Asia Pac. J. Math. 2024 **11**:60

INTUITIONISTIC HESITANT FUZZY SET THEORY APPLIED TO UP (BCC)-ALGEBRAS

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Received May 3, 2024

Abstract. Using the theory of hesitant fuzzy sets (HFSs) as a basis, the ideas of intuitionistic hesitant fuzzy BCC-subalgebras, BCC-ideals, and strong BCC-ideals of BCC-algebras are presented, some of their properties are described, and their extensions are proved. These intuitionistic hesitant fuzzy set (IHFS)'s necessary conditions are given, along with their relationship to their complement. Additionally, the idea of weakly prime and prime IHFSs was investigated. The relationships between intuitionistic hesitant fuzzy BCC-subalgebras (BCC-ideals, strong BCC-ideals) and their level subsets are also discussed. The homomorphic pre-images of intuitionistic hesitant fuzzy BCC-subalgebras (BCC-ideals, strong BCC-ideals), as well as other associated features, are also explored in BCC-algebras.

2020 Mathematics Subject Classification. 03G25; 03E72.

Keywords and phrases: BCC-algebra; intuitionistic hesitant fuzzy BCC-subalgebra; intuitionistic hesitant fuzzy BCC-ideal; intuitionistic hesitant fuzzy strong BCC-ideal.

1. Introduction

Zadeh [19] first proposed the idea of fuzzy sets. The theory of fuzzy sets has several applications in real-life situations, and many scholars have researched fuzzy set theory. After introducing the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets. The integration between fuzzy sets and some uncertainty approaches, such as soft sets and rough sets, has been discussed in [1,2,4]. In 2009–2010, Torra and Narukawa [17,18] introduced the notion of HFSs, a function from a reference set to a power set of the unit interval. The notion of HFSs is the other generalization of fuzzy sets. The HFS theories developed by Torra and others have found many applications in mathematics and elsewhere. After the introduction of the notion of HFSs by

DOI: 10.28924/APJM/11-60

Torra and Narukawa [17, 18], several researchers conducted research on the generalizations of the notion of HFSs and their application to many logical algebras, such as in 2012, Zhu, Xu and Xia [20] introduced the notion of dual HFSs, which is a new extension of fuzzy sets. In 2014, Jun, Ahn and Muhiuddin [11] introduced the notions of hesitant fuzzy soft subalgebras and (closed) hesitant fuzzy soft ideals in BCK/BCI-algebras. Jun and Song [13] introduced the notions of (Boolean, prime, ultra, and good) hesitant fuzzy filters and hesitant fuzzy MV-filters of MTL-algebras. In 2023, Iampan et al. [8, 9] introduced the concepts of intuitionistic hesitant fuzzy subalgebras, ideals, and deductive systems of Hilbert algebras. Iampan [6] introduced a new algebraic structure called a UP-algebra, and Mosrijai et al. [15] introduced the notion of HFSs on UP-algebras. The notions of hesitant fuzzy subalgebras, hesitant fuzzy filters, and hesitant fuzzy UP-ideals play an important role in studying the many logical algebras. The concepts of UP-algebras (see [6]) and BCC-algebras (see [14]) are the same concept, as shown by Jun et al. [12] in 2022. In this publication and following investigations, our research team will refer to it as BCC rather than UP out of respect for Komori, who first characterized it in 1984.

This paper introduces the ideas of intuitionistic hesitant fuzzy BCC-subalgebras, BCC-ideals, and strong BCC-ideals of BCC-algebras. It also talks about some of their features and shows how they can be extended using the theory of HFSs as a base. The necessary conditions for those IHFSs and their relation to their complement are provided. The concept of prime and weakly prime IHFSs was also introduced and studied. We also discuss the connections between intuitionistic hesitant fuzzy BCC-subalgebras (BCC-ideals, strong BCC-ideals) and their level subsets. The homomorphic pre-images of intuitionistic hesitant fuzzy BCC-subalgebras (BCC-ideals, strong BCC-ideals) in BCC-algebras are also studied, and some related properties are investigated.

2. Preliminaries

The concept of BCC-algebras (see [14]) can be redefined without the condition (2.6) as follows:

An algebra $X = (X, \cdot, 0)$ of type $(2, 0)$ is called a *BCC-algebra* (see [10]) if it satisfies the following conditions:

$$(\forall x, y, z \in X)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0) \tag{2.1}$$

$$(\forall x \in X)(0 \cdot x = x) \tag{2.2}$$

$$(\forall x \in X)(x \cdot 0 = 0) \tag{2.3}$$

$$(\forall x, y \in X)(x \cdot y = 0 = y \cdot x \ \Rightarrow \ x = y) \tag{2.4}$$

After this, we assign X instead of a BCC-algebra $(X, \cdot, 0)$ until otherwise specified.

We define a binary relation $\leq$ on X as follows:

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x \cdot y = 0) \tag{2.5}$$

In X, the following assertions are valid (see [6]).

$$(\forall x \in X)(x \leq x) \tag{2.6}$$

$$(\forall x, y, z \in X)(x \leq y, y \leq z \Rightarrow x \leq z) \tag{2.7}$$

$$(\forall x, y, z \in X)(x \leq y \Rightarrow z \cdot x \leq z \cdot y) \tag{2.8}$$

$$(\forall x, y, z \in X)(x \leq y \Rightarrow y \cdot z \leq x \cdot z) \tag{2.9}$$

$$(\forall x, y, z \in X)(x \leq y \cdot x, \text{ in particular, } y \cdot z \leq x \cdot (y \cdot z)) \tag{2.10}$$

$$(\forall x, y \in X)(y \cdot x \leq x \Leftrightarrow x = y \cdot x) \tag{2.11}$$

$$(\forall x, y \in X)(x \leq y \cdot y) \tag{2.12}$$

$$(\forall a, x, y, z \in X)(x \cdot (y \cdot z) \leq x \cdot ((a \cdot y) \cdot (a \cdot z))) \tag{2.13}$$

$$(\forall a, x, y, z \in X)(((a \cdot x) \cdot (a \cdot y)) \cdot z \leq (x \cdot y) \cdot z) \tag{2.14}$$

$$(\forall x, y, z \in X)((x \cdot y) \cdot z \leq y \cdot z) \tag{2.15}$$

$$(\forall x, y, z \in X)(x \leq y \Rightarrow x \leq z \cdot y) \tag{2.16}$$

$$(\forall x, y, z \in X)((x \cdot y) \cdot z \leq x \cdot (y \cdot z)) \tag{2.17}$$

$$(\forall a, x, y, z \in X)((x \cdot y) \cdot z \leq y \cdot (a \cdot z)) \tag{2.18}$$

Definition 2.1. [5,6,16] A nonempty subset S of X is called

(1) a *BCC-subalgebra* of X if

$$(\forall x, y \in S)(x \cdot y \in S), \tag{2.19}$$

(2) a *BCC-ideal* of X if

$$0 \in S, \tag{2.20}$$

$$(\forall x, y, z \in X)(x \cdot (y \cdot z), y \in S \Rightarrow x \cdot z \in S), \tag{2.21}$$

(3) a *strong BCC-ideal* of X if (2.20) and

$$(\forall x, y, z \in X)((z \cdot y) \cdot (z \cdot x), y \in S \Rightarrow x \in S). \tag{2.22}$$

Definition 2.2. [17] A *hesitant fuzzy set* (HFS) on a reference set X is defined in term of a function h that when applied to X return a subset of $[0, 1]$, that is, $h : X \to \mathcal{P}([0, 1])$.

Definition 2.3. [3] An *intuitionistic hesitant fuzzy set* (IHFS) on a reference set X is defined in the form $\mathcal{H} = (h, k)$, where h and k are functions that when applied to X return a subset of $[0, 1]$, that is, $h, k : X \to \mathcal{P}([0, 1])$ with the conditions that

$$(\forall x \in X) \left(\begin{array}{l} \sup h(x) + \inf k(x) \leq 1 \\ \inf h(x) + \sup k(x) \leq 1 \end{array} \right). \tag{2.23}$$

Definition 2.4. [15] A HFS h on X is said to be

(1) a *hesitant fuzzy BCC-subalgebra* of X if it satisfies the following property:

$$(\forall x, y \in X)(h(x \cdot y) \supseteq h(x) \cap h(y)) \tag{2.24}$$

(2) a *hesitant fuzzy BCC-ideal* of X if the following conditions hold:

$$(\forall x \in X)(h(0) \supseteq h(x)) \tag{2.25}$$

$$(\forall x, y, z \in X)(h(x \cdot z) \supseteq h(x \cdot (y \cdot z)) \cap h(y)) \tag{2.26}$$

(3) a *hesitant fuzzy strong BCC-ideal* of X if (2.25) and the following condition hold:

$$(\forall x, y, z \in X)(h(x) \supseteq h((z \cdot y) \cdot (z \cdot x)) \cap h(y)) \tag{2.27}$$

Definition 2.5. [17] The *complement* of a HFS h in a reference set X is the HFS $\overline{h}$ defined by $\overline{h}(x) = [0, 1] - h(x)$ for all $x \in X$.

Definition 2.6. [17] The *complement* of an IHFS $\mathcal{H} = (h, k)$ on a reference set X is the IHFS $\overline{\mathcal{H}} = (\overline{k}, \overline{h})$.

3. Intuitionistic hesitant fuzzy BCC-subalgebras/BCC-ideals/strong BCC-ideals

Definition 3.1. An IHFS $\mathcal{H} = (h, k)$ on X is called an *intuitionistic hesitant fuzzy BCC-subalgebra* of X if it satisfies the following property:

$$(\forall x, y \in X) \left(\begin{array}{l} h(x \cdot y) \supseteq h(x) \cap h(y) \\ k(x \cdot y) \subseteq k(x) \cup k(y) \end{array} \right) \tag{3.1}$$

Example 3.2. Let $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	0
2	0	1	0	0	4
3	0	1	2	0	4
4	0	4	2	3	0

Then X is a BCC-algebra. We define an IHFS $\mathcal{H} = (h, k)$ on X as follows:

$$h = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ \{0.1, 0.2, 0.3\} & \{0.1\} & \{0.2\} & \{0.3\} & \{0.3\} \end{pmatrix},$$

$$k = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ \{0.1\} & \{0.1\} & \{0.1, 0.2\} & \{0.1, 0.2, 0.3\} & \{0.1, 0.2, 0.3, 0.4\} \end{pmatrix}$$

Then $\mathcal{H}$ is an intuitionistic hesitant fuzzy BCC-subalgebra of X.

Proposition 3.3. *If $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy BCC-subalgebra of X, then the following property holds:*

$$(\forall x \in X) \begin{pmatrix} h(0) \supseteq h(x) \\ k(0) \subseteq k(x) \end{pmatrix} \tag{3.2}$$

Proof. For any $x \in X$, we have

$$h(0) = h(x \cdot x) \supseteq h(x) \cap h(x) = h(x),$$

$$k(0) = k(x \cdot x) \subseteq k(x) \cup k(x) = k(x).$$

□

Definition 3.4. The *characteristic intuitionistic hesitant fuzzy set* (CIHFS) of a subset A of a set X is defined to be the structure $\chi_A = (h_{\chi_A}, k_{\chi_A})$, where

$$h_{\chi_A}(x) = \begin{cases} [0,1] & \text{if } x \in A \\ \emptyset & \text{otherwise} \end{cases} \text{ and } k_{\chi_A}(x) = \begin{cases} \emptyset & \text{if } x \in A \\ [0,1] & \text{otherwise.} \end{cases}$$

Lemma 3.5. [7] *The constant 0 of X is in a nonempty subset B of X if and only if $h_{\chi_B}(0) \supseteq h_{\chi_B}(x)$ and $k_{\chi_B}(0) \subseteq k_{\chi_B}(x)$ for all $x \in X$.*

Theorem 3.6. *A nonempty subset S of X is a BCC-subalgebra of X if and only if the CIHFS χ_S is an intuitionistic hesitant fuzzy BCC-subalgebra of X.*

Proof. Assume that S is a BCC-subalgebra of X. Let $x, y \in X$.

Case 1: If $x, y \in S$, then $h_{\chi_S}(x) = [0,1]$ and $h_{\chi_S}(y) = [0,1]$. Thus $h_{\chi_S}(x) \cap h_{\chi_S}(y) = [0,1]$. Since S is a BCC-subalgebra of X, $x \cdot y \in S$ and so $h_{\chi_S}(x \cdot y) = [0,1]$. Then $h_{\chi_S}(x \cdot y) = [0,1] \supseteq [0,1] = h_{\chi_S}(x) \cap h_{\chi_S}(y)$. Also, $k_{\chi_S}(x) = \emptyset$ and $k_{\chi_S}(y) = \emptyset$. Thus $k_{\chi_S}(x) \cup k_{\chi_S}(y) = \emptyset$. Since S is a BCC-subalgebra of X, $x \cdot y \in S$ and so $k_{\chi_S}(x \cdot y) = \emptyset$. Then $k_{\chi_S}(x \cdot y) = \emptyset \subseteq \emptyset = k_{\chi_S}(x) \cup k_{\chi_S}(y)$.

Case 2: If $x \in S$ and $y \notin S$, then $h_{\chi_S}(x) = [0,1]$ and $h_{\chi_S}(y) = \emptyset$. Thus $h_{\chi_S}(x) \cap h_{\chi_S}(y) = \emptyset$. Then $h_{\chi_S}(x \cdot y) \supseteq \emptyset = h_{\chi_S}(x) \cap h_{\chi_S}(y)$. Also, $k_{\chi_S}(x) = \emptyset$ and $k_{\chi_S}(y) = [0,1]$. Thus $k_{\chi_S}(x) \cup k_{\chi_S}(y) = [0,1]$. Then $k_{\chi_S}(x \cdot y) \subseteq [0,1] = k_{\chi_S}(x) \cup k_{\chi_S}(y)$.

Case 3: If $x \notin S$ and $y \in S$, then $h_{\chi_S}(x) = \emptyset$ and $h_{\chi_S}(y) = [0,1]$. Thus $h_{\chi_S}(x) \cap h_{\chi_S}(y) = \emptyset$. Then $h_{\chi_S}(x \cdot y) \supseteq \emptyset = h_{\chi_S}(x) \cap h_{\chi_S}(y)$. Also, $k_{\chi_S}(x) = [0,1]$ and $k_{\chi_S}(y) = \emptyset$. Thus $k_{\chi_S}(x) \cup k_{\chi_S}(y) = [0,1]$. Then $k_{\chi_S}(x \cdot y) \subseteq [0,1] = k_{\chi_S}(x) \cup k_{\chi_S}(y)$.

Case 4: If $x \notin S$ and $y \notin S$, then $h_{\chi_S}(x) = \emptyset$ and $h_{\chi_S}(y) = \emptyset$. Thus $h_{\chi_S}(x) \cap h_{\chi_S}(y) = \emptyset$. Hence, $h_{\chi_S}(x \cdot y) \supseteq \emptyset = h_{\chi_S}(x) \cap h_{\chi_S}(y)$. Also, $k_{\chi_S}(x) = [0,1]$ and $k_{\chi_S}(y) = [0,1]$. Thus $k_{\chi_S}(x) \cup k_{\chi_S}(y) = [0,1]$. Then $k_{\chi_S}(x \cdot y) \subseteq [0,1] = k_{\chi_S}(x) \cup k_{\chi_S}(y)$.

Hence, χ_S is an intuitionistic hesitant fuzzy BCC-subalgebra of X.

Conversely, assume that χ_S is an intuitionistic hesitant fuzzy BCC-subalgebra of X. Let $x, y \in S$. Then $h_{\chi_S}(x) = [0,1]$ and $h_{\chi_S}(y) = [0,1]$. Thus $h_{\chi_S}(x \cdot y) \supseteq h_{\chi_S}(x) \cap h_{\chi_S}(y) = [0,1]$, so $h_{\chi_S}(x \cdot y) = [0,1]$. Hence, $x \cdot y \in S$, that is, S is a BCC-subalgebra of X. □

Definition 3.7. An IHFS $\mathcal{H} = (h,k)$ on X is said to be an *intuitionistic hesitant fuzzy BCC-ideal* of X if (3.2) and the following condition hold:

$$(\forall x, y, z \in X) \begin{pmatrix} h(x \cdot z) \supseteq h(x \cdot (y \cdot z)) \cap h(y) \\ k(x \cdot z) \subseteq k(x \cdot (y \cdot z)) \cup k(y) \end{pmatrix} \tag{3.3}$$

Example 3.8. Let $X = \{0,1,2,3\}$ with the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	3
3	0	0	0	0

Then X is a BCC-algebra. We define an IHFS $\mathcal{H} = (h,k)$ on X as follows:

$$h = \begin{pmatrix} 0 & 1 & 2 & 3 \\ \{0.1, 0.2, 0.3\} & \{0.1, 0.2\} & \{0.3\} & \{0.3\} \end{pmatrix},$$

$$k = \begin{pmatrix} 0 & 1 & 2 & 3 \\ \{0.1\} & \{0.1, 0.2\} & \{0.1, 0.2, 0.3\} & \{0.1, 0.2, 0.3\} \end{pmatrix}$$

Then $\mathcal{H}$ is an intuitionistic hesitant fuzzy BCC-ideal of X.

Definition 3.9. An IHFS $\mathcal{H} = (h,k)$ on X is said to be an *intuitionistic hesitant fuzzy strong BCC-ideal* of X if (3.2) and the following condition hold:

$$(\forall x, y, z \in X) \begin{pmatrix} h(x) \supseteq h((z \cdot y) \cdot (z \cdot x)) \cap h(y) \\ k(x) \subseteq k((z \cdot y) \cdot (z \cdot x)) \cup k(y) \end{pmatrix} \tag{3.4}$$

Example 3.10. Let $X = \{0,1,2,3\}$ with the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	0
3	0	1	2	0

Then X is a BCC-algebra. We define an IHFS $\mathcal{H} = (h, k)$ on X as follows:

$$h = \begin{pmatrix} 0 & 1 & 2 & 3 \\ \{0.1, 0.2\} & \{0.1, 0.2\} & \{0.1, 0.2\} & \{0.1, 0.2\} \end{pmatrix},$$

$$k = \begin{pmatrix} 0 & 1 & 2 & 3 \\ \{0.1\} & \{0.1\} & \{0.1\} & \{0.1\} \end{pmatrix}$$

Then $\mathcal{H}$ is an intuitionistic hesitant fuzzy strong BCC-ideal of X.

Theorem 3.11. *Every intuitionistic hesitant fuzzy strong BCC-ideal of X is an intuitionistic hesitant fuzzy BCC-ideal.*

Proof. Let $\mathcal{H} = (h, k)$ be an intuitionistic hesitant fuzzy strong BCC-ideal of X. Then (3.2) holds. Let $x, y, z \in X$. Then

$$\begin{aligned} h(x \cdot z) &\supseteq h((z \cdot y) \cdot (z \cdot (x \cdot z))) \cap h(y) \\ &= h((z \cdot y) \cdot 0) \cap h(y) \\ &\supseteq h(0) \cap h(y) \\ &= h(y) \\ &\supseteq h(x \cdot (y \cdot z)) \cap h(y), \end{aligned}$$

$$\begin{aligned} k(x \cdot z) &\subseteq k((z \cdot y) \cdot (z \cdot (x \cdot z))) \cup k(y) \\ &= k((z \cdot y) \cdot 0) \cup k(y) \\ &\subseteq k(0) \cup k(y) \\ &= k(y) \\ &\subseteq k(x \cdot (y \cdot z)) \cup k(y). \end{aligned}$$

Hence, $\mathcal{H}$ is an intuitionistic hesitant fuzzy BCC-ideal of X. □

The following example shows that the converse of Theorem 3.11 is generally untrue.

Example 3.12. Consider the BCC-algebra X from Example 3.8. We define an IHFS $\mathcal{H} = (h, k)$ on X as follows:

$$h = \begin{pmatrix} 0 & 1 & 2 & 3 \\ \{0.1, 0.2, 0.3\} & \{0.1, 0.2\} & \{0.2\} & \{0.2\} \end{pmatrix},$$

$$k = \begin{pmatrix} 0 & 1 & 2 & 3 \\ \{0.1\} & \{0.1, 0.2\} & \{0.1, 0.2, 0.3\} & \{0.1, 0.2, 0.3\} \end{pmatrix}$$

Then $\mathcal{H}$ is an intuitionistic hesitant fuzzy BCC-ideal of X but not an intuitionistic hesitant fuzzy strong BCC-ideal of X.

Theorem 3.13. *An IHFS $\mathcal{H} = (h, k)$ on X is an intuitionistic hesitant fuzzy strong BCC-ideal of X if and only if h and k are constant HFSs on X.*

Proof. Let $\mathcal{H}$ be an intuitionistic hesitant fuzzy strong BCC-ideal of X. For any $x \in X$, we have

$$\begin{aligned} h(x) &\supseteq h((x \cdot 0) \cdot (x \cdot x)) \cap h(0) \\ &= h(0 \cdot 0) \cap h(0) \\ &= h(0) \cap h(0) \\ &= h(0) \\ &\supseteq h(x), \end{aligned}$$

$$\begin{aligned} k(x) &\subseteq k((x \cdot 0) \cdot (x \cdot x)) \cup k(0) \\ &= k(0 \cdot 0) \cup k(0) \\ &= k(0) \cup k(0) \\ &= k(0) \\ &\subseteq k(x). \end{aligned}$$

Hence, h and k are constant HFSs on X.

Conversely, assume that h and k are constant HFSs on X. Then obviously $\mathcal{H}$ is an intuitionistic fuzzy strong BCC-ideal of X. □

The following two theorems can be proved similarly to Theorem 3.6.

Theorem 3.14. *A nonempty subset S of X is a BCC-ideal of X if and only if the CIHFS χ_S is an intuitionistic hesitant fuzzy BCC-ideal of X.*

Theorem 3.15. *A nonempty subset S of X is a strong BCC-ideal of X if and only if the CIHFS χ_S is an intuitionistic hesitant fuzzy strong BCC-ideal of X.*

Definition 3.16. An IHFS $\mathcal{H} = (h, k)$ on X is called a *prime IHFS* on X if it satisfies the following property:

$$(\forall x, y \in X) \begin{pmatrix} h(x \cdot y) \subseteq h(x) \cup h(y) \\ k(x \cdot y) \supseteq k(x) \cap k(y) \end{pmatrix} \tag{3.5}$$

Definition 3.17. [5] A nonempty subset S of X is called a *prime subset* of X if it satisfies the following property:

$$(\forall x, y \in X)(x \cdot y \in S \Rightarrow x \in S \text{ or } y \in S)$$

Theorem 3.18. [7] *A nonempty subset S of X is a prime subset of X if and only if the CIHFS χ_S is a prime IHFS on X.*

Theorem 3.19. *Let $\mathcal{H} = (h, k)$ be an IHFS on X. Then the following statements are equivalent:*

(1) *$\mathcal{H}$ is a prime intuitionistic hesitant fuzzy BCC-subalgebra (resp., prime intuitionistic hesitant fuzzy BCC-ideal, prime intuitionistic hesitant fuzzy strong BCC-ideal) of X,*

(2) *h and k are constant HFSs on X,*

(3) *$\mathcal{H}$ is an intuitionistic hesitant fuzzy strong BCC-ideal of X.*

Proof. (1)⇔(2): Assume that $\mathcal{H}$ is a prime intuitionistic hesitant fuzzy BCC-subalgebra of X. Then $h(0) \supseteq h(x)$ and $k(0) \subseteq k(x)$ for all $x \in X$. By (2.6), we have $h(0) = h(x \cdot x) \subseteq h(x) \cup h(x) = h(x)$ and $k(0) = k(x \cdot x) \supseteq k(x) \cup k(x) = k(x)$ for all $x \in X$ and so $h(x) = h(0)$ and $k(x) = k(0)$ for all $x \in X$. Hence, h and k are constant HFSs on X.

Conversely, assume that h and k are constant HFSs on X. Hence, we can easily show that $\mathcal{H}$ is a prime intuitionistic hesitant fuzzy BCC-subalgebra of X.

(2)⇔(3): It is straightforward by Theorem 3.13. □

Definition 3.20. [5] A nonempty subset S of X is called a *weakly prime subset* of X if it satisfies the following property:

$$(\forall x, y \in X, x \neq y)(x \cdot y \in S \Rightarrow x \in S \text{ or } y \in S)$$

Definition 3.21. [5] A BCC-subalgebra (resp., BCC-ideal, strong BCC-ideal) S of X is called a *weakly prime BCC-subalgebra* (resp., weakly prime BCC-ideal, weakly prime strong BCC-ideal) of X if S is a weakly prime subset of X.

Definition 3.22. An IHFS $\mathcal{H} = (h, k)$ on X is called a *weakly prime IHFS* on X if it satisfies the following property:

$$(\forall x, y \in X, x \neq y)\begin{pmatrix} h(x \cdot y) \subseteq h(x) \cup h(y) \\ k(x \cdot y) \supseteq k(x) \cap k(y) \end{pmatrix} \tag{3.6}$$

Definition 3.23. An intuitionistic hesitant fuzzy BCC-subalgebra (resp., intuitionistic hesitant fuzzy BCC-ideal, intuitionistic hesitant fuzzy strong BCC-ideal) $\mathcal{H} = (h, k)$ of X is called a *weakly prime intuitionistic hesitant fuzzy BCC-subalgebra* (resp., weakly prime intuitionistic hesitant fuzzy BCC-ideal, weakly prime intuitionistic hesitant fuzzy strong BCC-ideal) of X if $\mathcal{H}$ is a weakly prime IHFS on X.

Theorem 3.24. *For BCC-algebras, the notions of weakly prime intuitionistic hesitant fuzzy strong BCC-ideals and prime intuitionistic hesitant fuzzy strong BCC-ideals coincide.*

Proof. It is straightforward by Theorem 3.13. □

Theorem 3.25. [7] *A nonempty subset S of X is a weakly prime subset of X if and only if the CIHFS χ_S is a weakly prime IHFS on X.*

Theorem 3.26. *A nonempty subset S of X is a weakly prime BCC-subalgebra of X if and only if the CIHFS χ_S is a weakly prime intuitionistic hesitant fuzzy BCC-subalgebra on X.*

Proof. It is straightforward by Theorems 3.6 and 3.25. □

Theorem 3.27. *A nonempty subset S of X is a weakly prime BCC-ideal of X if and only if the CIHFS χ_S is a weakly prime intuitionistic hesitant fuzzy BCC-ideal on X.*

Proof. It is straightforward by Theorems 3.14 and 3.25. □

Theorem 3.28. *A nonempty subset S of X is a weakly prime strong BCC-ideal of X if and only if the CIHFS χ_S is a weakly prime intuitionistic hesitant fuzzy strong BCC-ideal on X.*

Proof. It is straightforward by Theorems 3.15 and 3.25. □

Theorem 3.29. *If an IHFS $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy strong BCC-ideal of X, then the HFSs $h, k, \overline{h}$, and $\overline{k}$ are fuzzy strong BCC-ideals of X.*

Proof. It is straightforward by Theorem 3.13. □

Theorem 3.30. *An IHFS $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy BCC-subalgebra of X if and only if the HFSs h and $\overline{k}$ are hesitant fuzzy BCC-subalgebras of X.*

Proof. Assume that $\mathcal{H}$ is an intuitionistic hesitant fuzzy BCC-subalgebra of X. Then for any $x, y \in X, h(x \cdot y) \supseteq h(x) \cap h(y)$. Hence, h is a hesitant fuzzy BCC-subalgebra of X. Now, for any $x, y \in X$, we have

$$\begin{aligned}\overline{k}(x \cdot y) &= [0,1] - k(x \cdot y) \\ &\supseteq [0,1] - (k(x) \cup k(y)) \\ &= [0,1] - k(x) \cap [0,1] - k(y) \\ &= \overline{k}(x) \cap \overline{k}(y).\end{aligned}$$

Hence, $\overline{k}$ is a hesitant fuzzy BCC-subalgebra of X.

Conversely, assume that the HFSs h and $\overline{k}$ are hesitant fuzzy BCC-subalgebras of X. Then for any $x, y \in X, h(x \cdot y) \supseteq h(x) \cap h(y)$. Now, for any $x, y \in X$, we have $\overline{k}(x \cdot y) \supseteq \overline{k}(x) \cap \overline{k}(y)$. Then

$$\begin{aligned}[0,1] - k(x \cdot y) &\supseteq [0,1] - k(x) \cap [0,1] - k(y) \\ &= [0,1] - (k(x) \cup k(y)), \\ k(x \cdot y) &\subseteq k(x) \cup k(y).\end{aligned}$$

Hence, $\mathcal{H}$ is an intuitionistic hesitant fuzzy BCC-subalgebra of X. □

The following two theorems can be proved similarly to Theorem 3.30.

Theorem 3.31. *An IHFS $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy BCC-ideal of X if and only if the HFSs h and $\overline{k}$ are hesitant fuzzy BCC-ideals of X.*

Theorem 3.32. *An IHFS $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy strong BCC-ideal of X if and only if the HFSs h and $\overline{k}$ are hesitant fuzzy strong BCC-ideals of X.*

Theorem 3.33. *An IHFS $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy strong BCC-ideal of X if and only if the IHFS $\overline{\mathcal{H}} = (\overline{k}, \overline{h})$ is an intuitionistic hesitant fuzzy strong BCC-ideal of X.*

Proof. It is straightforward from Theorem 3.13. □

Theorem 3.34. *An IHFS $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy BCC-ideal of X if and only if the IHFS $\overline{\mathcal{H}} = (\overline{k}, \overline{h})$ is an intuitionistic hesitant fuzzy BCC-ideal of X.*

Proof. Assume that $\mathcal{H}$ is an intuitionistic fuzzy BCC-ideal of X. Then for any $x, y, z \in X$, we have $h(0) \supseteq h(x)$ and $h(x \cdot z) \supseteq h(x \cdot (y \cdot z)) \cap h(y)$. Hence, for any $x, y, z \in X$, we have $\overline{h}(0) = [0, 1] - h(0) \subseteq [0, 1] - h(x) = \overline{h}(x)$ and

$$\begin{aligned}
\overline{h}(x \cdot z) &= [0, 1] - h(x \cdot z) \\
&\subseteq [0, 1] - (h(x \cdot (y \cdot z)) \cap h(y)) \\
&= [0, 1] - h(x \cdot (y \cdot z)) \cup [0, 1] - h(y) \\
&= \overline{h}(x \cdot (y \cdot z)) \cup \overline{h}(y).
\end{aligned}$$

Now, for any $x, y, z \in X$, we have $k(0) \subseteq k(x)$ and $k(x \cdot z) \subseteq k(x \cdot (y \cdot z)) \cup k(y)$. Hence, for any $x, y, z \in X$, we have $\overline{k}(0) = [0, 1] - k(0) \supseteq [0, 1] - k(x) = \overline{k}(x)$ and

$$\begin{aligned}
\overline{k}(x \cdot z) &= [0, 1] - k(x \cdot z) \\
&\supseteq [0, 1] - (k(x \cdot (y \cdot z)) \cup k(y)) \\
&= [0, 1] - k(x \cdot (y \cdot z)) \cap [0, 1] - k(y) \\
&= \overline{k}(x \cdot (y \cdot z)) \cap \overline{k}(y).
\end{aligned}$$

Hence, $\overline{\mathcal{H}} = (\overline{k}, \overline{h})$ is an intuitionistic hesitant fuzzy BCC-ideal of X.

Conversely, assume that the IHFS $\overline{\mathcal{H}} = (\overline{k}, \overline{h})$ is an intuitionistic hesitant fuzzy BCC-ideal of X. Then for any $x, y, z \in X$, we have $\overline{k}(0) \supseteq \overline{k}(x)$ and $\overline{k}(x \cdot z) \supseteq \overline{k}(x \cdot (y \cdot z)) \cap \overline{k}(y)$. Then $[0, 1] - k(0) \supseteq [0, 1] - k(x)$ and $[0, 1] - k(x \cdot z) \supseteq [0, 1] - (k(x \cdot (y \cdot z)) \cup k(y))$, so $k(0) \subseteq k(x)$ and $k(x \cdot z) \subseteq k(x \cdot (y \cdot z)) \cup k(y)$. Now, for any $x, y, z \in X$, we have $\overline{h}(0) \subseteq \overline{h}(x)$ and $\overline{h}(x \cdot z) \subseteq \overline{h}(x \cdot (y \cdot z)) \cup \overline{h}(y)$. Then $[0, 1] - h(0) \subseteq [0, 1] - h(x)$ and $[0, 1] - h(x \cdot z) \supseteq [0, 1] - (h(x \cdot (y \cdot z)) \cup h(y))$, so $h(0) \supseteq h(x)$ and $h(x \cdot z) \supseteq h(x \cdot (y \cdot z)) \cap h(y)$. Hence, $\mathcal{H}$ is an intuitionistic hesitant fuzzy BCC-ideal of X. □

The following two theorems can be proved similarly to Theorem 3.34.

Theorem 3.35. *An IHFS $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy BCC-subalgebra of X if and only if the IHFS $\overline{\mathcal{H}} = (\overline{k}, \overline{h})$ is an intuitionistic hesitant fuzzy BCC-subalgebra of X.*

Theorem 3.36. *An IHFS $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy strong BCC-ideal of X if and only if the IHFS $\overline{\mathcal{H}} = (\overline{k}, \overline{h})$ is an intuitionistic hesitant fuzzy strong BCC-ideal of X.*

Definition 3.37. Let $\mathcal{H} = (h, k)$ be an IHFS on a set X. The IHFSs $\oplus\mathcal{H}$ and $\otimes\mathcal{H}$ are defined as $\oplus\mathcal{H} = (h, \overline{h})$ and $\otimes\mathcal{H} = (\overline{k}, k)$.

Theorem 3.38. *An IHFS $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy BCC-subalgebra of X if and only if the IHFSs $\oplus\mathcal{H}$ and $\otimes\mathcal{H}$ are intuitionistic hesitant fuzzy BCC-subalgebras of X.*

Proof. Let $x, y \in X$. Then

$$\begin{aligned} \overline{h}(x \cdot y) &= [0,1] - h(x \cdot y) \\ &\subseteq [0,1] - (h(x) \cap h(y)) \\ &= ([0,1] - h(x)) \cup ([0,1] - h(y)) \\ &= \overline{h}(x) \cup \overline{h}(y). \end{aligned}$$

Hence, $\oplus\mathcal{H}$ is an intuitionistic hesitant fuzzy BCC-subalgebra of X. Let $x, y \in X$. Then

$$\begin{aligned} \overline{k}(x \cdot y) &= [0,1] - k(x \cdot y) \\ &\supseteq [0,1] - (k(x) \cup k(y)) \\ &= ([0,1] - k(x)) \cap ([0,1] - k(y)) \\ &= \overline{k}(x) \cap \overline{k}(y). \end{aligned}$$

Hence, $\otimes\mathcal{H}$ is an intuitionistic hesitant fuzzy BCC-subalgebra of X.

Conversely, assume that $\oplus\mathcal{H}$ and $\otimes\mathcal{H}$ are intuitionistic hesitant fuzzy BCC-subalgebras of X. Then for any $x, y \in X$, we have $h(x \cdot y) \supseteq h(x) \cap h(y)$ and $k(x \cdot y) \subseteq k(x) \cup k(y)$. Hence, $\mathcal{H}$ is an intuitionistic hesitant fuzzy BCC-subalgebra of X. □

Theorem 3.39. *If $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy BCC-subalgebra of X, then the sets $X_h = \{x \in X \mid h(x) = h(0)\}$ and $X_k = \{x \in X \mid k(x) = k(0)\}$ are BCC-subalgebras of X.*

Proof. Let $x, y \in X_h$. Then $h(x) = h(0) = h(y)$ and so $h(x \cdot y) \supseteq h(x) \cap h(y) = h(0)$. By using Proposition 3.3, we have $h(x \cdot y) = h(0)$; hence, $x \cdot y \in X_h$. Again, let $x, y \in X_k$. Then $k(x) = k(0) = k(y)$ and so $k(x \cdot y) \subseteq k(x) \cup k(y) = k(0)$. Again, by Proposition 3.3, we have $k(x \cdot y) = k(0)$; hence, $x \cdot y \in X_k$. Hence, X_h and X_k are BCC-subalgebras of X. □

The following two theorems can be proved similarly to Theorem 3.38.

Theorem 3.40. *An IHFS $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy BCC-ideal of X if and only if the IHFSs $\oplus\mathcal{H}$ and $\otimes\mathcal{H}$ are intuitionistic hesitant fuzzy BCC-ideals of X.*

Theorem 3.41. *An IHFS $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy strong BCC-ideal of X if and only if the IHFSs $\oplus\mathcal{H}$ and $\otimes\mathcal{H}$ are intuitionistic hesitant fuzzy strong BCC-ideals of X.*

The following two theorems can be proved similarly to Theorem 3.39.

Theorem 3.42. *If $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy BCC-ideal of X, then the sets $X_h = \{x \in X \mid h(x) = h(0)\}$ and $X_k = \{x \in X \mid k(x) = k(0)\}$ are BCC-ideals of X.*

Theorem 3.43. *If $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy strong BCC-ideal of X, then the sets $X_h = \{x \in X \mid h(x) = h(0)\}$ and $X_k = \{x \in X \mid k(x) = k(0)\}$ are strong BCC-ideals of X.*

Lemma 3.44. *If $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy BCC-ideal of X, then*

$$(\forall x, y, z, w \in X)\left(x \leq w \cdot (y \cdot z) \Rightarrow \begin{cases} h(x \cdot z) \supseteq h(w) \cap h(y) \\ k(x \cdot z) \subseteq k(w) \cup k(y) \end{cases} \right). \tag{3.7}$$

Proof. Let $x, y, z, w \in X$ be such that $x \leq w \cdot (y \cdot z)$. Then $x \cdot (w \cdot (y \cdot z)) = 0$. Thus

$$\begin{aligned} h(x \cdot z) &\supseteq h(x \cdot (y \cdot z)) \cap h(y) \\ &\supseteq h(x \cdot (w \cdot (y \cdot z))) \cap h(w) \cap h(x) \\ &= h(0) \cap h(w) \cap h(y) \\ &= h(w) \cap h(y), \end{aligned}$$

$$\begin{aligned} k(x \cdot z) &\subseteq k(x \cdot (y \cdot z)) \cup k(y) \\ &\subseteq k(x \cdot (w \cdot (y \cdot z))) \cup h(w) \cup h(x) \\ &= k(0) \cup k(w) \cup k(y) \\ &= k(w) \cup k(y). \end{aligned}$$

□

Lemma 3.45. *If $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy BCC-ideal of X, then*

$$(\forall x, y, z \in X)\left(x \leq y \cdot z \Rightarrow \begin{cases} h(x \cdot z) \supseteq h(y) \\ k(x \cdot z) \subseteq k(y) \end{cases} \right). \tag{3.8}$$

Proof. Let $x, y, z \in X$ be such that $x \leq y \cdot z$. By Lemma 3.44, put $w = 0$. Then $x \leq 0 \cdot (y \cdot z)$. Hence

$$h(x \cdot z) \supseteq h(0) \cap h(y) = h(y),$$

$$k(x \cdot z) \subseteq k(0) \cup k(y) = k(y).$$

□

Definition 3.46. Let $h : X \to \mathcal{P}([0, 1])$. For any $\pi \in \mathcal{P}([0, 1])$, the sets $U(h, \pi) = \{x \in X \mid h(x) \supseteq \pi\}$ and $U^+(h, \pi) = \{x \in X \mid h(x) \supset \pi\}$ are called an upper π-level subset and an upper π-strong level subset of h, respectively. The sets $L(h, \pi) = \{x \in X \mid h(x) \subseteq \pi\}$ and $L^-(h, \pi) = \{x \in X \mid h(x) \subset \pi\}$ are called a lower π-level subset and a lower π-strong level subset of h, respectively. The set $E(h, \pi) = \{x \in X \mid h(x) = \pi\}$ is called an equal π-level subset of h. Then $U(h, \pi) = U^+(h, \pi) \cup E(h, \pi)$ and $L(h, \pi) = L^-(h, \pi) \cup E(h, \pi)$.

Theorem 3.47. *An IHFS $\mathcal{H} = (h, k)$ on X is an intuitionistic hesitant fuzzy BCC-subalgebra of X if and only if for all $\pi \in \mathcal{P}([0,1])$, the nonempty subsets $U(h,\pi)$ and $L(k,\pi)$ of X are BCC-subalgebras.*

Proof. Assume that $\mathcal{H}$ is an intuitionistic hesitant fuzzy BCC-subalgebra of X. Let $\pi \in P([0,1])$ be such that $U(h,\pi) \neq \emptyset$ and let $x, y \in U(h,\pi)$. Then $h(x) \supseteq \pi$ and $h(y) \supseteq \pi$. Since H is an intuitionistic hesitant fuzzy BCC-subalgebra of X, we have $h(x \cdot y) \supseteq h(x) \cap h(y) \supseteq \pi$ and thus $x \cdot y \in U(h,\pi)$. So, $U(h,\pi)$ is a BCC-subalgebra of X. Let $\pi \in P([0,1])$ be such that $L(k,\pi) \neq \emptyset$ and let $x, y \in L(k,\pi)$. Then $k(x) \subseteq \pi$ and $k(y) \subseteq \pi$. Since H is an intuitionistic hesitant fuzzy BCC-subalgebra of X, we have $k(x \cdot y) \subseteq k(x) \cup k(y) \subseteq \pi$ and thus $x \cdot y \in L(k,\pi)$. So, $L(k,\pi)$ is a BCC-subalgebra of X.

Conversely, assume that for all $\pi \in P([0,1])$, the nonempty subsets $U(h,\pi)$ and $L(k,\pi)$ of X are BCC-subalgebras of X. Let $x, y \in X$. Choose $\pi = h(x) \cap h(y) \in P([0,1])$. Then $h(x) \supseteq \pi$ and $h(y) \supseteq \pi$. Thus $x, y \in U(h,\pi) \neq \emptyset$. By assumption, $U(h,\pi)$ is a BCC-subalgebra of X and thus $x \cdot y \in U(h,\pi)$. So, $h(x \cdot y) \supseteq \pi = h(x) \cap h(y)$. Let $x, y \in X$. Choose $\pi_1 = k(x) \cup k(y) \in P([0,1])$. Then $k(x) \subseteq \pi_1$ and $k(y) \subseteq \pi_1$. Thus $x, y \in L(k,\pi_1) \neq \emptyset$. By assumption, $L(k,\pi_1)$ is a BCC-subalgebra of X and thus $x \cdot y \in L(k,\pi_1)$. So, $k(x \cdot y) \subseteq \pi_1 = k(x) \cup k(y)$. Hence $\mathcal{H}$ is an intuitionistic hesitant fuzzy BCC-subalgebra of X. □

The following two theorems can be proved similarly to Theorem 3.47.

Theorem 3.48. *An IHFS $\mathcal{H} = (h, k)$ on X is an intuitionistic hesitant fuzzy BCC-ideal of X if and only if for all $\pi \in \mathcal{P}([0,1])$, the nonempty subsets $U(h,\pi)$ and $L(k,\pi)$ of X are BCC-ideals.*

Theorem 3.49. *An IHFS $\mathcal{H} = (h, k)$ on X is an intuitionistic hesitant fuzzy strong BCC-ideal of X if and only if for all $\pi \in \mathcal{P}([0,1])$, the nonempty subsets $U(h,\pi)$ and $L(k,\pi)$ of X are strong BCC-ideals.*

Definition 3.50. Let $\{\mathcal{H}_\alpha \mid \alpha \in \Delta\}$ be a family of IHFSs on a reference set X. We define the IHFS $\bigcap_{\alpha\in\Delta} \mathcal{H}_\alpha = (\bigcap_{\alpha\in\Delta} h_\alpha, \bigcup_{\alpha\in\Delta} k_\alpha)$ by $(\bigcap_{\alpha\in\Delta} h_\alpha)(x) = \bigcap_{\alpha\in\Delta} h_\alpha(x)$ and $(\bigcup_{\alpha\in\Delta} k_\alpha)(x) = \bigcup_{\alpha\in\Delta} k_\alpha(x)$ for all $x \in X$, which is called the *intuitionistic hesitant intersection of IHFSs*.

Proposition 3.51. *If $\{\mathcal{H}_\alpha \mid \alpha \in \Delta\}$ is a family of intuitionistic hesitant fuzzy BCC-ideals of X, then $\bigcap_{\alpha\in\Delta} \mathcal{H}_\alpha$ is an intuitionistic hesitant fuzzy BCC-ideal of X.*

Proof. Let $\{\mathcal{H}_\alpha \mid \alpha \in \Delta\}$ be a family of intuitionistic hesitant fuzzy BCC-ideals of X. Let $x \in X$. Then

$$(\bigcap_{\alpha\in\Delta} h_\alpha)(0) = \bigcap_{\alpha\in\Delta} h_\alpha(0) \supseteq \bigcap_{\alpha\in\Delta} h_\alpha(x) = (\bigcap_{\alpha\in\Delta} h_\alpha)(x),$$

$$(\bigcup_{\alpha\in\Delta} k_\alpha)(0) = \bigcup_{\alpha\in\Delta} k_\alpha(0) \subseteq \bigcup_{\alpha\in\Delta} k_\alpha(x) = (\bigcup_{\alpha\in\Delta} k_\alpha)(x).$$

Let $x, y, z \in X$. Then

$$\begin{aligned}(\bigcap_{\alpha\in\Delta} h_\alpha)(x\cdot z) &= \bigcap_{\alpha\in\Delta} h_\alpha(x\cdot z)\\ &\supseteq \bigcap_{\alpha\in\Delta}(h_\alpha(x\cdot(y\cdot z))\cap h_\alpha(y))\\ &= (\bigcap_{\alpha\in\Delta} h_\alpha(x\cdot(y\cdot z)))\cap(\bigcap_{\alpha\in\Delta} h_\alpha(y))\\ &= (\bigcap_{\alpha\in\Delta} h_\alpha)(x\cdot(y\cdot z))\cap(\bigcap_{\alpha\in\Delta} h_\alpha)(y),\end{aligned}$$

$$\begin{aligned}(\bigcup_{\alpha\in\Delta} k_\alpha)(x\cdot z) &= \bigcup_{\alpha\in\Delta} k_\alpha(x\cdot z)\\ &\subseteq \bigcup_{\alpha\in\Delta}(k_\alpha(x\cdot(y\cdot z))\cup k_\alpha(y))\\ &= \bigcup_{\alpha\in\Delta} k_\alpha(x\cdot(y\cdot z))\cup \bigcup_{\alpha\in\Delta} k_\alpha(y)\\ &= (\bigcup_{\alpha\in\Delta} k_\alpha)(x\cdot(y\cdot z))\cup(\bigcap_{\alpha\in\Delta} k_\alpha)(y).\end{aligned}$$

Hence, $\bigcap_{\alpha\in\Delta}\mathcal{H}_\alpha$ is an intuitionistic hesitant fuzzy BCC-ideal of X. □

The following two propositions can be proved similarly to Proposition 3.51.

Proposition 3.52. *If $\{\mathcal{H}_\alpha \mid \alpha\in\Delta\}$ is a family of intuitionistic hesitant fuzzy strong BCC-ideals of X, then $\bigcap_{\alpha\in\Delta}\mathcal{H}_\alpha$ is an intuitionistic hesitant fuzzy strong BCC-ideal of X.*

Proposition 3.53. *If $\{\mathcal{H}_\alpha \mid \alpha\in\Delta\}$ is a family of intuitionistic hesitant fuzzy BCC-subalgebras of X, then $\bigcap_{\alpha\in\Delta}\mathcal{H}_\alpha$ is an intuitionistic hesitant fuzzy BCC-subalgebra of X.*

Definition 3.54. Let $A = (h_A, k_A)$ and $B = (h_B, k_B)$ be IHFSs on sets X and Y, respectively. The Cartesian product $A \times B = (h, k)$ defined by $h(x, y) = h_A(x) \cap h_B(y)$ and $k(x, y) = k_A(x) \cup k_B(y)$, where $h : X \times Y \to \mathcal{P}([0, 1])$ and $k : X \times Y \to \mathcal{P}([0, 1])$ for all $x \in X$ and $y \in Y$.

Remark 3.55. Let $(X, \cdot, 0_X)$ and $(Y, \star, 0_Y)$ be BCC-algebras. Then $(X \times Y, \diamond, (0_X, 0_Y))$ is a BCC-algebra defined by $(x, y) \diamond (u, v) = (x \cdot u, y \star v)$ for every $x, u \in X$ and $y, v \in Y$.

Proposition 3.56. *If $A = (h_A, k_A)$ and $B = (h_B, k_B)$ are two intuitionistic hesitant fuzzy BCC-subalgebras of BCC-algebras X and Y, respectively, then the Cartesian product $A \times B$ is also an intuitionistic hesitant fuzzy BCC-subalgebra of $X \times Y$.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then

$$\begin{aligned}h((x_1,y_1)\diamond(x_2,y_2)) &= h(x_1\cdot x_2, y_1\star y_2)\\ &= h_A(x_1\cdot x_2)\cap h_B(y_1\star y_2)\\ &\supseteq (h_A(x_1)\cap h_A(x_2))\cap(h_B(y_1)\cap h_B(y_2))\\ &= (h_A(x_1)\cap h_B(y_1))\cap(h_A(x_2)\cap h_B(y_2))\\ &= h(x_1,y_1)\cap h(x_2,y_2),\end{aligned}$$

$$\begin{aligned} k((x_1, y_1) \diamond (x_2, y_2)) &= k(x_1 \cdot x_2, y_1 \star y_2) \\ &= k_A(x_1 \cdot x_2) \cup k_B(y_1 \star y_2) \\ &\subseteq (k_A(x_1) \cup k_A(x_2)) \cup (k_B(y_1) \cup k_B(y_2)) \\ &= (k_A(x_1) \cup k_B(y_1)) \cup (k_A(x_2) \cup k_B(y_2)) \\ &= k(x_1, y_1) \cup k(x_2, y_2). \end{aligned}$$

Hence, $A \times B$ is an intuitionistic hesitant fuzzy BCC-subalgebra of $X \times Y$. □

Theorem 3.57. *Two IHFSs $A = (h_A, k_A)$ and $B = (h_B, k_B)$ are intuitionistic hesitant fuzzy BCC-subalgebras of BCC-algebras X and Y, respectively if and only if the IHFSs $\oplus(A \times B)$ and $\otimes(A \times B)$ are intuitionistic hesitant fuzzy BCC-subalgebras of $X \times Y$.*

Proof. It follows from Proposition 3.56 and Theorem 3.38. □

The following two propositions can be proved similarly to Proposition 3.56.

Proposition 3.58. *If $A = (h_A, k_A)$ and $B = (h_B, k_B)$ are two intuitionistic hesitant fuzzy BCC-ideals of BCC-algebras X and Y, respectively, then the Cartesian product $A \times B$ is also an intuitionistic hesitant fuzzy BCC-ideal of $X \times Y$.*

Proposition 3.59. *If $A = (h_A, k_A)$ and $B = (h_B, k_B)$ are two intuitionistic hesitant fuzzy strong BCC-ideals of BCC-algebras X and Y, respectively, then the Cartesian product $A \times B$ is also an intuitionistic hesitant fuzzy strong BCC-ideal of $X \times Y$.*

The following two theorems can be proved similarly to Theorem 3.57.

Theorem 3.60. *Two IHFSs $A = (h_A, k_A)$ and $B = (h_B, k_B)$ are intuitionistic hesitant fuzzy BCC-ideals of BCC-algebras X and Y, respectively if and only if the IHFSs $\oplus(A \times B)$ and $\otimes(A \times B)$ are intuitionistic hesitant fuzzy BCC-ideals of $X \times Y$.*

Theorem 3.61. *Two IHFSs $A = (h_A, k_A)$ and $B = (h_B, k_B)$ are intuitionistic hesitant fuzzy strong BCC-ideals of BCC-algebras X and Y, respectively if and only if the IHFSs $\oplus(A \times B)$ and $\otimes(A \times B)$ are intuitionistic hesitant fuzzy strong BCC-ideals of $X \times Y$.*

A mapping $f : (X, \cdot, 0_X) \to (Y, \star, 0_Y)$ of BCC-algebras is called a *homomorphism* if $f(x \cdot y) = f(x) \star f(y)$ for all $x, y \in X$. Note that if $f : X \to Y$ is a homomorphism of BCC-algebras, then $f(0_X) = 0_Y$.

Definition 3.62. Let f be a function from a nonempty set X to a nonempty set Y. If $\mathcal{H} = (h, k)$ is an IHFS on Y, then the IHFS $f^{-1}(\mathcal{H}) = (h \circ f, k \circ f)$ on X is called the *pre-image of $\mathcal{H}$ under f*.

Theorem 3.63. *Let $f : (X, \cdot, 0_X) \to (Y, \star, 0_Y)$ be a homomorphism of BCC-algebras. If $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy BCC-ideal of Y, then $f^{-1}(\mathcal{H}) = (h \circ f, k \circ f)$ is an intuitionistic hesitant fuzzy BCC-ideal of X.*

Proof. By assumption, $h(f(0_X)) = h(0_X) \supseteq h(y)$ for every $y \in Y$. In particular, $(h \circ f)(0_X) = h(f(0_X)) \supseteq h(f(x)) = (h \circ f)(x)$ for all $x \in X$. Also, $k(f(0_X)) = k(0_Y) \subseteq k(y)$ for every $y \in Y$. In particular, $(k \circ f)(0_X) = k(f(0_X)) \subseteq k(f(x)) = (k \circ f)(x)$ for all $x \in X$. Let $x, y, z \in X$. Then

$$\begin{aligned}
(h \circ f)(x \cdot z) &= h(f(x \cdot z)) \\
&= h((f(x) \star f(z)) \\
&\supseteq h(f(x) \star (f(y) \star f(z))) \cap h(f(y)) \\
&= h(f(x \cdot (y \cdot z))) \cap h(f(y)) \\
&= (h \circ f)(x \cdot (y \cdot z)) \cap (h \circ f)(y),
\end{aligned}$$

$$\begin{aligned}
(k \circ f)(x \cdot z) &= k(f(x \cdot z)) \\
&= k((f(x) \star f(z)) \\
&\subseteq k(f(x) \star (f(y) \star f(z))) \cup k(f(y)) \\
&= k(f(x \cdot (y \cdot z))) \cup k(f(y)) \\
&= (k \circ f)(x \cdot (y \cdot z)) \cup (k \circ f)(y).
\end{aligned}$$

Hence, $f^{-1}(\mathcal{H})$ is an intuitionistic hesitant fuzzy BCC-ideal of X. □

The following two theorems can be proved similarly to Theorem 3.63.

Theorem 3.64. *Let $f : (X, \cdot, 0_X) \to (Y, \star, 0_Y)$ be a homomorphism of BCC-algebras. If $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy BCC-subalgebra of Y, then $f^{-1}(\mathcal{H}) = (h \circ f, k \circ f)$ is an intuitionistic hesitant fuzzy BCC-subalgebra of X.*

Theorem 3.65. *Let $f : (X, \cdot, 0_X) \to (Y, \star, 0_Y)$ be a homomorphism of BCC-algebras. If $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy strong BCC-ideal of Y, then $f^{-1}(\mathcal{H}) = (h \circ f, k \circ f)$ is an intuitionistic hesitant fuzzy strong BCC-ideal of X.*

4. Conclusion

In the present paper, we have introduced the concepts of intuitionistic hesitant fuzzy BCC-subalgebras, BCC-ideals, and strong BCC-ideals of BCC-algebras. The relationship between intuitionistic hesitant fuzzy BCC-subalgebras (BCC-ideals, strong BCC-ideals) and their level subsets is described. Moreover, the homomorphic pre-images of intuitionistic hesitant fuzzy BCC-subalgebras (BCC-ideals, strong BCC-ideals) in BCC-algebras are also studied, and some related properties are investigated.

Acknowledgment

This research was supported by the University of Phayao and the Thailand Science Research and Innovation Fund (Fundamental Fund 2024).

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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