Asia Pac. J. Math. 2024 **11**:61

FORMULAS FOR THE NTH DERIVATIVE OF THE FUNCTION x^{ax}

MALINEE CHAIYA, SOMJATE CHAIYA*

Department of Mathematics, Silpakorn University, Thailand

*Corresponding author: chaiya_s@su.ac.th

Received Apr. 20, 2024

Abstract. In this paper, we provide two closed-form formulas for the nth derivative of the power–exponential function x^{ax}. Some applications are also presented.

2020 Mathematics Subject Classification. 26A06.

Key words and phrases. derivative; power-exponential function; closed-form formulas.

1. Introduction

For non-negative integers n and k, Comtet introduced in [2, pp. 139–140] a sequence of numbers $b(n,k)$ that satisfy the identity

$$\frac{[(1+T)\ln(1+T)]^k}{k!} = \sum_{n=k}^{\infty} b(n,k)\frac{T^k}{n!}. \tag{1}$$

Comtet also presented the explicit formula

$$b(n,k) = \sum_{l=k}^{n} \binom{l}{k} k^{l-k} s(n,l),$$

where $s(n,l)$ are the Stirling numbers of the first kind. The numbers $b(n,k)$ were later called *Comtet's numbers* (see [1,3]). A formula for the nth derivative of the functions x^{ax} was first given in [2, p. 140] by

$$(x^{ax})^{(n)} = a^n x^{ax} \sum_{j=0}^{n} \binom{n}{j} (\ln x)^j \sum_{k=0}^{n-j} \frac{b(n-j, n-k-j)}{(ax)^k}. \tag{2}$$

In [1], two new formulas for the nth derivative of the power–exponential function x^x were given. Here we will establish two new formulas for the nth derivative of the functions x^{ax}, for any $a \neq 0$, by using the tools given in [1].

DOI: 10.28924/APJM/11-61

2. Preliminaries

For any $z \in \mathbb{C}$ and $m \in \mathbb{N}$, the binomial coefficient $\binom{z}{m}$ is defined by

$$\binom{z}{m} = \frac{z(z-1)\cdots(z-m+1)}{m!}$$

and $\binom{z}{0} = 1$. Hence, for integers $0 \le j \le n$, we have

$$\binom{j-n-1}{j} = (-1)^j \binom{n}{j}. \tag{3}$$

The Faà di Bruno formula (see [2, Theorem C, p. 139] or [1, p. 3]) is a formula for the nth derivative of a composition function $f \circ u(x)$ given by

$$\frac{d^n}{dx^n} f \circ u(x) = \sum_{k=0}^{n} f^{(k)}(u(x)) B_{n,k}\left(u'(x), u''(x), \ldots, u^{(n-k+1)}(x)\right), \tag{4}$$

where $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ are the partial Bell polynomials. For integers $n \ge k \ge 0$, the partial Bell polynomials $B_{n,k}$ satisfy the following identities:

$$B_{n,k}\left(\alpha\beta x_1, \alpha\beta^2 x_2, \ldots, \alpha\beta^{n-k+1} x_{n-k+1}\right) = \alpha^k \beta^n B_{n,k}\left(x_1, x_2, \ldots, x_{n-k+1}\right) \tag{5}$$

and

$$B_{n,k}\left(x_1, x_2, \ldots, x_{n-k+1}\right) = \sum_{l=0}^{k} \binom{n}{l} x_1^l B_{n-l,k-l}(0, x_2, \ldots, x_{n-k+1}). \tag{6}$$

The proofs of both indentities can be found in [2, pp. 135–136].

Lemma 1. [1, *Lemma 1, p.4*] *For integers $n \ge k \ge 0$, we have*

$$B_{n,k}\left(0, 0!, 1!, 2!, \ldots, (n-k-1)!\right) = (-1)^{n-k} \sum_{j=0}^{k} \frac{(-1)^j}{(k-j)!} \sum_{l=0}^{n-k} \frac{s(l+j,j)}{(l+j)!} \binom{j}{n-k-l} \tag{7}$$

and

$$B_{n,k}\left(0, 0!, 1!, 2!, \ldots, (n-k-1)!\right) = (-1)^{n-k} \sum_{j=0}^{k} (-1)^j \binom{n}{k-j} b(n-k+j, j). \tag{8}$$

3. Two new formulas for the nth derivative of x^{ax}

Theorem 2. *Let $a \neq 0, n \in \mathbb{N}$ and $x > 0$. We have*

$$\left(x^{ax}\right)^{(n)} = n! x^{ax-n} \sum_{k=0}^{n} (ax)^k \sum_{j=0}^{k} \frac{(\ln x)^{k-j}}{(k-j)!} \left[\sum_{m=0}^{n-k} \frac{s(m+j,j)}{(m+j)!} \binom{j}{n-k-m}\right] \tag{9}$$

and

$$\left(x^{ax}\right)^{(n)} = x^{ax-n} \sum_{k=0}^{n} (ax)^k \sum_{j=0}^{k} \binom{n}{j} b(n-j, k-j) \left(\ln x\right)^j. \tag{10}$$

Proof. Let $u(x) = ax \ln x$. By Faà di Bruno's formula, we obtain

$$(x^{ax})^{(n)} = \left(e^{ax \ln x}\right)^{(n)} = \sum_{k=0}^{n} (e^{u})^{(k)} B_{n,k}\left(u'(x), u''(x), \ldots, u^{(n-k+1)}(x)\right)$$
$$= e^{ax \ln x} \sum_{k=0}^{n} B_{n,k}\left(a(1+\ln x), \frac{a}{x}, -\frac{a}{x^2}, \frac{2!a}{x^3}, \ldots, a(-1)^{n-k+1}\frac{(n-k-1)!}{x^{n-k}}\right).$$

By the identity (6), we have

$$(x^{ax})^{(n)} = x^{ax} \sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{l} a^{l}(1+\ln x)^{l} B_{n-l,k-l}\left(0,, \frac{a}{x}, -\frac{a}{x^2}, \frac{2!a}{x^3}, \ldots, a(-1)^{n-k+1}\frac{(n-k-1)!}{x^{n-k}}\right).$$

By the identity (5) with $x_1 = 0, x_j = (j-2)!$ for $j = 2, \ldots, n-k+1$, $\alpha = ax$ and $\beta = -\frac{1}{x}$, we derive

$$(x^{ax})^{(n)} = x^{ax} \sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{l} a^{l}(1+\ln x)^{l}(ax)^{k-l}\left(-\frac{1}{x}\right)^{n-l} B_{n-l,k-l}(0, 0!, 1!, 2!, \ldots, (n-k-1)!)$$
$$= (-1)^{n} x^{ax-n} \sum_{k=0}^{n}(ax)^{k} \sum_{l=0}^{k}\binom{n}{l}(-1)^{l}(1+\ln x)^{l} B_{n-l,k-l}(0, 0!, 1!, 2!, \ldots, (n-k-1)!). \quad (11)$$

Applying the identity (7) into the identity (11), it becomes

$$(x^{ax})^{(n)} = x^{ax-n} \sum_{k=0}^{n}(ax)^{k} \sum_{l=0}^{k}\binom{n}{l}(-1)^{l}(1+\ln x)^{l}(-1)^{n-k}(n-l)!$$
$$\times \sum_{j=0}^{k-l}\frac{(-1)^{j}}{(k-l-j)!}\sum_{m=0}^{n-k}\frac{s(m+j,j)}{(m+j)!}\binom{j}{n-k-m}$$
$$= n! x^{ax-n} \sum_{k=0}^{n}(-ax)^{k} \sum_{l=0}^{k}\frac{(-1)^{l}(1+\ln x)^{l}}{l!}\sum_{j=0}^{k-l}\frac{(-1)^{j}}{(k-l-j)!}\sum_{m=0}^{n-k}\frac{s(m+j,j)}{(m+j)!}\binom{j}{n-k-m}.$$

By rearranging the order of summations with indices l and j, and using the fact that

$$\sum_{l=0}^{k-j}\binom{k-j}{l}(-1)^{l}(1+\ln x)^{l} = (-1)^{k-j}(\ln x)^{k-j}$$

we obtain

$$(x^{ax})^{(n)} = n! x^{ax-n} \sum_{k=0}^{n}(ax)^{k}(-1)^{k}\sum_{j=0}^{k}(-1)^{j}\sum_{m=0}^{n-k}\frac{s(m+j,j)}{(m+j)!}\binom{j}{n-k-m}\sum_{l=0}^{k-j}\frac{(-1)^{l}(1+\ln x)^{l}}{(k-j-l)!l!}$$
$$= n! x^{ax-n} \sum_{k=0}^{n}(ax)^{k}(-1)^{k}\sum_{j=0}^{k}(-1)^{j}\sum_{m=0}^{n-k}\frac{s(m+j,j)}{(m+j)!}\binom{j}{n-k-m}\frac{(-1)^{k-j}}{(k-j)!}(\ln x)^{k-j}$$
$$= n! x^{ax-n} \sum_{k=0}^{n}(ax)^{k}\sum_{j=0}^{k}\frac{(\ln x)^{k-j}}{(k-j)!}\sum_{m=0}^{n-k}\frac{s(m+j,j)}{(m+j)!}\binom{j}{n-k-m}.$$

This proves the formula (9). To derive the formula (10), we apply the identity (6) to the identity (11) and rearranging the order of summations. We get

$$\begin{aligned}(x^{ax})^{(n)} &= (-1)^n x^{ax-n}\sum_{k=0}^{n}(ax)^k \sum_{l=0}^{k}\binom{n}{l}(-1)^l\,(1+\ln x)^l\,(-1)^{n-k}\sum_{j=0}^{k-l}(-1)^l\binom{n-l}{k-l-j}b(n-k+j,j)\\
&= x^{ax-n}\sum_{k=0}^{n}(ax)^k(-1)^k\sum_{j=0}^{k}(-1)^j b(n-k+j,j)\sum_{l=0}^{k-j}(-1)^l\binom{n}{l}\binom{n-l}{n-(k+j)}\,(1+\ln x)^l\\
&= x^{ax-n}\sum_{k=0}^{n}(ax)^k(-1)^k\sum_{j=0}^{k}(-1)^j b(n-k+j,j)\binom{n}{k-j}(-1)^{k-j}\,(\ln x)^{k-j}\\
&= x^{ax-n}\sum_{k=0}^{n}(ax)^k\sum_{j=0}^{k}\binom{n}{j}b(n-j,k-j)\,(\ln x)^j\,,\end{aligned}$$

as desired. □

Corollary 3. *For $a \neq 0$, we have*

$$x^{ax} = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}a^k\sum_{m=k}^{n}\frac{s(m,k)}{m!}\binom{k}{n-m}\right](x-1)^n, \quad |x-1|<1 \tag{12}$$

and

$$x^{ax} = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}a^k b(n,k)\right]\frac{(x-1)^n}{n!}, \quad |x-1|<1. \tag{13}$$

Proof. From the formulas (9) and (10), by taking $x \to 1$, the Taylor expansion of x^{ax} are of the forms (12) and (13) respectively. □

4. Applications

In this section we present a nice application of the Taylor series in Corollary 3. Let $a = -1$. Then, from the formula (12), the Taylor series expansion of $f(x) = x^{-x}$ around 1 is

$$f(x) = \sum_{j=0}^{\infty}\left[\sum_{k=0}^{j}(-1)^k\sum_{m=k}^{j}\frac{s(m,k)}{m!}\binom{k}{j-m}\right](x-1)^j, \quad |x-1|<1. \tag{14}$$

For any integer $n \geq 2$, we have

$$f\left(n^{-1}\right) = n^{\frac{1}{n}} = \sqrt[n]{n}.$$

Therefore, $\sqrt[n]{n}$ can be expressed in the form

$$\sqrt[n]{n} = \sum_{j=0}^{\infty}(-1)^j\left[\sum_{k=0}^{j}(-1)^k\sum_{m=k}^{j}\frac{s(m,k)}{m!}\binom{k}{j-m}\right]\left(\frac{n-1}{n}\right)^j. \tag{15}$$

Especially when $n = 2$, we get

$$\sqrt{2} = \sum_{j=0}^{\infty} \frac{(-1)^j}{2^j} \left[\sum_{k=0}^{j} (-1)^k \sum_{m=k}^{j} \frac{s(m,k)}{m!} \binom{k}{j-m} \right]. \tag{16}$$

Next we will compare the rate of convergence of the series (16) to other forms that were derived from Taylor series expansions of other well-known functions. The first function which is mostly mentioned to when ones want to estimate the value of $\sqrt{2}$ by using Taylor series approximation is the function $\sqrt{x+1}$. Ones can directly derive that the Maclaurin series expansions of the function $\sqrt{x+1}$ is

$$\sqrt{x+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}(2n)!}{4^n (n!)^2 (2n-1)} x^n. \tag{17}$$

Plugging $x = 1$ into (17), we obtain

$$\sqrt{2} = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}(2n)!}{4^n (n!)^2 (2n-1)} = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{(-1)^{n-1}}{4^n (2n-1)}. \tag{18}$$

The function $(1-x)^{-1/2}$ is another function that is easy to find its Maclaurin series expansion. We have

$$(1-x)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2} x^n, \quad |x| < 1. \tag{19}$$

Plugging $x = \frac{1}{2}$ into (19), it yields

$$\sqrt{2} = \sum_{n=0}^{\infty} \frac{(2n)!}{8^n (n!)^2} = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{8^n}. \tag{20}$$

To compare the rate of convergent, let

$$a_N = \left| \sqrt{2} - \sum_{n=0}^{N} \frac{(-1)^n}{2^n} \left[\sum_{k=0}^{n} (-1)^k \sum_{m=k}^{n} \frac{s(m,k)}{m!} \binom{k}{n-m} \right] \right|,$$

$$b_N = \left| \sqrt{2} - \sum_{n=0}^{N} \binom{2n}{n} \frac{(-1)^{n-1}}{4^n (2n-1)} \right|, \text{ and}$$

$$c_N = \left| \sqrt{2} - \sum_{n=0}^{N} \binom{2n}{n} \frac{1}{8^n} \right|.$$

The table below shows how fast each formula converges to $\sqrt{2}$.

N	a_N	b_N	c_N
2	0.085786438	0.039213562	0.070463562
3	0.023286438	0.023286438	0.031401062
4	0.002453105	0.015776062	0.014311218
5	0.000151062	0.011567688	0.006620789
6	0.000020854	0.008940124	0.003096008
7	0.000044250	0.007173157	0.001459503
8	0.000028750	0.005918884	0.000692391
9	0.000012861	0.004991150	0.000330144
10	0.000005401	0.004282379	0.000158077

Table 1. The errors of the approximations of $\sqrt{2}$ by the formulas (16), (18), and (20)

As we can see, the series (16) gives the best estimate of $\sqrt{2}$ among those three series. However, the disadvantages of the formula (16) are that it is hard to compute and time consumption.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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