 2\|g(y_n)-g(p)\|\ \|g(p)-p\|\big] \\
&\le (1-\eta_n)\|y_n-p\|^2 + 2\eta_n\big[\mu^2\|y_n-p\|^2 + \|g(p)-p\|^2\big]. \qquad (15)
\end{aligned}$$

On substituting (12) and (13) in (15), we get

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le (1-(1-2\mu^2)\eta_n)\big[\|u_n-p\|^2 - \theta_n^{(1)}(1-\theta_n^{(1)})\|D_n^{(1)}u_n-u_n\|^2 \\
&\cdots - \theta_n^{(m)}(1-\theta_n^{(m)})\|D_n^{(m-1)}\psi_n^{(m-1)} - \psi_n^{(m-1)}\|^2\big] + 2\eta_n\|g(p)-p\|^2 \qquad (16) \\
&(1-(1-2\mu^2)\eta_n)\|x_n-p\|^2 + 2\eta_n\|g(p)-p\|^2 \\
&= (1-(1-2\mu^2)\eta_n)\|x_n-p\|^2 + (1-2\mu^2)\eta_n\frac{2\|g(p)-p\|^2}{1-2\mu^2} \\
&\le \max\Big\{\|x_n-p\|^2, \frac{2\|g(p)-p\|^2}{1-2\mu^2}\Big\} \\
&\vdots \\
&\le \max\Big\{\|x_1-p\|^2, \frac{2\|g(p)-p\|^2}{1-2\mu^2}\Big\}.
\end{aligned}$$

Hence, $\{x_n\}$ is bounded. Consequently, $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ are bounded sequences.
By applying Lemma 2.1 in (8), (12) and (14), we obtain that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\eta_n g(y_n) + (\beta_{n,0}-\eta_n)y_n + \sum_{i=1}^{m}\beta_{n,i}z_n^i - p\|^2 \\
&= \|(\beta_{n,0}-\eta_n)(y_n-p) + \sum_{i=1}^{m}\beta_{n,i}(z-n^i-p) + \eta_n(g(y_n)-p)\|^2
\end{aligned}$$

$$\leq \|(\beta_{n,0}-\eta_n)(y_n-p)+\sum_{i=1}^{m}\beta_{n,i}(z_n^i-p)\|^2+\eta_n\langle x_{n+1}-p,g(y_n)-p\rangle$$

$$\leq \left[(\beta_{n,0}-\eta_n)\|y_n-p\|+\sum_{i=1}^{m}\beta_{n,i}\|z_n^i-p\|^2\right]^2+2\eta_n\langle x_{n+1}-p,g(y_n)-p\rangle$$

$$\leq (1-\eta_n)\|y_n-p\|^2+2\eta_n\langle x_{n+1}-p,g(y_n)-p\rangle$$

$$\leq (1-\eta_n)\|u_n-p\|^2-(1-\eta_n)\theta_n^{(1)}(1-\theta_n^{(1)})\|D_n^{(1)}u_n-u_n\|^2$$

$$-\cdots-(1-\eta_n)\theta_n^{(m)}(1-\theta_n^{(m)})\|D_n^{(m-1)}\psi_n^{(m-1)}-\psi_n^{(m-1)}\|^2+2\eta_n\langle x_{n+1}-p,g(y_n)-p\rangle$$

$$\leq (1-\eta_n)\|x_n-p\|^2-(1-\eta_n)\theta_n^{(1)}(1-\theta_n^{(1)})\|D_n^{(1)}u_n-u_n\|^2$$

$$-\cdots-(1-\eta_n)\theta_n^{(m)}(1-\theta_n^{(m)})\|D_n^{(m-1)}\psi_n^{(m-1)}-\psi_n^{(m-1)}\|^2$$

$$-\delta_n(1-\delta_n)(1-\eta_n)\|U_nx_n-x_n\|^2+2\eta_n\langle x_{n+1}-p,g(y_n)-p\rangle. \quad (17)$$

Suppose that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ satisfying the condition

$$\limsup_{k\to\infty}\left\{\|x_{n_k}-p\|^2-\|x_{n_{k+1}}-p\|\right\}\leq 0. \quad (18)$$

Consider such a sequence, then we obtain from (17) that

$$\begin{aligned}\limsup_{k\to\infty}\left(\delta_{n_k}(1-\delta_{n_k})(1-\eta_{n_k})\|U_{n_k}x_{n_k}-x_{n_k}\|^2\right) &\leq \limsup_{k\to\infty}\left((1-\eta_{n_k})\|x_{n_k}-p\|^2-\|x_{n_{k+1}}-p\|^2\right.\\ &\quad\left.+2\eta_{n_k}\langle x_{n_{k+1}}-p,g(y_{n_k})-p\rangle\right)\\ &\leq \limsup_{k\to\infty}\left(\|x_{n_k}-p\|^2-\|x_{n_{k+1}}-p\|^2\right)\\ &= -\liminf_{k\to\infty}\left(\|x_{n_{k+1}}-p\|^2-\|x_{n_k}-p\|^2\right)\\ &\leq 0. \end{aligned} \quad (19)$$

Since $\delta_{n_k}=\frac{1+\gamma_{n_k}\|B\|^2}{2}$, then by the condition on γ_{n_k}, we obtain that

$$\lim_{k\to\infty}\|U_{n_k}x_{n_k}-x_{n_k}\|=0. \quad (20)$$

Following the same approach as in (19), we have

$$\begin{aligned}&\limsup_{k\to\infty}\left((1-\eta_{n_k})\theta_{n_k}^{(1)}(1-\theta_{n_k}^{(1)})\|D_{n_k}^{(1)}u_{n_k}-u_{n_k}\|^2+\right.\\ &\left.\cdots+(1-\eta_{n_k})\theta_{n_k}^{(m)}(1-\theta_{n_k}^{(m)})\|D_{n_k^{(m-1)}}\psi_{n_k}^{(m-1)}-\psi_{n_k}^{(m-1)}\|^2\right)\\ &\leq \limsup_{k\to\infty}\left(\|x_{n_k}-p\|^2-\|x_{n_{k+1}}-p\|^2\right)\\ &-\liminf_{k\to\infty}\left(\|x_{n_{k+1}}-p\|^2-\|x_{n_k}-p\|^2\right)\\ &\leq 0.\end{aligned} \quad (21)$$

It is obvious from (21) that

$$\lim_{k\to\infty}\left(\|D_{n_k}^{(1)}u_{n_k}-u_{n_k}\|=\|D_{n_k}^{(2)}\psi_{n_k}^{(1)}-\psi_{n_k}^{(1)}\|=\cdots=\|D_{n_k}^{(m)}\psi_{n_k}^{(m-1)}-\psi_{n_k}^{(m-1)}\|\right)=0. \tag{22}$$

From (10) and (22), we have that

$$\lim_{k\to\infty}\left(\|\psi_{n_k}^{(1)}-u_{n_k}\|=\|\psi_{n_k}^{(2)}-\psi_{n_k}^{(1)}\|=\cdots=\|y_{n_k}-\psi_{n_k}^{(m-1)}\|\right)=0. \tag{23}$$

On summing up (22) and (23), we obtain

$$\lim_{k\to\infty}\|y_{n_k}-u_{n_k}\|=0. \tag{24}$$

From Lemma 2.1 and Lemma 2.4, we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\eta_n g(y_n)+(\beta_{n,0}-\eta_n)y_n+\sum_{i=1}^m\beta_{n,i}z_n^i-p\|^2\\
&= \|\beta_{n,0}(y_n-p)+\sum_{i=1}^m\beta_{n,i}(z_n^i-p)\eta_n(g(y_n)-y_n)\|^2\\
&= \|\beta_{n,0}(y_n-p)+\sum_{i=1}^m\beta_{n,i}(z_n^i-p)\|^2+\eta_n^2\|g(y_n)-y_n\|^2\\
&+2\eta_n\langle g(y_n)-y_n,\beta_{n,0}(y_n-p)+\sum_{i=1}^m\beta_{n,i}(z_n^i-p)\rangle\\
&\le \|y_n-p\|^2-\beta_{n,0}\beta_{n,i}h(\|y_n-z_n^{(i)})+\eta_n^2\|g(y_n)-y_n\|^2\\
&+2\eta_n\langle g(y_n)-y_n,\beta_{n,0}(y_n-p)+\sum_{i=1}^m\beta_{n,i}(z_n^i-p)\rangle\\
&\le \|u_n-p\|^2-\beta_{n,0}\beta_{n,i}h(\|y_n-z_n^i\|)+\eta_n\big[\eta_n\|g(y_n)-y_n\|^2\\
&+2\langle g(y_n)-y_n,\beta_{n,0}(y_n-p)+\sum_{i=1}^m\beta_{n,i}(z_n^i-p)\rangle\big]\\
&\le \|x_n-p\|^2-\beta_{n,0}\beta_{n,i}h(\|y_n-z_n^i\|)+\eta_n\big[\eta_n\|g(y_n)-y_n\|^2\\
&+2\langle g(y_n)-y_n,\beta_{n,0}(y_n-p)+\sum_{i=1}^m\beta_{n,i}(z_n^i-p)\rangle\big].
\end{aligned} \tag{25}$$

Thus, by applying (18) and using the same approach as in (21), we have

$$\begin{aligned}
&\limsup_{k\to\infty}\left[\beta_{n_k,0}\beta_{n_k,i}h\|y_{n_k}-z_{n_k}^i\|\right] \qquad (26)\\
&\le \limsup_{k\to\infty}\Big[\|x_{n_k}-p\|^2+\eta_{n_k}\big[\eta_{n_k}\|g(y_{n_k})-y_{n_k}\|+2\langle g(y_{n_k})-y_{n_k},\beta_{n_k,0}(y_{n_k}-p)\\
&+\sum_{i=1}^m\beta_{n_k,i}(z_{n_k}^i-p)\rangle-\|x_{n_{k+1}}-p\|^2\Big]\\
&\le \lim_{k\to\infty}\left[\|x_{n_k}-p\|^2-\|x_{n_{k+1}}-p\|^2\right]
\end{aligned}$$

$$= -\liminf_{k\to\infty}\left[\|x_{n_{k+1}}-p\|^2-\|x_{n_k}-p\|^2\right]$$
$$\leq 0. \tag{27}$$

Thus,

$$\lim_{k\to\infty}\beta_{n_k,0}\beta_{n_k,i}h\|y_{n_k}-z^i_{n_k}\|=0. \tag{28}$$

Hence, by condition (2) of (8), we get

$$\lim_{k\to\infty}h\|y_{n_k}-z^i_{n_k}\|=0. \tag{29}$$

Using the condition of h as stated in Lemma 2.4, we obtain that

$$\lim_{k\to\infty}\|y_{n_k}-z^i_{n_k}\|=0. \tag{30}$$

Thus

$$\lim_{k\to\infty}d(y_{n_k},V_iy_{n_k})\leq\lim_{k\to\infty}h\|y_{n_k}-z^i_{n_k}\|=0. \tag{31}$$

Also from (9) and (20), we have

$$\lim_{k\to\infty}\|u_{n_k}-x_{n_k}\|=\lim_{k\to\infty}\delta_{n_k}\|U_{n_k}x_{n_k}-x_{n_k}\|=0, \tag{32}$$

and from condition (1) of (8), we get

$$\lim_{k\to\infty}\|x_{n_{k+1}}-y_{n_k}\|=\lim_{k\to\infty}\eta_{n_k}\|g(y_{n_k})-y_{n_k}\|=0. \tag{33}$$

From (24) and (32), we obtain

$$\lim_{k\to\infty}\|y_{n_k}-x_{n_k}\|=0. \tag{34}$$

Similarly, from (33) and (34), we obtain that

$$\lim_{k\to\infty}\|x_{n_{k+1}}-x_{n_k}\|=0. \tag{35}$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_l}}\}$ which converges weakly to $q\in H_1$. Also, since $\{u_{n_k}\}$ and $\{y_{n_k}\}$ are bounded, there exist subsequences $\{u_{n_{k_l}}\}$ of $\{u_{n_k}\}$ and $\{y_{n_{k_l}}\}$ of $\{y_{n_k}\}$ which converge weakly to q respectively. Now, applying (31) and Lemma 2.6 for $i=1,2,\cdots,m$, we obtain that $q\in F(V_i)$ and thus $q\in\bigcap_{i=1}^{m}F(V_i)$. In addition, using (24) and Lemma 2.6, we obtain that $q\in S_i^{-1}(0), i=1,2,\cdots,m$. By Lemma 2.5 and the fact that $\{B^*(I-U_\gamma)Bx_{n_k}\}$ is bounded. It then follows from the firmly nonexpansive property of P_C that

$$||P_C(I-\gamma_{n_{k_l}}B^*(I-U_\gamma)B)x_{n_{k_l}}-P_C(I-\overline{\gamma}B^*(I-U_\gamma)B)x_{n_{k_l}}||$$
$$\leq\|\gamma_{n_{k_l}}-\overline{\gamma}\|\,||B^*(I-U_\gamma)Bx_{n_{k_l}}||\to 0,\ l\to\infty. \tag{36}$$

Thus, we obtain from (32) that

$$\lim_{k\to\infty} ||u_{n_{k_l}} - P_C(I - \overline{\gamma}B^*(I - U_\gamma)B)x_{n_{k_l}}|| = 0,$$

which implies from Lemma 2.6 that $q \in F(P_C(I - \overline{\gamma}B^*(I - U_\gamma)B))$. Hence, by Lemma 2.3, we get

$$q \in C \cap B^{-1}F(U_\gamma),$$

which implies that $Bz \in F(U_\gamma) = F(T)$. Therefore, we conclude that $q \in \Delta$.

Next, we show that $\limsup_{k\to\infty}\langle x_{n_{k+1}} - q, g(y_{n_{k_l}}) - q\rangle \leq 0$, where $q = P_\Delta g(q)$. Now, since $\{x_{n_{k_l}}\}$ converges weakly to p, we obtain from the property of P_Δ and (34) that

$$\begin{aligned}\limsup_{k\to\infty}\langle x_{n_{k_l}+1} - q, g(y_{n_{k_l}}) - q\rangle &= \lim_{l\to\infty}\langle x_{n_{k_l}+1} - q, g(x_{n_{k_l}}) - q\rangle \\ &= \langle p - q, g(p) - q\rangle.\end{aligned}$$

Hence, we obtain that

$$\limsup_{k\to\infty}\langle x_{n_{k_l}+1} - q, g(y_{n_{k_l}}) - q\rangle \leq \langle p - q, g(p) - q\rangle \leq 0. \tag{37}$$

From (17), we have that

$$\|x_{n+1} - q\|^2 \leq (1 - \eta_n)\|x_n - q\|^2 + 2\eta_n\langle x_{n+1} - q, g(y_n) - q\rangle. \tag{38}$$

On substituting (37) into (38) and applying Lemma 2.9, we conclude that $\{x_n\}$ converges strongly to q as required.

We now state some consequences of our main result.

Corollary 3.2 Let H_1 and H_2 be a real Hilbert spaces and C be a nonempty, closed and convex subset of H_1. Let $T : H_2 \to H_2$ be a nonexpansive mapping and $B : H_1 \to H_2$ be a bounded linear operator with B^* being the adjoint of B such that $B \neq 0$. Let $V_i : H_1 \to CB(H_1)$ be a finite family of multi-valued quasi-nonexpansive mappings and for each $i \geq 1$, V_i is demiclosed at 0 and $\bigcap_{i=1}^{m} F(V_i) \neq \emptyset$. For $i = 1, 2, \cdots, m, S : H_1 \to H_1$ be σ-inverse strongly monotone mapping. Assume that $\Delta := \{q \in \bigcap_{j=1}^{m} F(V_i) \cap S^{-1}(0) : Bq \in F(T)\}$ is nonempty and $g : H_1 \to H_1$ is a contraction mapping with coefficient $\mu \in (0, 1)$, then the sequence $\{x_n\}$ generated for arbitrary $x_1, u \in H_1$ is defined by:

$$\begin{cases} u_n = P_C(x_n - \gamma_n B^*(I - T)Bx_n), \\ y_n = (I - \alpha\lambda_n S_m)u_n \\ x_{n+1} = \eta_n g(y_n) + (\beta_{n,0} - \eta_n)y_n + \sum_{i=1}^{m}\beta_{n,i}z_n^i, \end{cases} \tag{39}$$

where $z_n^i \in V_i y_n, \{\gamma_n\} \subset [a, b]$ for some $a, b \in \left(0, \frac{1}{\|B\|^2}\right)$ and $\{\eta_n\}$, $\{\beta_{n,i}\}$ are sequences in (0,1) that satisfy the following conditions:

(i) $\lim_{n\to\infty}\eta_n = 0$ and $\sum_{n=1}^{\infty}\eta_n = \infty$,

(ii) for each $i \geq 1$, $\liminf_{n\to\infty} \beta_{n,0}\beta_{n,i} > 0$ and $\eta_n < \beta_{n,0}$ for each $i \geq 1$,

(iii) for each $p \in \bigcap_{i=1}^{m} F(S_j)$, $S_j p = \{p\}$, for each $i \geq 1$,

(iv) $\{\alpha\lambda_n\} \subset [a, b] \subset (0, 2\sigma)$.

Then the sequence $\{x_n\}$ converges strongly to $q = P_\Delta g(q)$.

Theorem 3.3 Let H_1 and H_2 be a real Hilbert spaces, C and Q be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $f : H_2 \to H_2$ be θ-inverse strongly monotone mapping and $B : H_1 \to H_2$ be a bounded linear operator with B^* being the adjoint of B such that $B \neq 0$. Let $V_i : H_1 \to CB(H_1)$ be a finite family of multi-valued quasi-nonexpansive mappings and for each $i \geq 1$, V_i is demiclosed at 0 and $\bigcap_{i=1}^{m} F(V_i) \neq \emptyset$. For $i = 1, 2, \cdots, m$, $S_i : H_1 \to H_1$ be finite σ_i-inverse strongly monotone mappings. Assume that $\Delta := \{q \in \bigcap_{j=1}^{m} (F(V_i) \cap S_i^{-1}(0)) : Bq \in VI(Q, f)\}$ is nonempty and $g : H_1 \to H_1$ is a contraction mapping with coefficient $\mu \in (0, 1)$, then the sequence $\{x_n\}$ generated for arbitrary $x_1, u \in H_1$ is defined by:

$$
\begin{cases}
u_n = P_C(x_n - \gamma_n B^*(I - P_Q(I - \theta f))Bx_n), \\
y_n = (I - \alpha^{(m)}\lambda_n^{(m)} S_m) \circ \cdots \circ (I - \alpha^{(2)}\lambda_n^{(2)} S_2) \circ (I - \alpha^{(1)}\lambda_n^{(1)} S_1)u_n \\
x_{n+1} = \eta_n g(y_n) + (\beta_{n,0} - \eta_n)y_n + \sum_{i=1}^{m} \beta_{n,i} z_n^i,
\end{cases}
\tag{40}
$$

where $z_n^i \in V_i y_n, \{\gamma_n\} \subset [a, b]$ for some $a, b \in \big(0, \frac{1}{\|B\|^2}\big)$ and $\{\eta_n\}$, $\{\beta_{n,i}\}$ are sequences in (0,1) that satisfy the following conditions:

(i) $\lim_{n\to\infty} \eta_n = 0$ and $\sum_{n=1}^{\infty} \eta_n = \infty$,

(ii) for each $i \geq 1$, $\liminf_{n\to\infty} \beta_{n,0}\beta_{n,i} > 0$ and $\eta_n < \beta_{n,0}$ for each $i \geq 1$,

(iii) for each $p \in \bigcap_{i=1}^{m} F(S_j)$, $S_j p = \{p\}$, for each $i \geq 1$,

(iv) $\{\alpha^{(i)}\lambda^{(i)}\} \subset [a_i, b_i] \subset (0, 2\sigma_i)$, for each $i \geq 1$.

Then the sequence $\{x_n\}$ converges strongly to $q = P_\Delta g(q)$.

Proof. Using Lemma 2.8, we can easily see that $q \in VI(Q, F)$ if and only if $z = P_Q(I - \theta f)$ for $\omega > 0$, and for $\theta \in (0, 2\omega)$, $P_Q(I - \theta f)$ is nonexpansive. On substituting $U_\gamma = P_Q(I - \theta f)$ in (8), the result follows.

4. Application

In this section, we apply our results to solve split minimization problem.

4.1. **Computing Split Minimization Problem.**

For a nonempty, closed and convex subset of a real Hilbert space H, the convex minimization problem is to find $p \in C$ such that

$$
\phi(p) = \min_{x \in C} \phi(x),
\tag{41}
$$

where ϕ is a real-valued convex function. We denote by $argmin_{x\in C}\phi(x)$, the solution set of (41).

Lemma 4.1 Let C be a nonempty, closed and convex subset of a real Hilbert space H and ϕ be a convex function of H into $\mathbb{R}$. If ϕ is differentiable, then q is a solution of (41) if and only if $q \in VI(C, \bigtriangledown f)$.

Proof. Let q be a solution of (41). For each $x \in C,\ q + \rho(x-q) \in C,\ \rho \in (0,1)$. Since ϕ is differentiable, we have

$$\langle \bigtriangledown \phi(q), x-q\rangle = \lim_{\rho\to o^+} \frac{\phi(q+\rho(x-q)) - \phi(q)}{\rho} \geq 0.$$

Conversely, if $q \in VI(C, \bigtriangledown\phi)$, i.e. $\langle \bigtriangledown\phi(q), x-q\rangle \geq 0,\ \forall\, x \in C$. Since ϕ is convex, we have

$$\phi(x) \geq \phi(q) + \langle \bigtriangledown\phi(q), x-q\rangle \geq \phi(q).$$

Hence, q is a solution of (41).

By applying Theorem 3.3 and Lemma 4.1, we obtain the following result:

Theorem 4.2 Let H_1 and H_2 be a real Hilbert spaces, C and Q be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $\phi : H_2 \to \mathbb{R}$ be a differentiable convex function and $\bigtriangledown\phi$ be θ-inverse strongly monotone mapping. Let $B : H_1 \to H_2$ be a bounded linear operator with B^* being the adjoint of B such that $B \neq 0$. Let $V_i : H_1 \to CB(H_1)$ be a finite family of multi-valued quasi-nonexpansive mappings and for each $i \geq 1,\ V_i$ is demiclosed at 0 and $\bigcap_{i=1}^{m} F(V_i) \neq \emptyset$. For $i = 1, 2, \cdots, m, S_i : H_1 \to H_1$ be finite σ_i-inverse strongly monotone mappings. Assume that $\Delta := \{q \in \bigcap_{j=1}^{m}(F(V_i) \cap S_i^{-1}(0)) : Bq \in argmin_{y\in Q}\phi(y)\}$ is nonempty and $g : H_1 \to H_1$ is a contraction mapping with coefficient $\mu \in (0,1)$, then the sequence $\{x_n\}$ generated for arbitrary $x_1, u \in H_1$ is defined by:

$$\begin{cases} u_n = P_C(x_n - \gamma_n B^*(I - P_Q(I - \theta \bigtriangledown \phi))Bx_n), \\ y_n = (I - \alpha^{(m)}\lambda_n^{(m)} S_m) \circ \cdots \circ (I - \alpha^{(2)}\lambda_n^{(2)} S_2) \circ (I - \alpha^{(1)}\lambda_n^{(1)} S_1)u_n \\ x_{n+1} = \eta_n g(y_n) + (\beta_{n,0} - \eta_n)y_n + \sum_{i=1}^{m} \beta_{n,i} z_n^i, \end{cases} \tag{42}$$

where $z_n^i \in V_i y_n,\ \theta \in (0, 2\omega), \{\gamma_n\} \subset [a,b]$ for some $a, b \in \left(0, \frac{1}{\|B\|^2}\right)$ and $\{\eta_n\},\ \{\beta_{n,i}\}$ are sequences in (0,1) that satisfy the following conditions:

(i) $\lim_{n\to\infty} \eta_n = 0$ and $\sum_{n=1}^{\infty} \eta_n = \infty$,

(ii) for each $i \geq 1$, $\liminf_{n\to\infty} \beta_{n,0}\beta_{n,i} > 0$ and $\eta_n < \beta_{n,0}$ for each $i \geq 1$,

(iii) for each $p \in \bigcap_{i=1}^{m} F(S_j), S_j p = \{p\}$, for each $i \geq 1$,

(iv) $\{\alpha^{(i)}\lambda^{(i)}\} \subset [a_i, b_i] \subset (0, 2\sigma_i)$, for each $i \geq 1$.

Then the sequence $\{x_n\}$ converges strongly to $q = P_\Delta g(q)$.

Proof. Putting $f = \bigtriangledown\phi$ in Theorem 3.3, we get the desired result by Lemma 4.1.

Remarks 4.3 In our result, it is worth-mentioning that we establish a strong convergence result which is more desirable to the weak convergence obtained in [7, 21]. Also, we were able to dispense the compactness condition during the course of establishing our strong convergence result. Lastly, the method of proof employed in our article looks simple and understanding.

Authors' Contributions

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] H.A. Abass, F.U. Ogbuisi, O.T. Mewomo, Common solution of split equilibrium problem with no prior knowledge of operator norm, U.P.B Sci. Bull., Ser. A. 80 (2018), 175–190.

[2] H.A. Abass, G.C. Ugwunnadi, O.K. Narain, V. Darvish, Inertial extragradient method for solving variational inequality and fixed point problems of a Bregman demigeneralized mapping in a reflexive Banach spaces, Numer. Funct. Anal. Optim. 43 (2022), 933–960. https://doi.org/10.1080/01630563.2022.2069813.

[3] H.A. Abass, O.K. Oyewole, K.O. Aremu, L.O. Jolaoso, Self-adaptive technique with double inertial steps for inclusion problem on Hadamard manifolds, J. Oper. Res. Soc. China. (2024). https://doi.org/10.1007/s40305-024-00537-0.

[4] F. U. Ogbuisi, O. T. Mewomo, Strong convergence result for solving split hierachical variational inequality problem for demicontractive mappings, Adv. Nonlinear Var. Ineq. 22 (2019), 24–39.

[5] C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, Inverse Probl. 20 (2003), 103–120. https://doi.org/10.1088/0266-5611/20/1/006.

[6] L.C. Ceng, Q.H. Ansari, J.C. Yao, Some iterative methods for finding fixed points and for solving constrained convex minimization problems, Nonlinear Anal.: Theory Meth. Appl. 74 (2011), 5286–5302. https://doi.org/10.1016/j.na.2011.05.005.

[7] Y. Censor, A. Gibali, S. Reich, The split variational inequality problem, arXiv:1009.3780v1 [math.OC], (2010). https://arxiv.org/abs/1009.3780v1.

[8] Y. Censor, T. Elfving, A multiprojection algorithm using Bregman projections in a product space, Numer. Algor. 8 (1994), 221–239. https://doi.org/10.1007/bf02142692.

[9] J.K. Kim, S.Y. Cho, X. Qin, Some results on generalized equilibrium problems involving strictly pseudocontractive mappings, Acta Math. Sci. 31 (2011), 2041–2057. https://doi.org/10.1016/s0252-9602(11)60380-9.

[10] C. Huang, X. Ma, On generalized equilibrium problems and strictly pseudocontractive mappings in Hilbert spaces, Fixed Point Theory Appl. 2014 (2014), 145. https://doi.org/10.1186/1687-1812-2014-145.

[11] Y. Censor, A. Gibali, S. Reich, Algorithms for the split variational inequality problem, Numer. Algor. 59 (2012), 301–323. https://doi.org/10.1007/s11075-011-9490-5.

[12] Y. Censor, T. Bortfeld, B. Martin, A. Trofimov, A unified approach for inversion problems in intensity-modulated radiation therapy, Phys. Med. Biol. 51 (2006), 2353–2365. https://doi.org/10.1088/0031-9155/51/10/001.

[13] G.C. Ugwunnadi, H.A. Abass, M. Aphane, O.K. Oyewole, Inertial Halpern-type method for solving split feasibility and fixed point problems via dynamical stepsize in real Banach spaces, Ann. Univ. Ferrara. (2023). https://doi.org/10.1007/s11565-023-00473-6.

[14] Y. Censor, A. Segal, The split common fixed point problem for directed operators, J. Convex Anal. 16 (2009), 587–600.

[15] A. Moudafi, A note on the split common fixed-point problem for quasi-nonexpansive operators, Nonlinear Anal.: Theory Meth. Appl. 74 (2011), 4083–4087. https://doi.org/10.1016/j.na.2011.03.041.

[16] O.K. Oyewole, H.A. Abass, O.T. Mewomo, A strong convergence algorithm for a fixed point constrained split null point problem, Rend. Circ. Mat. Palermo, II. Ser. 70 (2020), 389–408. https://doi.org/10.1007/s12215-020-00505-6.

[17] O.K. Oyewole, H.A. Abass, O.T. Mewomo, A totally relaxed self-adaptive subgradient extragradient scheme for equilibrium problem and fixed point problem in a Banach space, Kragujevac J. Math. 49 (2025), 181–202.

[18] S. Plubtieng, K. Ungchittrakool, Hybrid iterative methods for convex feasibility problems and fixed point problems of relatively nonexpansive mappings in Banach spaces, Fixed Point Theory Appl. 2008 (2009), 583082. https://doi.org/10.1155/2008/583082.

[19] S. Saejung, P. Yotkaew, Approximation of zeros of inverse strongly monotone operators in Banach spaces, Nonlinear Anal.: Theory Meth. Appl. 75 (2012), 742–750. https://doi.org/10.1016/j.na.2011.09.005.

[20] Y. Shehu, P. Cholamjiak, Another look at the split common fixed point problem for demicontractive operators, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM. 110 (2015), 201–218. https://doi.org/10.1007/s13398-015-0231-9.

[21] M. Tian, B.N. Jiang, Weak convergence theorem for a class of split variational inequality problems and applications in a Hilbert space, J. Ineq. Appl. 2017 (2017), 123. https://doi.org/10.1186/s13660-017-1397-9.

[22] H.K. Xu, Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces, Inverse Probl. 26 (2010), 105018. https://doi.org/10.1088/0266-5611/26/10/105018.

[23] W. Takahashi, H.K. Xu, J.C. Yao, Iterative Methods for generalized split feasibility problems in Hilbert spaces, Set-Valued Var. Anal. 23 (2014), 205–221. https://doi.org/10.1007/s11228-014-0285-4.

[24] H.K. Xu, Viscosity approximation methods for nonexpansive mappings, J. Math. Anal. Appl. 298 (2004), 279–291. https://doi.org/10.1016/j.jmaa.2004.04.059.

[25] H. Zhou, Convergence theorems of fixed points for k-strict pseudo-contractions in Hilbert spaces, Nonlinear Anal.: Theory Meth. Appl. 69 (2008), 456–462. https://doi.org/10.1016/j.na.2007.05.032.

[26] C. Chidume, Geometric properties of Banach spaces and nonlinear iterations, Springer London, London, 2009. https://doi.org/10.1007/978-1-84882-190-3.