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APPROXIMATING SOLUTION FOR FAMILY OF SPLIT EQUILIBRIUM BY VISCOSITY

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Abstract. This paper introduces an iterative scheme for finding a common element of the set of solutions of a family of split equilibrium problems, the set of solutions of finite family of variational inclusion problems involving inverse strongly monotone mappings and multi-valued maximal monotone mappings, and the set of fixed points of nonexpansive mappings in Hilbert space. Strong convergence analysis for approximating these common elements are established and the convergence of the iterates generated under suitable conditions is shown. These results are improvement and extension of the results obtained in [3] and the references therein. In addition, this paper give a numerical example of the set of solutions of a family of split equilibrium problems, the set of solutions of finite family of variational inclusion problems and the set of fixed points of nonexpansive mappings derived from our generalization.
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1. Introduction

Throughout this paper, denote that $\mathcal{H}$ to be a real Hilbert space with a nonempty closed convex subsets $\mathcal{C}, \mathcal{Q}$ and $\mathcal{R}$ to be the set of all real numbers. For all $z, \bar{z} \in \mathcal{C}$ a mapping $\mathcal{S} : \mathcal{C} \to \mathcal{C}$ is said to be *nonexpansive* if

$$\|\mathcal{S}z - \mathcal{S}\bar{z}\| \leq \|z - \bar{z}\|.$$

It is known that, the set of fixed points of $\mathcal{S}$ (to be denoted by $Fix(\mathcal{S})$) is a closed and convex set.

The *equilibrium problem* for a bifunction $\mathcal{F} : \mathcal{C} \times \mathcal{C} \to \mathcal{R}$ is formulated as a problem of determining $z \in \mathcal{C}$ such that

$$\mathcal{F}(z, \bar{z}) \geq 0 \tag{1.1}$$

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for all $\bar{z} \in \mathcal{C}$. Let $EP(\mathcal{F})$ denotes the set of all solutions of (1.1), that is, $EP(\mathcal{F}) = \{z \in \mathcal{C} : \mathcal{F}(z, \bar{z}) \geq 0, \forall \bar{z} \in \mathcal{C}\}$. The equilibrium problem (1.1) is an important formulation which is considered as a generalization of some related problems, such as, variational inequality problems, optimization, fixed point problems, control systems, game theory and other problems. In 2005, Combettes and Hirstoaga [4] introduced several iterative schemes for finding approximation to the solutions of equilibrium problems and established a strong convergence result when $EP(\mathcal{F})$ is nonempty. Several iterative methods have also been proposed for approximations of common solution of (1.1) and fixed point problem of nonexpansive mapping. In particular, Plubtieng and Punpaeng in [8] proposed the following iterative scheme. Suppose $z_1 \in \mathcal{H}$ and $\{z_n\}$ and $\{\bar{z}_n\}$ be iterates generated by

$$\begin{aligned} &\mathcal{F}(\bar{z}_n, z) + \frac{1}{r_n}\langle z - \bar{z}_n, \bar{z}_n - z_n\rangle \geq 0,\ \forall z \in \mathcal{H} \\ &z_{n+1} = \eta_n \gamma h(z_n) + (I - \eta_n \mathcal{A})\mathcal{S}\bar{z}_n,\ \forall n \in \mathbb{N}. \end{aligned} \tag{1.2}$$

They proved that, if the sequence of parameters $\{\eta_n\}$ and $\{r_n\}$ satisfy some suitable conditions, then the iterates $\{z_n\}$ and $\{\bar{z}_n\}$ both converge strongly to the unique solution of the problem

$$\langle(\mathcal{A} - \gamma h)u, u - z\rangle \geq 0,\ \forall z \in Fix(\mathcal{S}) \cap EP(\mathcal{F}), \tag{1.3}$$

which is considered as the optimality condition for the problem

$$\min_{z \in Fix(\mathcal{S}) \cap EP(\mathcal{F})} \frac{1}{2}\langle \mathcal{A}z, z\rangle - f(z), \tag{1.4}$$

where f is a potential function for γh. see [6], [7].

Another important problem is the *variational inclusion*, which concerns finding $z^* \in \mathcal{H}$ such that

$$\mathbf{0} \in \mathcal{B}(z^*) + \mathcal{M}(z^*), \tag{1.5}$$

where **0** is the zero vector in $\mathcal{H}$, $\mathcal{M} : \mathcal{H} \to 2^{\mathcal{H}}$ is a set-valued mapping and $\mathcal{B} : \mathcal{H} \to \mathcal{H}$ is a single valued nonlinear mapping. Let $I(\mathcal{B}, \mathcal{M})$ denotes the set of solutions of (1.5). If $\mathcal{B} \equiv 0$, then (1.5) reduces to the inclusion problem introduced in [9]. Moreover, if $\mathcal{M} = \partial\delta_{\mathcal{C}}$ where $\delta_{\mathcal{C}} : \mathcal{H} \to [0, \infty]$ is the indicator function of $\mathcal{C}$, defined as, $\delta_{\mathcal{C}}(z) = 0$, if $z \in \mathcal{C}$ and $\delta_{\mathcal{C}}(z) = +\infty$, if $z \neq \mathcal{C}$, then problem (1.5) is equivalent to the variational inequality problem. In addition, it is known that problem (1.5) establishes a suitable system for a consolidated study of optimal solutions in many related areas of optimization such as variational inequalities, optimal control, mathematical programming, economics, complementarity and so on. Several generalizations and extensions of the problem (1.5) have been studied; see [10] and the references there in.

In 2009, Plubtieng and Sripard [12] proposed and studied an iterative approximation method for finding common element for set of solutions to problem (1.5) involving an inverse strongly monotone mapping and a multi-valued maximal monotone mapping, the set of solutions of problem (1.1) and the set of fixed points of a nonexpansive mapping in Hilbert spaces. The proposed scheme generates the sequences $\{z_n\}, \{y_n\}$ and $\{\bar{z}_n\}$ starting with an arbitrary point $z_1 \in \mathcal{H}$ by

$$\begin{aligned} \mathcal{F}(\bar{z}_n, y) + \frac{1}{r_n}\langle y - \bar{z}_n, \bar{z}_n - z_n\rangle \geq 0, \qquad \forall y \in \mathcal{H}, \\ y_n = \mathcal{J}_{\mathcal{M},\lambda}(\bar{z}_n - \lambda B\bar{z}_n), \qquad \forall n > 0, \\ z_{n+1} = \eta_n\gamma h(z_n) + (I - \eta_n D)\mathcal{S}_n y_n, \end{aligned} \tag{1.6}$$

where $\lambda \in (0, 2\eta], \{\eta_n\} \subset [0, 1], \{r_n\} \subset (0, \infty), \{\mathcal{S}_n\}$ is a sequence of nonexpansive mappings on $\mathcal{H}$ and $\mathcal{D}$ is a strongly positive bounded linear operator. It is proved that under some suitable conditions, the iterates $\{z_n\}, \{y_n\}$ and $\{\bar{z}_n\}$ generated by (1.6) converge strongly to $z^* \in \cap_{i=1}^{\infty} Fix(\mathcal{S}_n) \cap I(\mathcal{B}, \mathcal{M}) \cap EP(\mathcal{F})$, where $z^* = P_{\cap_{i=1}^{\infty} Fix(\mathcal{S}_n)\cap I(\mathcal{B},\mathcal{M})\cap EP(\mathcal{F})} h(z^*)$.

On the other hand, the problem of determining a solution $z^* \in \mathcal{C}$, of a particular problem formulated in $\mathcal{H}_1$, such that $\bar{z}^* = \mathcal{A}z^* \in \mathcal{Q}$ is a solution of another problem formulated in $\mathcal{H}_2$, where $\mathcal{C}, \mathcal{Q}$ are nonempty closed and convex subsets of real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively and $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator has been studied and considered by several researchers. In particular, Kazmi and Rizvi [3] studied a class of split equilibrium problems involving bifunctions $\mathcal{F}_1 : \mathcal{C} \times \mathcal{C} \to \mathcal{R}$ and $\mathcal{F}_2 : \mathcal{Q} \times \mathcal{Q} \to \mathcal{R}$ which is determine by finding $z^* \in \mathcal{H}_1$ such that

$$\mathcal{F}_1(z^*, z) \geq 0 \tag{1.7}$$

for all $z \in \mathcal{C}$ and such that,

$$\bar{z}^* = \mathcal{A}z^* \in Q \text{ solves } \mathcal{F}_2(\bar{z}^*, \bar{z}) \geq 0 \tag{1.8}$$

for all $\bar{z} \in \mathcal{Q}$. The set of all solutions of the problem (1.7) and (1.8) is denoted by Γ, i.e.,

$$\Gamma = \{z \in \mathcal{C} : z \in EP(\mathcal{F}_1) \text{ such that } \mathcal{A}z \in EP(\mathcal{F}_2)\}. \tag{1.9}$$

Kazmi and Rizvi [3] further proposed an iterative approximation for a common solution of a split equilibrium problem, a fixed point problem for a nonexpansive mapping and a variational inequality problem in real Hilbert spaces. The proposed iterative approximation is given by: For a given arbitrary $z_0 = v \in \mathcal{C}$, let $\{u_n\}, \{z_n\}$ and $\{\bar{z}_n\}$ be generated by

$$\begin{aligned} u_n &= \mathcal{T}_{r_n}^{\mathcal{F}_1}(z_n + \gamma\mathcal{A}^*(\mathcal{T}_{r_n}^{\mathcal{F}_2} - I)\mathcal{A}z_n), \\ \bar{z}_n &= P_{\mathcal{C}}(u_n - \lambda_n\mathcal{D}u_n), \\ z_{n+1} &= \alpha_n v + \sigma_n z_n + \gamma_n\mathcal{S}\bar{z}_n, \end{aligned} \tag{1.10}$$

where the sequences $\{\gamma_n\}, \{\alpha_n\}$ and $\{\sigma_n\}$ are in $(0, 1)$. Under some appropriate conditions, for instance, $\lambda_n \in (0, 2\tau), r_n \subset (0, \infty)$ and $\gamma \in \frac{1}{L}$ with L as the spectral radius of the operator $\mathcal{A}^*\mathcal{A}$ and $\mathcal{A}^*$ is the adjoint of $\mathcal{A}$. The sequence $\{z_n\}$ generated by (1.10) converges strongly to a common solution $z \in Fix(\mathcal{S}) \cap \Gamma \cap VI(\mathcal{C}, \mathcal{D})$, where $z = P_{Fix(\mathcal{S})\cap\Gamma\cap VI(\mathcal{C},\mathcal{D})} v$.

Motivated and inspired by the work of Plubtieng and Sripard [12] and Kazmi and Rizvi [3]. In this paper, the author consider an iterative approximation for determining a common element of the set of solutions of a finite family of variational inclusions involving for each $i = 1, 2, \dots, N$, inverse strongly monotone mappings and multivalued maximal monotone mappings, which concerns with determining a point $z^* \in \mathcal{H}$ such that

$$\mathbf{0} \in \mathcal{B}_i(z^*) + \mathcal{M}_i(z^*). \tag{1.11}$$

Given $\cap_{i=1}^{N} I(\mathcal{B}_i, \mathcal{M}_i)$ to be the set of the solutions of (1.11), the set of solutions of a family of split equilibrium problems and the set of fixed points of nonexpansive mappings in Hilbert space. The problem (1.11) extends the concepts in the problem (1.5) to finite family of variational inclusions covering. Particularly, several types of feasibility problem can be derived from the problem (1.11); see [11]. The author consider the following iterative scheme: starting with an arbitrary point $z_1 \in \mathcal{H}$ and define sequences $\{z_n\}, \{\bar{z}_n\}$ and $\{u_n\}$ by

$$\begin{aligned} u_{i,n} &= \mathcal{T}_{r_n}^{\mathcal{F}}(I - \gamma \mathcal{A}_i^*(I - \mathcal{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_i)z_n, \ i = 1, 2, \dots, N_1, \\ \bar{z}_n &= \mathcal{J}_{\mathcal{M}_N, \lambda_{N,n}}(I - \lambda_{N,n}\mathcal{B}_N) \dots \mathcal{J}_{\mathcal{M}_1, \lambda_{1,n}}(I - \lambda_{1,n}\mathcal{B}_1)u_{i,n}, \\ z_{n+1} &= \epsilon_n \xi f(z_n) + \sigma_n z_n + ((1 - \sigma_n)I - \epsilon_n \mathcal{D})\mathcal{S}_n \bar{z}_n, \end{aligned}$$

for all $n \in \mathbb{N}$, where $\sigma_n \in (0,1), \lambda_{i,n} \in (0, 2\alpha_i], i \in \{1, 2, \dots, N\}, \{\epsilon_n\} \subset [0,1], \{r_n\} \subset (r, \infty)$ with $r > 0, \gamma \subset (0, 1/L^2), L = \max\{L_1, \dots, L_{N_1}\}$ and for each $i \in \{1, \dots, N_1\}$, L_i denotes the spectral radius of $\mathcal{A}_i^*\mathcal{A}_i$ with $\mathcal{A}_i^*$ as an adjoint of $\mathcal{A}_i$, $\mathcal{D}$ is a strongly positive bounded linear operator on $\mathcal{H}$ and $\{\mathcal{S}_n\}$ is a sequence of nonexpansive mappings on $\mathcal{H}$. This paper present some strong convergence theorems for approximating these common elements under some standard conditions. The proposed results can be seen as an improvements and extensions of the results proposed in [3] and some references therein.

2. Preliminaries

In this section, the author present some notations and lemmas that will be required for the convergence analysis of our method. Throughout this paper use $z_n \to z$ and $z_n \rightharpoonup z$ to indicate that the sequence $\{z_n\}$ converges to $z \in \mathcal{H}$ strongly and weakly respectively. It is also well known [13] that Hilbert space $\mathcal{H}$ satisfies $Opail's\ condition$; that is, for any sequence $\{z_n\}$ with $z_n \rightharpoonup z$, the inequality

$$\limsup_{n\to\infty} \|z_n - z\| < \limsup_{n\to\infty} \|z_n - \bar{z}\| \tag{2.1}$$

holds for every $\bar{z} \in \mathcal{H}$ with $\bar{z} \neq z$.

Lemma 2.1. *In a real Hilbert space $\mathcal{H}$, for all $z, \bar{z} \in \mathcal{H}$, the followings are satisfied:*

(1) $\|z - \bar{z}\|^2 = \|z\|^2 - \|\bar{z}\|^2 - 2\langle z - \bar{z}, \bar{z}\rangle$,

(2) $\|z + \bar{z}\|^2 \leq \|z\|^2 + 2\langle \bar{z}, z + \bar{z}\rangle$,

(3) $\|\eta z + (1-\eta)\bar{z}\|^2 = \eta\|z\|^2 + (1-\eta)\|\bar{z}\|^2 - \eta(1-\eta)\|z-\bar{z}\|^2$.

Recall that a mapping $h : \mathcal{H} \to \mathcal{H}$ is said to be contractive if there exists a constant $\eta \in (0,1)$ such that $\|f(z) - f(\bar{z})\| \leq \eta\|z-\bar{z}\|$ for all $z, \bar{z} \in \mathcal{H}$. An operator $\mathcal{S} : \mathcal{H} \to \mathcal{H}$ is said to be η-inverse strongly monotone if there exists a positive η such that

$$\langle \mathcal{S}z - \mathcal{S}\bar{z}, z - \bar{z}\rangle \geq \eta\|\mathcal{S}z - \mathcal{S}\bar{z}\|^2,\ \forall z, \bar{z} \in \mathcal{H}. \tag{2.2}$$

It has been shown that if $\mathcal{S}$ satisfied (2.2), then it is $\frac{1}{\eta}$-Lipschitz continuous and monotone mapping. Furthermore, if $0 < \zeta \leq 2\eta$, for some positive number ζ then the operator $I - \zeta\mathcal{S}$ is a nonexpansive mapping where I is the identity mapping on $\mathcal{H}$.

A set-valued $\mathcal{M} : \mathcal{H} \to 2^{\mathcal{H}}$ is called a maximal monotone if for all $z, \bar{z} \in \mathcal{H}, j \in \mathcal{M}z$ and $g \in \mathcal{M}\bar{z}$ imply $\langle z - \bar{z}, j - g\rangle \geq 0$ for every $(\bar{z}, g) \in \mathcal{G}(\mathcal{H})$ imply $j \in \mathcal{M}z$. Furthermore, for a positive number ζ, the resolvent operator $\mathcal{J}_{\mathcal{M},\zeta}$ associated with $\mathcal{M}$ and ζ is defined as follows:

$$\mathcal{J}_{\mathcal{M},\zeta}(z) = (I + \zeta\mathcal{M})^{-1}(z), \forall z \in \mathcal{H}. \tag{2.3}$$

It is important to mention that the resolvent operator $\mathcal{J}_{\mathcal{M},\zeta}$ is single-valued, nonexpansive and 1-inverse strongly monotone (see [15]). Additionally, it has been shown that a solution of problem (1.5) is a fixed point of the operator $\mathcal{J}_{\mathcal{M},\lambda}(I - \zeta\mathcal{B})$ for all $\zeta > 0$ (see [16]). Consequently, for each $k \in \{1, \dots, N\}$, and $\zeta > 0$, the solution of a finite family of variational inclusion problems (1.11) is a common fixed point of $\mathcal{J}_{\mathcal{M},\zeta}(I - \zeta\mathcal{B}_k)$.

Lemma 2.2. [15] *Let $\mathcal{M} : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone mapping and $\mathcal{B} : \mathcal{H} \to 2^{\mathcal{H}}$ be a Lipschitz-continuous mapping. Then, the mapping $\mathcal{M} + \mathcal{B} : \mathcal{H} \to 2^{\mathcal{H}}$ is a maximal monotone mapping.*

Lemma 2.3. [17] *Let $h : \mathcal{H} \to \mathcal{H}$ be a contraction with coefficient $0 < \eta < 1$, and $\mathcal{D}$ be a strongly positive linear bounded operator with $\bar{\zeta} > 0$. Then,*

(1) *if $0 < \zeta < \frac{\bar{\zeta}}{\eta}$, then $\langle z - \bar{z}, (\mathcal{D} - \zeta h)z - (\mathcal{D} - \zeta h)\bar{z}\rangle \geq (\bar{\zeta} - \zeta\eta)\|z - \bar{z}\|^2, z, \bar{z} \in \mathcal{H}$;*
(2) *if $0 < \nu < \|\mathcal{D}\|^{-1}$, then $\|I - \nu\mathcal{B}\| \leq 1 - \nu\bar{\zeta}$.*

Lemma 2.4. *Suppose $\mathcal{E}$ is a Banach space for $\eta > 0, \nu > 0$ and $z \in \mathcal{E}$,*

$$\mathcal{J}_\eta z = \mathcal{J}_\nu\left(\frac{\nu}{\eta}z + \left(1 - \frac{\nu}{\eta}\right)\mathcal{J}_\eta z\right). \tag{2.4}$$

Lemma 2.5. [18] *Let $\mathcal{M}_i : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone operators, $\mathcal{B}_i : \mathcal{H} \to \mathcal{H}$ be α_i-inverse strongly monotone operators for each $i = 1, 2, \dots, N$ and $\{w_n\}$ be a bounded sequence in $\mathcal{H}$. Suppose that for each $j = 1, 2, \dots, N, \lambda_{j,n} > 0$ satisfy*

(H1) $\lim_{n\to\infty} \sum_{n=1}^{\infty} |\lambda_{j,n} - \lambda_{j,n+1}| < \infty$,
(H2) $\liminf_{n\to\infty} \lambda_{j,n} > 0$.

Set $\Theta_n^k = \mathcal{J}_{\mathcal{M}_k,\lambda_{k,n}}(I-\lambda_{k,n}\mathcal{B}_k)\ldots\mathcal{J}_{\mathcal{M}_1,\lambda_{1,n}}(I-\lambda_{1,n}\mathcal{B}_1)$ *and* $\Theta_n^0 = I$*. Then, for* $k\in\{1,2,\ldots,N\}$*,*

$$\sum_{i=1}^{\infty}\|\Theta_{i+1}^k w_i - \Theta_i^k w_i\| < \infty. \tag{2.5}$$

Proof. For each $k\in\{1,2,\ldots,N\}$, get from Lemma 2.5,

$$\begin{aligned} & \|\mathcal{J}_{\mathcal{M}_k,\lambda_{k,n+1}}(I-\lambda_{k,n+1}\mathcal{B}_k)w_n - \mathcal{J}_{\mathcal{M}_k,\lambda_{k,n}}(I-\lambda_{k,n}\mathcal{B}_k)w_n\| \\ \le\ & \left|1-\frac{\lambda_{k,n}}{\lambda_{k,n+1}}\right|(\|\mathcal{J}_{\mathcal{M}_k,\lambda_{k,n+1}}(I-\lambda_{k,n+1}\mathcal{B}_k)w_n\| + \|w_n\|). \end{aligned} \tag{2.6}$$

Furthermore, it follows from Θ_n^k, that

$$\Theta_n^k = \mathcal{J}_{\mathcal{M}_k,\lambda_{k,n}}(I-\lambda_{k,n}\mathcal{B}_k)\Theta_n^{k-1}. \tag{2.7}$$

Combining (2.6) and (2.7), obtain

$$\begin{aligned} & \|\Theta_{n+1}^k w_n - \Theta_n^k w_n\| \\ \le\ & \|\mathcal{J}_{\mathcal{M}_k,\lambda_{k,n+1}}(I-\lambda_{k,n+1}\mathcal{B}_k)\Theta_{n+1}^{k-1}w_n - \mathcal{J}_{\mathcal{M}_k,\lambda_{k,n}}(I-\lambda_{k,n}\mathcal{B}_k)\Theta_n^{k-1}w_n\| \\ \le\ & \|\mathcal{J}_{\mathcal{M}_k,\lambda_{k,n+1}}(I-\lambda_{k,n+1}\mathcal{B}_k)\Theta_{n+1}^{k-1}w_n - \mathcal{J}_{\mathcal{M}_k,\lambda_{k,n+1}}(I-\lambda_{k,n+1}\mathcal{B}_k)\Theta_n^{k-1}w_n\| \\ & + \|\mathcal{J}_{\mathcal{M}_k,\lambda_{k,n+1}}(I-\lambda_{k,n+1}\mathcal{B}_k)\Theta_n^{k-1}w_n - \mathcal{J}_{\mathcal{M}_k,\lambda_{k,n}}(I-\lambda_{k,n}\mathcal{B}_k)\Theta_n^{k-1}w_n\| \\ \le\ & \|\Theta_{n+1}^{k-1}w_n - \Theta_n^{k-1}w_n\| + \left|1-\frac{\lambda_{k,n}}{\lambda_{k,n+1}}\right|\Big(\|\mathcal{J}_{\mathcal{M}_k,\lambda_{k,n+1}}(I-\lambda_{k,n+1}\mathcal{B}_k)\Theta_n^{k-1}w_n\| + \|w_n\|\Big) \\ \le\ & \|\Theta_{n+1}^{k-1}w_n - \Theta_n^{k-1}w_n\| + \left|1-\frac{\lambda_{k,n}}{\lambda_{k,n+1}}\right|M_1 \\ \le\ & \ldots \\ \le\ & \|\Theta_{n+1}^0 w_n - \Theta_n^0 w_n\| + \sum_{l=1}^{k}\left|1-\frac{\lambda_{l,n}}{\lambda_{l,n+1}}\right|M_1 \\ =\ & \sum_{l=1}^{k}\left|1-\frac{\lambda_{l,n}}{\lambda_{l,n+1}}\right|M_1, \end{aligned} \tag{2.8}$$

where $M_1 = \sup\{\|w_n\| + \sum_{k=1}^N\|\mathcal{J}_{\mathcal{M}_k,\lambda_{k,n+1}}(I-\lambda_{k,n+1}\mathcal{B}_k)\Theta_n^{k-1}w_n\|\}$. According to $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, (2.6) holds. □

Lemma 2.6. [19] *Let* $\{z_n\}$ *and* $\{\bar{z}_n\}$ *be bounded sequences in a Banach space* $\mathcal{E}$ *and* $\{\alpha_n\}$ *be a sequence in* $[0,1]$ *with* $0<\liminf_{n\to\infty}\alpha_n\le\limsup_{n\to\infty}\alpha_n<1$*. Suppose* $z_{n+1}=(1-\alpha_n)\bar{z}_n+\alpha_n z_n$ *for all integers* $n\ge 0$ *and* $\limsup_{n\to\infty}(\|\bar{z}_{n+1}-\bar{z}_n\|-\|z_{n+1}-z_n\|)\le 0$*. Then,* $\lim_{n\to\infty}\|\bar{z}_n-z_n\|=0$*.*

Lemma 2.7. [20] *Let* $\{\zeta_n\}$ *be a sequence of nonnegative real numbers such that* $\zeta_{n+1}\le(1-\gamma_n)\zeta_n+\delta_n, n\ge 0$ *where* $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\gamma_n\}$ *is a sequence in* $\mathcal{R}$ *such that (i)* $\sum_{n=1}^{\infty}\gamma_n=\infty$ *(ii)* $\limsup_{n\to\infty}\frac{\delta_n}{\gamma_n}\le 0$ *or (iii)* $\sum_{n=1}^{\infty}\gamma_n<\infty$*. Then,* $\lim_{n\to\infty}\zeta_n=0$*.*

Assumption 2.8. *Let* $\mathcal{F}:\mathcal{C}\times\mathcal{C}\to\mathcal{R}$ *be a bifunction satisfying the following assumptions:*

(1) $\mathcal{F}(z,z) \geq 0,\ \forall z \in \mathcal{C}$,

(2) $\mathcal{F}$ *is monotone, i.e.,* $\mathcal{F}(z,\bar{z}) + F(\bar{z},z) \leq 0,\ \forall z,\bar{z} \in \mathcal{C}$,

(3) $\mathcal{F}$ *is upper hemi-continuous, i.e., for each* $z,\bar{z},w \in \mathcal{C}, \limsup_{t\to 0} F(tw+(1-t)z,\bar{z}) \leq F(z,\bar{z})$,

(4) *For each* $z \in \mathcal{C}$ *fixed, the function* $z \to F(z,\bar{z})$ *is convex and lower semi-continuous;*

Lemma 2.9. [4] *Suppose that the bifunctions* $\mathcal{F} : \mathcal{C}\times\mathcal{C} \to \mathcal{R}$ *satisfy Assumption 2.8. For any* $z \in \mathcal{H}$ *and* $r > 0$*, define a mapping* $\mathcal{T}_r^{\mathcal{F}} : \mathcal{H} \to \mathcal{C}$ *as follows:*

$$\mathcal{T}_r^{\mathcal{F}} z = \{\bar{z} \in \mathcal{C} : \mathcal{F}(\bar{z},w) + \frac{1}{r}\langle w-\bar{z},\bar{z}-z\rangle \geq 0, \forall w \in \mathcal{C}\}$$

Then, the following holds:

(1) $\mathcal{T}_r^{\mathcal{F}}$ *is single-valued;*

(2) $\mathcal{T}_r^{\mathcal{F}}$ *is firmly nonexpansive, i.e.,*

$$\|\mathcal{T}_r^{\mathcal{F}} z - \mathcal{T}_r^{\mathcal{F}}\bar{z}\|^2 \leq \langle \mathcal{T}_r^{\mathcal{F}} z - \mathcal{T}_r^{\mathcal{F}}\bar{z}, z-\bar{z}\rangle,\ z,\bar{z} \in \mathcal{F},$$

(3) $Fix(\mathcal{T}_r^{\mathcal{F}}) = EP(\mathcal{F})$.

(4) $EP(\mathcal{F})$ *is closed and convex.*

Lemma 2.10. [21] *Let* $\mathcal{F} : \mathcal{C}\times\mathcal{C} \to \mathcal{R}$ *be a bifunction satisfying Assumption 2.8 and let* $\mathcal{T}_r^{\mathcal{F}}$ *and* $\mathcal{T}_s^{\mathcal{F}}$ *be defined as in Lemma 2.9, for* $r,s > 0$*. Letting* $z,\bar{z} \in \mathcal{H}$*, one has*

$$\|\mathcal{T}_r^{\mathcal{F}} z - \mathcal{T}_s^{\mathcal{F}}\bar{z}\| \leq \|z-\bar{z}\| + \left|1-\frac{s}{r}\right| \|\mathcal{T}_r^{\mathcal{F}} z - z\|.$$

Lemma 2.11. [22] *Let* $\mathcal{F} : \mathcal{C}\times\mathcal{C} \to \mathcal{R}$ *be a bifunction satisfying Assumption 2.9 and let* $\mathcal{T}_s^{\mathcal{F}}, \mathcal{T}_t^{\mathcal{F}}$ *be defined as in Lemma 2.10, for* $s,t > 0$*. Then, the following holds:*

$$\|\mathcal{T}_s^{\mathcal{F}} z - \mathcal{T}_t^{\mathcal{F}} z\|^2 \leq \frac{s-t}{s}\langle \mathcal{T}_s^{\mathcal{F}} z - \mathcal{T}_t^{\mathcal{F}} z, \mathcal{T}_s^{\mathcal{F}} z - z\rangle, \forall z \in \mathcal{H}.$$

Lemma 2.12. (*Demiclosedness principle*) *Let* $\mathcal{T} : \mathcal{C} \longrightarrow \mathcal{C}$ *be a nonexpansive operator, where* $\mathcal{C} \subset \mathcal{H}_1$ *is closed convex. If* $\mathcal{T}$ *admits a fixed point, then the operator* $I-\mathcal{T}$ *is demiclosed, in other word, if* $\{z_n\}$ *weakly converges to* z *and* $\{(I-\mathcal{T})z_n\}$ *converges strongly to* $\bar{z}$*, then it follows that* $(I-T)z = \bar{z}$.

3. Main Result

Theorem 3.1. *Let* $\mathcal{H}_1, \mathcal{H}_2$ *be two real Hilbert spaces and* $\mathcal{C} \subset \mathcal{H}_1, \mathcal{Q} \subset \mathcal{H}_2$ *be nonempty closed convex subsets. Let* $\mathcal{A}_i : \mathcal{H}_1 \to \mathcal{H}_2$ *be a bounded linear operator for each* $i = 1,2,\ldots,N_1$ *with* $N_1 \in \mathbb{N}$*. Assume that* $\mathcal{F} : \mathcal{C}\times\mathcal{C} \to \mathcal{R}$ *and* $\mathcal{F}_i : \mathcal{Q}\times\mathcal{Q} \to \mathcal{R}$ $(i = 1,2,\ldots,N_1)$ *satisfy* $(1)-(4)$ *of Assumption 2.8. Let* $\mathcal{B}_i : \mathcal{C} \to \mathcal{H}_1$ *be* α_i*-inverse strongly monotone mappings for each* $i = 1,2,\ldots,N_2$ *with* $N_2 \in \mathbb{N}$ *and let* $\mathcal{M}_i : \mathcal{H} \to 2^{\mathcal{H}}, i = 1,2,\ldots,N$ *be maximal monotone mappings such that* $\Omega := (\cap_{n=1}^{\infty} Fix(\mathcal{S}_n)) \cap SEP \cap (\cap_{n=1}^{N} I(\mathcal{B}_i,\mathcal{M}_i)) \neq \emptyset$*, where* $SEP = \{z \in \mathcal{C} : z \in EP(\mathcal{F})$ *and* $\mathcal{A}_i z \in EP(\mathcal{F}_i), i = 1,2,\ldots,N_1\}$*. Let* f *be a contraction of* $\mathcal{H}$ *into itself* *with constant* $\alpha \in (0,1)$ *and let* $\mathcal{D}$ *be a strongly positive bounded linear operator on* $\mathcal{H}$ *with coefficient* $\bar{\xi} > 0$ *and* $0 < \xi < \frac{\bar{\xi}}{\alpha}$*. Let* $\{z_n\}, \{\bar{z}_n\}$ *and* $\{u_n\}$ *be sequences generated by* $z_1 \in \mathcal{H}$ *and*

$$\begin{cases} u_{i,n} = \mathcal{T}^{\mathcal{F}}_{r_n}(I - \gamma \mathcal{A}_i^*(I - \mathcal{T}^{\mathcal{F}_i}_{r_n})\mathcal{A}_i)z_n, \ i = 1, 2, \ldots, N_1, \\ \bar{z}_n = \mathcal{J}_{\mathcal{M}_N, \lambda_{N,n}}(I - \lambda_{N,n}\mathcal{B}_N) \ldots \mathcal{J}_{\mathcal{M}_1, \lambda_{1,n}}(I - \lambda_{1,n}\mathcal{B}_1)u_{i,n}, \\ z_{n+1} = \epsilon_n \xi f(z_n) + \sigma_n z_n + ((1-\sigma_n)I - \epsilon_n \mathcal{D})\mathcal{S}_n \bar{z}_n, \end{cases} \tag{3.1}$$

where $\sigma_n \in (0,1), \lambda_{i,n} \in (0, 2\alpha_i], i \in \{1, 2, \ldots, N\}$ *satisfy* $(H1) - (H2), \{\epsilon_n\} \subset [0,1], \{r_n\} \subset (r, \infty)$ *with* $r > 0, \gamma \subset (0, 1/L^2), L = \max\{L_1, \ldots, L_{N_1}\}$ *and* L_i *is the spectral radius of the operator* $\mathcal{A}_i^*\mathcal{A}_i$ *and* $\mathcal{A}_i^*$ *is the adjoint of* $\mathcal{A}_i$ *for each* $i \in \{1, \ldots, N_1\}$ *and assume that the control sequences* $\{\epsilon_n\}, \{\sigma_n\}$ *and* $\{r_n\}$ *satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \epsilon_n = 0$;
(C2) $\sum_{n=1}^{\infty} \epsilon_n = \infty$;
(C3) $\sum_{n=1}^{\infty} |\epsilon_{n+1} - \epsilon_n| < \infty$;
(C4) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;
(C5) $\lim_{n\to\infty} \sigma_n = 0$.

Suppose that $\sum_{n=1}^{\infty} \sup\{\|\mathcal{S}_{n+1}z - \mathcal{S}_n z\| : z \in C\} < \infty$*. Let* $\mathcal{S}$ *be a mapping of* $\mathcal{H}_1$ *into itself defined by* $\mathcal{S}z = \lim_{n\to\infty} \mathcal{S}_n z$*, for all* $z \in \mathcal{H}_1$ *and suppose that* $Fix(\mathcal{S}) = \cap_{n=1}^{\infty} Fix(\mathcal{S}_n)$*. Then, the sequence* $\{z_n\}$ *generated by (3.1) converges strongly to a point* z*, where* $z = P_\Omega(I - \mathcal{D} + \xi f)(z)$ *is a unique solution of the variational inequality*

$$\langle (\mathcal{D} - \xi f)z, z - \bar{z} \rangle \leq 0, \ \bar{z} \in \Omega. \tag{3.2}$$

Proof. Given $\epsilon_n \to 0$, suppose that $\epsilon_n \leq (1-\sigma_n)\|\mathcal{D}\|^{-1}$ and $1 - \epsilon_n(\bar{\xi} - \alpha\xi) > 0$. Notice that if $\|u\| = 1$, then

$$\langle ((1-\sigma_n)I - \epsilon_n \mathcal{D})u, u \rangle = (1-\sigma_n) - \epsilon_n \langle \mathcal{D}u, u \rangle \geq (1 - \sigma_n - \epsilon_n \|\mathcal{D}\|) \geq 0. \tag{3.3}$$

From Lemma 2.3, obtain

$$\|(1-\sigma_n)I - \epsilon_n \mathcal{D}\| \leq (1-\sigma_n) - \epsilon_n \bar{\xi}. \tag{3.4}$$

By Lemma 2.5, it get $\bar{z}_n = \Theta_n^N u_{i,n}$. Next, consider several steps for the proof.

Step 1. The sequence $\{z_n\}$ is bounded.

Since $\epsilon_n \to 0$, suppose that $\epsilon_n \leq \|D\|^{-1}$. Let $p \in \Omega$. Considering that for $k \in \{1, 2, \ldots, N\}$, $\mathcal{J}_{\mathcal{M}_k, \lambda_{k,n}}(I - \lambda_{k,n}\mathcal{B}_k)$, is nonexpansive and $p = \mathcal{J}_{\mathcal{M}_k, \lambda_{k,n}}(I - \lambda_{k,n}\mathcal{B}_k)p$, get

$$\|\bar{z}_n - p\| = \|\Theta_n^N u_{i,n} - \Theta_n^N p\| \leq \|u_{i,n} - p\|. \tag{3.5}$$

Since $p \in \Omega$, $p \in \mathcal{T}_{r_n}^{\mathcal{F}_i} p$ and $(I - \gamma \mathcal{A}_i^*(I - \mathcal{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_i)p = p$. Thus,

$$\begin{aligned}
\|u_{i,n} - p\| &= \|\mathcal{T}_{r_n}^{\mathcal{F}}(I - \gamma \mathcal{A}_i^*(I - \mathcal{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_i)z_n - \mathcal{T}_{r_n}^{\mathcal{F}}(I - \gamma \mathcal{A}_i^*(I - \mathcal{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_i)p\| \\
&\leq \|(I - \gamma \mathcal{A}_i^*(I - \mathcal{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_i)z_n - (I - \gamma \mathcal{A}_i^*(I - \mathcal{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_i)p\| \\
&\leq \|z_n - p\|. \qquad (3.6)
\end{aligned}$$

Substituting (3.6) in (3.5), get

$$\|\bar{z}_n - p\| = \|\Theta_n^N u_{i,n} - \Theta_n^N p\| \leq \|u_{i,n} - p\| \leq \|z_n - p\|. \qquad (3.7)$$

Then,

$$\begin{aligned}
\|z_{n+1} - p\| &= \Big\|((1 - \sigma_n)I - \epsilon_n \mathcal{D})(\mathcal{S}_n \Theta_n^N u_{i,n} - \mathcal{S}_n \Theta_n^N p) + \epsilon_n \xi (f(z_n) - f(p)) \\
&\quad + \epsilon_n(\xi f(p) - \mathcal{D}p) + \sigma_n(z_n - p)\Big\| \\
&\leq (1 - \epsilon_n(\bar{\xi} - \alpha\xi))\|z_n - p\| + \epsilon_n(\bar{\xi} - \alpha\xi)\frac{\|\xi f(p) - \mathcal{D}p\|}{\bar{\xi} - \alpha\xi}. \qquad (3.8)
\end{aligned}$$

It follows by induction and (3.8) that

$$\|z_n - p\| \leq \max\left\{\|x_1 - p\|, \frac{\|\xi f(p) - \mathcal{D}p\|}{\bar{\xi} - \alpha\xi}\right\}, \ \forall n \geq 1. \qquad (3.9)$$

Hence, $\{z_n\}$ is bounded, and consequently, $\{u_{i,n}\}, \{\bar{z}_n\}, \{f(z_n)\}$, and $\{\mathcal{S}_n \bar{z}_n\}$ are also bounded.

Step 2. Show that $\|z_{n+1} - z_n\| \to 0$.

Define $z_{n+1} = \sigma_n z_n + (1 - \sigma_n)v_n$ for each $n \geq 0$. Then obtain,

$$\begin{aligned}
v_{n+1} - v_n &= \frac{1}{1 - \sigma_{n+1}}(z_{n+2} - \sigma_{n+1}z_{n+1}) - \frac{1}{1 - \sigma_n}(z_{n+1} - \sigma_n z_n) \\
&= \frac{\epsilon_{n+1}\xi f(z_{n+1}) + ((1 - \sigma_{n+1})I - \epsilon_{n+1}\mathcal{D})\mathcal{S}_{n+1}\bar{z}_{n+1}}{1 - \sigma_{n+1}} \\
&\quad - \frac{\epsilon_n \xi f(z_n) + ((1 - \sigma_n)I - \epsilon_n \mathcal{D})\mathcal{S}_n \bar{z}_n}{1 - \sigma_n} \\
&= \frac{\epsilon_{n+1}\xi f(z_{n+1})}{1 - \sigma_{n+1}} - \frac{\epsilon_n \xi f(z_n)}{1 - \sigma_n} + \mathcal{S}_{n+1}\bar{z}_{n+1} - \mathcal{S}_n \bar{z}_n \\
&\quad + \frac{\epsilon_n}{1 - \sigma_n}\mathcal{D}\mathcal{S}_n \bar{z}_n - \frac{\epsilon_{n+1}}{1 - \sigma_{n+1}}\mathcal{D}\mathcal{S}_{n+1}\bar{z}_{n+1} \\
&= \frac{\epsilon_{n+1}}{1 - \sigma_{n+1}}(\xi f(z_{n+1}) - \mathcal{D}\mathcal{S}_{n+1}\bar{z}_{n+1}) + \frac{\epsilon_n}{1 - \sigma_n}(\xi f(z_n) - \mathcal{D}\mathcal{S}_n \bar{z}_n) \\
&\quad + \mathcal{S}_{n+1}\bar{z}_{n+1} - \mathcal{S}_{n+1}\bar{z}_n + \mathcal{S}_{n+1}\bar{z}_n - \mathcal{S}_n \bar{z}_n. \qquad (3.10)
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|v_{n+1} - v_n\| - \|z_{n+1} - z_n\| \\
&\leq \frac{\epsilon_{n+1}}{1 - \sigma_{n+1}}(\|\xi f(z_{n+1})\| + \|\mathcal{D}\mathcal{S}_{n+1}\bar{z}_{n+1}\|)
\end{aligned}$$

$$
\begin{aligned}
& +\frac{\epsilon_n}{1-\sigma_n}(\|\xi f(z_n)\| + \|\mathcal{D}\mathcal{S}_n\bar{z}_n\|) + \|\mathcal{S}_{n+1}\bar{z}_{n+1} - \mathcal{S}_{n+1}\bar{z}_n\| \\
& +\|\mathcal{S}_{n+1}\bar{z}_n - \mathcal{S}_n\bar{z}_n\| - \|z_{n+1} - z_n\| \\
\leq\ & \frac{\epsilon_{n+1}}{1-\sigma_{n+1}}(\|\xi f(z_{n+1})\| + \|\mathcal{D}\mathcal{S}_{n+1}\bar{z}_{n+1}\|) \\
& +\frac{\epsilon_n}{1-\sigma_n}(\|\xi f(z_n)\| + \|\mathcal{D}\mathcal{S}_n\bar{z}_n\|) + \|\bar{z}_{n+1} - \bar{z}_n\| \\
& +\|\mathcal{S}_{n+1}\bar{z}_n - \mathcal{S}_n\bar{z}_n\| - \|z_{n+1} - z_n\|. \qquad (3.11)
\end{aligned}
$$

From (3.1), get

$$
\begin{aligned}
\|\bar{z}_{n+1} - \bar{z}_n\| &= \|\mathcal{J}_{\mathcal{M}_N,\lambda_{N,n}}(I-\lambda_{N,n}\mathcal{B}_N)\ldots\mathcal{J}_{\mathcal{M}_1,\lambda_{1,n}}(I-\lambda_{1,n}\mathcal{B}_1)u_{i,n+1} \\
&\quad -\mathcal{J}_{\mathcal{M}_N,\lambda_{N,n}}(I-\lambda_{N,n}\mathcal{B}_N)\ldots\mathcal{J}_{\mathcal{M}_1,\lambda_{1,n}}(I-\lambda_{1,n}\mathcal{B}_1)u_{i,n}\| \\
&\leq \|u_{i,n+1} - u_{i,n}\|. \qquad (3.12)
\end{aligned}
$$

Since the mapping $I - \gamma\mathcal{A}_i^*(I - \mathcal{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_i$ is nonexpansive by Lemma 2.10 and 2.11, then

$$
\begin{aligned}
& \|u_{i,n+1} - u_{i,n}\| \\
=\ & \|\mathcal{T}_{r_{n+1}}^{\mathcal{F}}(I - \gamma\mathcal{A}_i^*(I - \mathcal{T}_{r_{n+1}}^{\mathcal{F}_i})\mathcal{A}_i)z_{n+1} - \mathcal{T}_{r_n}^{\mathcal{F}}(I - \gamma\mathcal{A}_i^*(I - \mathcal{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_i)z_n\| \\
\leq\ & \|(I - \gamma\mathcal{A}_i^*(I - \mathcal{T}_{r_{n+1}}^{\mathcal{F}_i})\mathcal{A}_i)z_{n+1} - (I - \gamma\mathcal{A}_i^*(I - \mathcal{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_i)z_n\| \\
& +\frac{|r_{n+1} - r_n|}{r_{n+1}}\|\mathcal{T}_{r_{n+1}}^{\mathcal{F}}(I - \gamma\mathcal{A}_i^*(I - \mathcal{T}_{r_{n+1}}^{\mathcal{F}_i})\mathcal{A}_i)z_{n+1} \\
& -(I - \gamma\mathcal{A}_i^*(I - \mathcal{T}_{r_{n+1}}^{\mathcal{F}_i})\mathcal{A}_i)z_{n+1}\| \\
\leq\ & \|z_{n+1} - z_n\| + \|(I - \gamma\mathcal{A}_i^*(I - \mathcal{T}_{r_{n+1}}^{\mathcal{F}_i})\mathcal{A}_i)z_n - (I - \gamma\mathcal{A}_i^*(I - \mathcal{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_i)z_n\| \\
& +\frac{|r_{n+1} - r_n|}{r_{n+1}}\delta_{n+1} \\
=\ & \|z_{n+1} - z_n\| + \|\gamma\mathcal{A}_i^*(\mathcal{T}_{r_{n+1}}^{\mathcal{F}_i}\mathcal{A}_i z_n - \mathcal{T}_{r_n}^{\mathcal{F}_i}\mathcal{A}_i z_n)\| + \frac{|r_{n+1} - r_n|}{r_{n+1}}\delta_{n+1} \\
\leq\ & \|z_{n+1} - z_n\| + \frac{|r_{n+1} - r_n|}{r}\delta_{n+1} \\
& +\gamma\|\mathcal{A}_i^*\|\left[\frac{|r_{n+1} - r_n|}{r_{n+1}}|\langle\mathcal{T}_{r_{n+1}}^{\mathcal{F}_i}\mathcal{A}_i z_n - \mathcal{T}_{r_n}^{\mathcal{F}_i}\mathcal{A}_i z_n, \mathcal{T}_{r_{n+1}}^{\mathcal{F}_i}\mathcal{A}_i z_n - \mathcal{A}_i z_n\rangle|\right]^{\frac{1}{2}} \\
\leq\ & \|z_{n+1} - z_n\| + \gamma\|\mathcal{A}_i^*\|\left[\frac{|r_{n+1} - r_n|}{r}\sigma_{n+1}\right]^{\frac{1}{2}} + \frac{|r_{n+1} - r_n|}{r}\delta_{n+1} \\
\leq\ & \|z_{n+1} - z_n\| + \eta_{i,n+1}, \qquad (3.13)
\end{aligned}
$$

where

$$
\begin{aligned}
& \sigma_{n+1} \\
=\ & \sup_{n\in\mathbb{N}}|\langle\mathcal{T}_{r_{n+1}}^{\mathcal{F}_i}\mathcal{A}_i z_n - \mathcal{T}_{r_n}^{\mathcal{F}_i}\mathcal{A}_i z_n, \mathcal{T}_{r_{n+1}}^{\mathcal{F}_i}\mathcal{A}_i z_n - \mathcal{A}_i z_n\rangle| \\
& \delta_{n+1}
\end{aligned}
$$

$$= \sup_{n\in\mathbb{N}} \|\mathcal{T}^{\mathcal{F}}_{r_{n+1}}(I-\gamma\mathcal{A}_i^*(I-\mathcal{T}^{\mathcal{F}_i}_{r_{n+1}})\mathcal{A}_i)z_{n+1} - (I-\gamma\mathcal{A}_i^*(I-\mathcal{T}^{\mathcal{F}_i}_{r_{n+1}})\mathcal{A}_i)z_{n+1}\|,$$

and

$$\eta_{i,n+1} = \gamma\|\mathcal{A}_i^*\|\left[\frac{|r_{n+1}-r_n|}{r}\sigma_{n+1}\right]^{\frac{1}{2}} + \frac{|r_{n+1}-r_n|}{r}\delta_{n+1}.$$

Substituting (3.13) in (3.12), obtain

$$\|\bar{z}_{n+1}-\bar{z}_n\| \le \|u_{i,n+1}-u_{i,n}\| \le \|z_{n+1}-z_n\| + \eta_{i,n+1}. \tag{3.14}$$

Supposing that $\sum_{n=1}^{\infty}\sup\{\|\mathcal{S}_{n+1}z-\mathcal{S}_nz\| : z\in\mathcal{C}\}<\infty$, obtain

$$\lim_{n\to\infty}\|\mathcal{S}_{n+1}\bar{z}_n - \mathcal{S}_n\bar{z}_n\| = 0. \tag{3.15}$$

From Lemma 2.5, obtain

$$\lim_{n\to\infty}\|\Theta^N_{n+1}u_{i,n+1} - \Theta^N_n u_{i,n+1}\| = 0. \tag{3.16}$$

By Θ^N_n and $I-\gamma\mathcal{A}_i^*(I-\mathcal{T}^{\mathcal{F}_i}_{r_n})\mathcal{A}_i$ are nonexpansive, get

$$\begin{aligned}\|\bar{z}_{n+1}-\bar{z}_n\| &= \|\Theta^N_{n+1}u_{i,n+1} - \Theta^N_n u_{i,n}\| \\ &\le \|\Theta^N_{n+1}u_{i,n+1} - \Theta^N_n u_{i,n+1}\| + \|\Theta^N_n u_{i,n+1} - \Theta^N_n u_{i,n}\| \\ &\le \|\Theta^N_{n+1}u_{i,n+1} - \Theta^N_n u_{i,n+1}\| + \|u_{i,n+1}-u_{i,n}\|. \end{aligned} \tag{3.17}$$

Substituting (3.13) in (3.17), obtain

$$\|\bar{z}_{n+1}-\bar{z}_n\| \le \|\Theta^N_{n+1}u_{i,n+1} - \Theta^N_n u_{i,n+1}\| + \|z_{n+1}-z_n\| + \eta_{i,n+1}. \tag{3.18}$$

Substituting (3.18) in (3.11), obtain

$$\begin{aligned} & \|v_{n+1}-v_n\| - \|z_{n+1}-z_n\| \\ \le\ & \frac{\epsilon_{n+1}}{1-\sigma_{n+1}}(\|\xi f(z_{n+1})\| + \|\mathcal{D}\mathcal{S}_{n+1}\bar{z}_{n+1}\|) \\ & + \frac{\epsilon_n}{1-\sigma_n}(\|\xi f(z_n)\| + \|\mathcal{D}\mathcal{S}_n\bar{z}_n\|) + \|\Theta^N_{n+1}u_{i,n+1} - \Theta^N_n u_{i,n+1}\| \\ & + \eta_{i,n+1} + \|\mathcal{S}_{n+1}\bar{z}_n - \mathcal{S}_n\bar{z}_n\| \end{aligned} \tag{3.19}$$

Based on (3.15), (3.16) and the conditions $(C1)$ and $(C5)$, deduce that

$$\lim_{n\to\infty}(\|v_{n+1}-v_n\| - \|z_{n+1}-z_n\|) = 0. \tag{3.20}$$

Thus, from Lemma 2.6, it has $\lim_{n\to\infty}\|v_n - z_n\| = 0$. Consequently, it now follows that

$$\lim_{n\to\infty}\|z_{n+1}-z_n\| = \lim_{n\to\infty}(1-\sigma_n)\|v_n-z_n\| = 0. \tag{3.21}$$

From (3.16), (3.18), (3.21) and the condition $(C4)$, obtain

$$\lim_{n\to\infty}\|\bar{z}_{n+1}-\bar{z}_n\| = 0. \tag{3.22}$$

Step 3. Show that for $k=1,2,\ldots,N,\ \lim_{n\to\infty}\|\Theta^k_n u_{i,n} - \Theta^{k-1}_n u_{i,n}\| = 0.$

In fact, suppose $p \in \Omega$, then it can be deduced from the strong nonexpansiveness of $\mathcal{J}_{\mathcal{M}_k,\lambda_{k,n}}(I - \lambda_{k,n}\mathcal{B}_k)$ that

$$\begin{aligned} & \|\Theta_n^k u_{i,n} - p\|^2 \\ = & \|\mathcal{J}_{\mathcal{M}_k,\lambda_{k,n}}(I - \lambda_{k,n}\mathcal{B}_k)\Theta_n^{k-1}u_{i,n} - \mathcal{J}_{\mathcal{M}_k,\lambda_{k,n}}(I - \lambda_{k,n}\mathcal{B}_k)p\|^2 \\ \leq & \langle \Theta_n^k u_{i,n} - p, \Theta_n^{k-1}u_{i,n} - p\rangle \\ = & \frac{1}{2}\Big(\|\Theta_n^k u_{i,n} - p\|^2 + \|\Theta_n^{k-1}u_{i,n} - p\|^2 - \|\Theta_n^k u_{i,n} - \Theta_n^{k-1}u_{i,n}\|^2\Big). \end{aligned} \tag{3.23}$$

Thus, obtain

$$\|\Theta_n^k u_{i,n} - p\|^2 \leq \|\Theta_n^{k-1}u_{i,n} - p\|^2 - \|\Theta_n^k u_{i,n} - \Theta_n^{k-1}u_{i,n}\|^2, \tag{3.24}$$

imply that for each $k \in \{1, 2, \ldots, N\}$,

$$\begin{aligned} \|\bar{z}_n - p\|^2 &= \|\Theta_n^N u_{i,n} - p\|^2 \leq \|\Theta_n^0 u_{i,n} - p\|^2 - \sum_{k=1}^{N}\|\Theta_n^k u_{i,n} - \Theta_n^{k-1}u_{i,n}\|^2 \\ &\leq \|u_{i,n} - p\|^2 - \|\Theta_n^k u_{i,n} - \Theta_n^{k-1}u_{i,n}\|^2. \end{aligned} \tag{3.25}$$

(3.6) imply that $\|u_{i,n} - p\|^2 \leq \|z_n - p\|^2$, thus

$$\|\bar{z}_n - p\|^2 \leq \|z_n - p\|^2 - \|\Theta_n^k u_{i,n} - \Theta_n^{k-1}u_{i,n}\|^2. \tag{3.26}$$

Set $\theta_n = \xi f(z_n) - \mathcal{D}\mathcal{S}_n\bar{z}_n$, and let $\lambda > 0$ be a constant such that

$$\lambda > \sup_{n,k}\{\|\theta_n\|, \|x_k - p\|\}. \tag{3.27}$$

Deduce from Lemma 2.2 and the fact that $\|\cdot\|^2$ is convex that

$$\begin{aligned} & \|z_{n+1} - p\|^2 \\ = & \|(1-\sigma_n)(\mathcal{S}_n\bar{z}_n - p) + \sigma_n(z_n - p) + \epsilon_n\theta_n\|^2 \\ \leq & \|(1-\sigma_n)(\mathcal{S}_n\bar{z}_n - p) + \sigma_n(z_n - p)\|^2 + 2\epsilon_n\langle\theta_n, z_{n+1} - p\rangle \\ \leq & (1-\sigma_n)\|\mathcal{S}_n\bar{z}_n - p\|^2 + \sigma_n\|z_n - p\|^2 + 2\lambda^2\epsilon_n \\ \leq & (1-\sigma_n)\|\bar{z}_n - p\|^2 + \sigma_n\|z_n - p\|^2 + 2\lambda^2\epsilon_n \\ \leq & (1-\sigma_n)\Big(\|z_n - p\|^2 - \|\Theta_n^k u_{i,n} - \Theta_n^{k-1}u_{i,n}\|^2\Big) + \sigma_n\|z_n - p\|^2 + 2\lambda^2\epsilon_n \\ \leq & \|z_n - p\|^2 - (1-\sigma_n)\|\Theta_n^k u_{i,n} - \Theta_n^{k-1}u_{i,n}\|^2 + 2\lambda^2\epsilon_n. \end{aligned} \tag{3.28}$$

By Condition $(C1)$ and Step 2, obtain

$$\begin{aligned} & \|\Theta_n^k u_{i,n} - \Theta_n^{k-1}u_{i,n}\|^2 \\ \leq & \frac{1}{1-\sigma_n}(\|z_n - p\|^2 - \|z_{n+1} - p\|^2 + 2\lambda^2\epsilon_n) \\ \leq & \frac{1}{1-\sigma_n}(2\lambda\|z_n - z_{n+1}\| + 2\lambda^2\epsilon_n) \to 0, \text{ as } n \to \infty. \end{aligned} \tag{3.29}$$

Step 4. Claim that $\lim_{n\to\infty}\|\mathcal{A}_i^*(I-\mathfrak{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_i z_n\|=0$ and $\lim_{n\to\infty}\|u_{i,n}-z_n\|=0$.

First, show that $\lim_{n\to\infty}\|u_{i,n}-z_n\|=0$ for each $i=\{1,2,\dots,N_1\}$. Since each $\mathcal{A}_i^*(I-\mathfrak{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_i$ is $\frac{1}{2L_i^2}$-inverse strongly monotone, by (3.1), we have

$$\begin{aligned}
&\|u_{i,n}-p\|^2\\
=\;&\|\mathfrak{T}_{r_n}^{\mathcal{F}}(I-\gamma\mathcal{A}_i^*(I-\mathfrak{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_i)z_n-\mathfrak{T}_{r_n}^{\mathcal{F}}(I-\gamma\mathcal{A}_i^*(I-\mathfrak{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_i)p\|^2\\
\le\;&\|(I-\gamma\mathcal{A}_i^*(I-\mathfrak{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_i)z_n-(I-\gamma\mathcal{A}_i^*(I-\mathfrak{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_i)p\|^2\\
=\;&\|(z_n-p)-\gamma(\mathcal{A}_i^*(I-\mathfrak{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_iz_n-\mathcal{A}_i^*(I-\mathfrak{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_ip)\|^2\\
=\;&\|z_n-p\|^2-2\gamma\langle z_n-p,\mathcal{A}_i^*(I-\mathfrak{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_iz_n-\mathcal{A}_i^*(I-\mathfrak{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_ip\rangle\\
&+\gamma^2\|\mathcal{A}_i^*(I-\mathfrak{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_iz_n-\mathcal{A}_i^*(I-\mathfrak{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_ip\|^2\\
\le\;&\|z_n-p\|^2-\frac{\gamma}{L_i^2}\|\mathcal{A}_i^*(I-\mathfrak{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_iz_n-\mathcal{A}_i^*(I-\mathfrak{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_ip\|^2\\
&+\gamma^2\|\mathcal{A}_i^*(I-\mathfrak{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_iz_n-\mathcal{A}_i^*(I-\mathfrak{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_ip\|^2\\
=\;&\|z_n-p\|^2+\gamma\left(\gamma-\frac{1}{L_i^2}\right)\|\mathcal{A}_i^*(I-\mathfrak{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_iz_n-\mathcal{A}_i^*(I-\mathfrak{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_ip\|^2\\
=\;&\|z_n-p\|^2+\gamma\left(\gamma-\frac{1}{L_i^2}\right)\|\mathcal{A}_i^*(I-\mathfrak{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_iz_n\|^2.
\end{aligned}\tag{3.30}$$

It follows that since (3.28) and (3.7) imply that $\|\bar{z}_n-p\|^2\le\|u_{i,n}-p\|^2$, obtain

$$\begin{aligned}
&\|z_{n+1}-p\|^2\\
\le\;&(1-\sigma_n)\|\bar{z}_n-p\|^2+\sigma_n\|z_n-p\|^2+2\lambda^2\epsilon_n\\
\le\;&(1-\sigma_n)\|u_{i,n}-p\|^2+\sigma_n\|z_n-p\|^2+2\lambda^2\epsilon_n\\
\le\;&(1-\sigma_n)\left[\|z_n-p\|^2+\gamma\left(\gamma-\frac{1}{L_i^2}\right)\|\mathcal{A}_i^*(I-\mathfrak{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_iz_n\|^2\right]\\
&+\sigma_n\|z_n-p\|^2+2\lambda^2\epsilon_n\\
\le\;&\|z_n-p\|^2+(1-\sigma_n)\gamma\left(\gamma-\frac{1}{L_i^2}\right)\|\mathcal{A}_i^*(I-\mathfrak{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_iz_n\|^2+2\lambda^2\epsilon_n.
\end{aligned}$$

Since $\gamma<\frac{1}{L^2}=\max\{\frac{1}{L_1^2},\dots,\frac{1}{L_{N_1}^2}\}$, get

$$\begin{aligned}
&(1-\sigma_n)\gamma\left(\frac{1}{L_i^2}-\gamma\right)\|\mathcal{A}_i^*(I-\mathfrak{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_iz_n\|^2\\
\le\;&\|z_n-p\|^2-\|z_{n+1}-p\|^2+2\lambda^2\epsilon_n\\
\le\;&\|z_n-z_{n+1}\|(\|z_n-p\|+\|z_{n+1}-p\|)+2\lambda^2\epsilon_n.
\end{aligned}$$

Since $\|z_n-z_{n+1}\|\to 0$ and the conditions $(C1)-(C5)$, get

$$\lim_{n\to\infty}\|\mathcal{A}_i^*(I-\mathfrak{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_iz_n\|=0\tag{3.31}$$

for each $i \in \{1, \ldots, N_1\}$. Hence,

$$\lim_{n\to\infty} \|(I - \mathcal{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_i z_n\| = 0 \tag{3.32}$$

for each $i \in \{1, \ldots, N_1\}$. Since $\mathcal{T}_{r_n}^{\mathcal{F}}$ is firmly nonexpansive and $I - \gamma\mathcal{A}_i^*(I - \mathcal{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_i$ is nonexpansive, by (3.1), get

$$\begin{aligned}
& \|u_{i,n} - p\|^2 \\
= \;& \|\mathcal{T}_{r_n}^{\mathcal{F}}(I - \gamma\mathcal{A}_i^*(I - \mathcal{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_i)z_n - \mathcal{T}_{r_n}^{\mathcal{F}}p\|^2 \\
\leq \;& \langle u_{i,n} - p, z_n + \gamma\mathcal{A}_i^*(\mathcal{T}_{r_n}^{\mathcal{F}_i} - I)\mathcal{A}_i z_n - p\rangle \\
= \;& \frac{1}{2}\{\|u_{i,n} - p\|^2 + \|z_n + \gamma\mathcal{A}_i^*(\mathcal{T}_{r_n}^{\mathcal{F}_i} - I)\mathcal{A}_i z_n - p\|^2 \\
& -\|u_{i,n} - p - [z_n + \gamma\mathcal{A}_i^*(\mathcal{T}_{r_n}^{\mathcal{F}_i} - I)\mathcal{A}_i z_n - p]\|^2\} \\
= \;& \frac{1}{2}\{\|u_{i,n} - p\|^2 + \|(I - \gamma\mathcal{A}_i^*(I - \mathcal{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_i)z_n - (I - \gamma\mathcal{A}_i^*(I - \mathcal{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_i)p\|^2 \\
& -\|u_{i,n} - z_n - \gamma\mathcal{A}_i^*(\mathcal{T}_{r_n}^{\mathcal{F}_i} - I)\mathcal{A}_i z_n\|^2\} \\
\leq \;& \frac{1}{2}\{\|u_{i,n} - p\|^2 + \|z_n - p\|^2 - \|u_{i,n} - z_n - \gamma\mathcal{A}_i^*(\mathcal{T}_{r_n}^{\mathcal{F}_i} - I)\mathcal{A}_i)z_n\|^2\} \\
= \;& \frac{1}{2}\{\|u_{i,n} - p\|^2 + \|z_n - p\|^2 - [\|u_{i,n} - z_n\|^2 + \gamma^2\|\mathcal{A}_i^*(\mathcal{T}_{r_n}^{\mathcal{F}_i} - I)\mathcal{A}_i)z_n\|^2 \\
& -2\gamma\langle u_{i,n} - z_n, \mathcal{A}_i^*(\mathcal{T}_{r_n}^{\mathcal{F}_i} - I)\mathcal{A}_i z_n\rangle]\}
\end{aligned}$$

imply that

$$\|u_{i,n} - p\|^2 \leq \|z_n - p\|^2 - \|u_{i,n} - z_n\|^2 + 2\gamma\|u_{i,n} - z_n\|\|\mathcal{A}_i^*(\mathcal{T}_{r_n}^{\mathcal{F}_i} - I)\mathcal{A}_i z_n\|. \tag{3.33}$$

Hence, by Lemma 2.5 and (3.33), obtain

$$\begin{aligned}
& \|z_{n+1} - p\|^2 \\
\leq \;& (1 - \sigma_n)\|\bar{z}_n - p\|^2 + \sigma_n\|z_n - p\|^2 + 2\lambda^2\epsilon_n \\
\leq \;& (1 - \sigma_n)\|u_{i,n} - p\|^2 + \sigma_n\|z_n - p\|^2 + 2\lambda^2\epsilon_n \\
\leq \;& (1 - \sigma_n)[\|z_n - p\|^2 - \|u_{i,n} - z_n\|^2 + 2\gamma\|u_{i,n} - z_n\|\|\mathcal{A}_i^*(\mathcal{T}_{r_n}^{\mathcal{F}_i} - I)\mathcal{A}_i z_n\|] \\
& +\sigma_n\|z_n - p\|^2 + 2\lambda^2\epsilon_n \\
\leq \;& \|z_n - p\|^2 - (1 - \sigma_n)\|u_{i,n} - z_n\|^2 \\
& +2\gamma(1 - \sigma_n)\|u_{i,n} - z_n\|\|\mathcal{A}_i^*(\mathcal{T}_{r_n}^{\mathcal{F}_i} - I)\mathcal{A}_i z_n\| + 2\lambda^2\epsilon_n,
\end{aligned} \tag{3.34}$$

and hence,

$$\begin{aligned}
& \|u_{i,n} - z_n\|^2 \\
\leq \;& \frac{1}{1 - \sigma_n}\Big(\|z_n - p\|^2 - \|z_{n+1} - p\|^2 + 2\gamma(1 - \sigma_n)\|u_{i,n} - z_n\|
\end{aligned}$$

$$\begin{aligned} & \times\|\mathcal{A}_i^*(\mathcal{T}_{r_n}^{\mathcal{F}_i}-I)\mathcal{A}_i z_n\| + 2\lambda^2\epsilon_n\Big) \\ \leq\ & \frac{1}{1-\sigma_n}\Big\{\|z_n - z_{n+1}\|(\|z_n - p\| - \|z_{n+1}-p\|) + 2\gamma(1-\sigma_n)\|u_{i,n} - z_n\| \\ & \times\|\mathcal{A}_i^*(\mathcal{T}_{r_n}^{\mathcal{F}_i}-I)\mathcal{A}_i z_n\| + 2\lambda^2\epsilon_n\Big\}. \end{aligned} \tag{3.35}$$

Since $\{z_n\}$ is bounded, $\liminf_{n\to\infty}\epsilon_n = 0, \lim_{n\to\infty}\|z_n - z_{n+1}\| = 0, \lim_{n\to\infty}\|(I-\mathcal{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_i z_n\| = 0$, and $\lambda > \sup_{n,k}\{\|\theta_n\|, \|x_k - p\|\}$, it follows that

$$\lim_{n\to\infty}\|u_{i,n} - z_n\| = 0 \tag{3.36}$$

for each $i \in \{1, \ldots, N_1\}$.

Step 5. Claim that $\lim_{n\to\infty}\|u_{i,n} - \bar{z}_n\| = 0, \lim_{n\to\infty}\|z_n - \bar{z}_n\| = 0$ and $\lim_{n\to\infty}\|S\bar{z}_n - \bar{z}_n\| = 0$.

Notice that

$$\begin{aligned} & \|u_{i,n} - \bar{z}_n\| \\ =\ & \|\Theta_n^N u_{i,n} - u_{i,n}\| \\ \leq\ & \|\Theta_n^N u_{i,n} - \Theta_n^{N-1}u_{i,n}\| + \|\Theta_n^{N-1}u_{i,n} - \Theta_n^{N-2}u_{i,n}\| \\ & + \ldots + \|\Theta_n^2 u_{i,n} - \Theta_n^1 u_{i,n}\| + \|\Theta_n^1 u_{i,n} - \Theta_n^0 u_{i,n}\| + \|u_{i,n} - u_{i,n}\|. \end{aligned} \tag{3.37}$$

From (2.5), obtain

$$\lim_{n\to\infty}\|u_{i,n} - \bar{z}_n\| = 0. \tag{3.38}$$

Moreover, from $\|z_n - \bar{z}_n\| \leq \|z_n - u_{i,n}\| + \|u_{i,n} - \bar{z}_n\|$, get

$$\lim_{n\to\infty}\|z_n - \bar{z}_n\| = 0. \tag{3.39}$$

Note from (3.1)

$$\begin{aligned} & \|z_n - \mathcal{S}_n\bar{z}_n\| \\ \leq\ & \|z_n - S_{n-1}\bar{z}_{n-1}\| + \|S_{n-1}\bar{z}_{n-1} - S_{n-1}\bar{z}_n\| + \|S_{n-1}\bar{z}_n - \mathcal{S}_n\bar{z}_n\| \\ \leq\ & \epsilon_{n-1}\|\xi f(x_{n-1}) - DS_{n-1}\bar{z}_{n-1}\| + \|\bar{z}_{n-1} - \bar{z}_n\| \\ & + \sigma_n\|x_{n-1} - S_{n-1}\bar{z}_{n-1}\| + \sup\{\|\mathcal{S}_{n+1}z - \mathcal{S}_n z\| : z \in \|\bar{z}_n\|\}. \end{aligned} \tag{3.40}$$

Since $\epsilon_n \to 0, \sigma_n \to 0$ as $n \to \infty, \lim_{n\to\infty}\|\bar{z}_{n+1} - \bar{z}_n\| = 0$ and $\sup\{\|\mathcal{S}_{n+1}z - \mathcal{S}_n z\| :\}$ $z \in \{\bar{z}_n\}\} \to 0$, obtain

$$\lim_{n\to\infty}\|z_n - \mathcal{S}_n\bar{z}_n\| = 0. \tag{3.41}$$

It now follows from (3.39), (3.40), and $\|\bar{z}_n - \mathcal{S}_n\bar{z}_n\| \leq \|\bar{z}_n - z_n\| + \|z_n - \mathcal{S}_n\bar{z}_n\|$ that $\lim_{n\to\infty}\|\bar{z}_n - \mathcal{S}_n\bar{z}_n\| = 0$. Since

$$\begin{aligned} \|S\bar{z}_n - \bar{z}_n\| &\leq \|S\bar{z}_n - \mathcal{S}_n\bar{z}_n\| + \|\mathcal{S}_n\bar{z}_n - \bar{z}_n\| \\ &\leq \sup\{\|Sz - \mathcal{S}_nz\| : z \in \{\bar{z}_n\}\} + \|\mathcal{S}_n\bar{z}_n - \bar{z}_n\|. \end{aligned} \tag{3.42}$$

Therefore,

$$\lim_{n\to\infty}\|S\bar{z}_n - \bar{z}_n\| = 0. \tag{3.43}$$

Step 6. This proof shows that $w \in (\cap_{n=1}^{\infty} Fix(\mathcal{S}_n)) \cap \Gamma \cap (\cap_{i=1}^{N} I(\mathcal{B}_i, \mathcal{M}_i))$.

First, show $w \in \cap_{n=1}^{\infty} Fix(\mathcal{S}_n)$. Assume that $w \notin \cap_{n=1}^{\infty} Fix(\mathcal{S}_n)$; then $w \neq \mathcal{S}w$. It follows from (2.1) and (3.43) that

$$\begin{aligned} \liminf_{n\to\infty}\|\bar{z}_n - w\| &< \liminf_{n\to\infty}\|\bar{z}_n - Sw\| \\ &\leq \liminf_{n\to\infty}\{\|\bar{z}_n - S\bar{z}_n\| + \|S\bar{z}_n - Sw\|\} \\ &\leq \liminf_{n\to\infty}\|\bar{z}_n - w\|. \end{aligned} \tag{3.44}$$

This is a contradiction. Hence, $w \in \cap_{n=1}^{\infty} Fix(\mathcal{S}_n)$.

Next, show that $w \in \Gamma$. Since $w \in \Gamma, w \in EP(\mathcal{F})$ and $\mathcal{A}_i w \in EP(\mathcal{F}_i)$ for each $i = 1, \ldots, N_1$. Let $w_{i,n} = (I - \mathcal{A}_i^*(I - \mathcal{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_i)z_n$ for each $i = 1, \ldots, N_1$. By (3.31) and (3.36), get that $w_{i,n} - z_n \to 0$ and $\mathcal{T}_{r_n}^{\mathcal{F}} w_{i,n} - w_{i,n} \to 0$ as $n \to \infty$. By Lemma 2.10, obtain $\|\mathcal{T}_{r_n}^{\mathcal{F}} w_{i,n} - \mathcal{T}_{r_n}^{\mathcal{F}} w_{i,n}\| \leq |1 - \frac{r}{r_n}|\|\mathcal{T}_{r_n}^{F} w_{i,n} - w_{i,n}\| \to 0$ as $n \to \infty$. Hence $\mathcal{T}_r^{\mathcal{F}} w_{i,n} - w_{i,n} \to 0$ as $n \to \infty$ for each $i = 1, \ldots, N_1$. Since $\mathcal{T}_{r_n}^{\mathcal{F}}$ is nonexpansive and $\{w_{i,n}\}$ converges weakly to w, by Lemma 2.12 obtain $w = \mathcal{T}_{r_n}^{\mathcal{F}} w$, that is $w \in EP(\mathcal{F})$. On the other hand, since $(I - \gamma\mathcal{A}_i^*(I - \mathcal{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_i)z_n - z_n \to 0$ (by (3.36)) and $I - \gamma\mathcal{A}_i^*(I - \mathcal{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_i$ is nonexpansive, from Lemma 2.10 and 2.12 it follows that $w = (I - \gamma\mathcal{A}_i^*(I - \mathcal{T}_{r_n}^{\mathcal{F}_i})\mathcal{A}_i)w$, *i.e.*, $w = \mathcal{T}_{r_n}^{\mathcal{F}}\mathcal{A}_i w$. Therefore, $w \in \Gamma$.

Now show that $w \in \cap_{i=1}^{N} I(\mathcal{B}_i, \mathcal{M}_i)$. In fact, since $\mathcal{B}_i$ is α_i-inverse strongly monotone, $\mathcal{B}_i$ is a $\frac{1}{\alpha_i}$-Lipschitz continuous monotone mapping and $D(\mathcal{B}_i) = \mathcal{H}_1$. It follows from Lemma 2.2 that $\mathcal{M}_i + \mathcal{B}_i, i = 1, 2, \ldots, N$ is maximal monotone. Let $(p, g) \in G(\mathcal{M}_i + \mathcal{B}_i), i = 1, 2, \ldots, N$ that is $g - \mathcal{B}_i p \in (\mathcal{M}_i p), i = 1, 2, \ldots, N$. Since $\Theta_n^k u_{i,n} = \mathcal{J}_{\mathcal{M}_N, \lambda_{N,n}}(I - \lambda_{N,n}\mathcal{B}_N)\Theta_n^{k-1} u_{i,n}$, get $\Theta_n^{k-1} u_{i,n} - \lambda_{k,n}\mathcal{B}_k\Theta_n^{k-1} u_{i,n} \in (I + \lambda_{k,n}\mathcal{M}_k)(\Theta_n^k u_{i,n})$, that is,

$$\frac{1}{\lambda_{k,n}}\left(\Theta_n^{k-1} u_{i,n} - \Theta_n^k u_{i,n} - \lambda_{k,n}\mathcal{B}_k\Theta_n^{k-1} u_{i,n}\right) \in \mathcal{M}_k\left(\Theta_n^k u_{i,n}\right). \tag{3.45}$$

By the maximal monotonicity of $\mathcal{M}_i + \mathcal{B}_i$, obtain

$$\left\langle p - \Theta_n^k u_{i,n}, g - \mathcal{B}_k p - \frac{1}{\lambda_{k,n}}\left(\Theta_n^{k-1} u_{i,n} - \Theta_n^k u_{i,n} - \lambda_{k,n}\mathcal{B}_k\Theta_n^{k-1} u_{i,n}\right)\right\rangle \geq 0, \tag{3.46}$$

which implies that

$$\begin{aligned}
& \langle p-\Theta_n^k u_{i,n}, g\rangle \\
\geq\ & \left\langle p-\Theta_n^k u_{i,n}, \mathcal{B}_k p+\frac{1}{\lambda_{k,n}}\left(\Theta_n^{k-1} u_{i,n}-\Theta_n^k u_{i,n}-\lambda_{k,n} \mathcal{B}_k \Theta_n^{k-1} u_{i,n}\right)\right\rangle \\
=\ & \left\langle p-\Theta_n^k u_{i,n}, \mathcal{B}_k p-\mathcal{B}_k \Theta_n^k u_{i,n}+\mathcal{B}_k \Theta_n^k u_{i,n}-\mathcal{B}_k \Theta_n^{k-1} u_{i,n}\right. \\
& \left.+\frac{1}{\lambda_{k,n}}\left(\Theta_n^{k-1} u_{i,n}-\Theta_n^k u_{i,n}\right)\right\rangle \\
\geq\ & 0+\left\langle p-\Theta_n^k u_{i,n}, \mathcal{B}_k \Theta_n^k u_{i,n}-\mathcal{B}_k \Theta_n^{k-1} u_{i,n}\right\rangle \\
& +\left\langle p-\Theta_n^k u_{i,n}, \frac{1}{\lambda_{k,n}}\left(\Theta_n^{k-1} u_{i,n}-\Theta_n^k u_{i,n}\right)\right\rangle .
\end{aligned} \tag{3.47}$$

for $k \in \{1,2,\ldots,N\}$. From $\lim_{n\to\infty}\|\Theta_n^k u_{i,n}-\Theta_n^{k-1}u_{i,n}\|=0$, especially $\Theta_{n_i}^k u_{i,n} \rightharpoonup w$. Since $\mathcal{B}_k, k = 1,2,\ldots,N$, are Lipschitz continuous operators, then $\|\mathcal{B}_k\Theta_n^{k-1}u_{i,n}-\mathcal{B}_k\Theta_n^k u_{i,n}\| \to 0$. Therefore, from (3.47), get

$$\lim_{i\to\infty}\langle p-\Theta_{n_i}^k u_{i,n_i}, g\rangle = \langle p-w, g\rangle \geq 0. \tag{3.48}$$

Since $\mathcal{B}_k+\mathcal{M}_k, k\in\{1,2,\ldots,N\}$ is maximal monotone, imply that $0\in(\mathcal{M}_k+\mathcal{B}_k)(w), k=\{1,2,\ldots,N\}$, that is, $w\in\cap_{i=1}^N I(\mathcal{B}_i,\mathcal{M}_i)$. Thus, obtain the desired result.

Next prove that

$$\limsup_{n\to\infty}\langle(\mathcal{D}-\xi f)z, z-z_{n_i}\rangle \leq 0, \tag{3.49}$$

where $z=P_\Omega(I-\mathcal{D}+\xi f)(z)$ is a unique solution of the problem (3.2). To that end, select a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that

$$\lim_{i\to\infty}\langle(\mathcal{D}-\xi f)z, z-z_{n_i}\rangle = \limsup_{n\to\infty}\langle(\mathcal{D}-\xi f)z, z-z_n\rangle. \tag{3.50}$$

From the claim of Step 5, obtain

$$\limsup_{n\to\infty}\langle(\mathcal{D}-\xi f)z, z-z_n\rangle = \lim_{i\to\infty}\langle(\mathcal{D}-\xi f)z, z-z_{n_i}\rangle = \langle(\mathcal{D}-\xi f)z, z-w\rangle \leq 0. \tag{3.51}$$

Step 7. Claim that $z_n \to w$.

By using Lemma 2.1 (2) and 2.3, obtain

$$\begin{aligned}
& \|z_{n+1}-w\|^2 \\
=\ & \|((1-\sigma_n)I-\epsilon_n\mathcal{D})(\mathcal{S}_n\bar{z}_n-w)+\sigma_n(z_n-w)+\epsilon_n(\xi f(z_n)-\mathcal{D}w)\|^2 \\
\leq\ & \|((1-\sigma_n)I-\epsilon_n\mathcal{D})(\mathcal{S}_n\bar{z}_n-w)+\sigma_n(z_n-w)\|^2 \\
& +2\epsilon_n\langle\xi f(z_n)-\mathcal{D}w, z_{n+1}-w\rangle \\
\leq\ & \|((1-\sigma_n)I-\epsilon_n\mathcal{D})(\mathcal{S}_n\bar{z}_n-w)\|^2+\sigma_n\|z_n-w\|^2 \\
& +2\epsilon_n\langle\xi f(z_n)-f(w), z_{n+1}-w\rangle+2\epsilon_n\langle\xi f(w)-\mathcal{D}w, z_{n+1}-w\rangle
\end{aligned}$$

$$\leq ((1-\sigma_n)-\epsilon_n\bar{\xi})\|\mathcal{S}_n\bar{z}_n - w\|^2 + \sigma_n\|z_n - w\|^2$$
$$+2\epsilon_n\xi\alpha\|z_n - w\|\|z_{n+1} - w\| + 2\epsilon_n\langle \xi f(w) - \mathcal{D}w, z_{n+1} - w\rangle$$
$$\leq ((1-\sigma_n)-\epsilon_n\bar{\xi})\|z_n - w\|^2 + \sigma_n\|z_n - w\|^2$$
$$+\epsilon_n\xi\alpha\Big(\|z_n - w\|^2 + \|z_{n+1} - w\|^2\Big) + 2\epsilon_n\langle \xi f(w) - \mathcal{D}w, z_{n+1} - w\rangle$$
$$\leq (1-\epsilon_n(\bar{\xi}-\xi\alpha))\|z_n - w\|^2 + \epsilon_n\xi\alpha\|z_{n+1} - w\|^2$$
$$+2\epsilon_n\langle \xi f(w) - \mathcal{D}w, z_{n+1} - w\rangle. \tag{3.52}$$

It follows that

$$\|z_{n+1} - w\|^2 \leq \left(1 - \frac{(\bar{\xi}-\alpha\xi)\epsilon_n}{1-\alpha\xi\epsilon_n}\right)\|z_n - w\|^2 + \frac{2\epsilon_n}{1-\alpha\xi\epsilon_n}\langle \xi f(w) - \mathcal{D}w, z_{n+1} - w\rangle. \tag{3.53}$$

Now, from conditions $(C1)-(C5)$, Step 6 and Lemma 2.7, get

$$\lim_{n\to\infty}\|z_n - w\| = 0.$$

This completes the proof. □

Corollary 3.2. *Let $\mathcal{H}_1, \mathcal{H}_2$ be two real Hilbert spaces and $\mathcal{C} \subset \mathcal{H}_1, \mathcal{Q} \subset \mathcal{H}_2$ be nonempty closed convex subsets. Let $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Assume that $\mathcal{F} : \mathcal{C}\times\mathcal{C} \to \mathcal{R}, \mathcal{F}_1 : \mathcal{Q}\times\mathcal{Q} \to \mathcal{R}$ are bifunctions satisfying $(A1)-(A4)$. Let $\mathcal{B}_i : \mathcal{C} \to \mathcal{H}_1$ be α_i-inverse strongly monotone mappings for each $i = 1, 2, \ldots, N_2$ with $N_2 \in \mathbb{N}$ and let $\mathcal{M}_i : H \to 2^H, i = 1, 2, \ldots, N$ be maximal monotone mappings such that $\Omega := (\cap_{n=1}^{\infty} Fix(\mathcal{S}_n)) \cap SEP \cap (\cap_{n=1}^{N} I(\mathcal{B}_i, \mathcal{M}_i)) \neq \emptyset$, where $SEP = \{z \in \mathcal{C} : z \in EP(\mathcal{F})$ and $\mathcal{A}z \in EP(\mathcal{F}_1)\}$. Let f be a contraction of $\mathcal{H}$ into itself with constant $\alpha \in (0,1)$ and let $\mathcal{D}$ be a strongly positive bounded linear operator on H with coefficient $\bar{\xi} > 0$ and $0 < \xi < \frac{\bar{\xi}}{\alpha}$. Let $\{z_n\}, \{\bar{z}_n\}$ and $\{u_n\}$ be sequences generated by $z_1 \in \mathcal{H}$ and*

$$\begin{cases} u_n = \mathcal{T}_{r_n}^{\mathcal{F}}(I - \gamma A^*(I - \mathcal{T}_{r_n}^{F_1})A)z_n, \\ \bar{z}_n = \mathcal{J}_{\mathcal{M}_N,\lambda_{N,n}}(I - \lambda_{N,n}\mathcal{B}_N)\ldots\mathcal{J}_{\mathcal{M}_1,\lambda_{1,n}}(I - \lambda_{1,n}\mathcal{B}_1)u_n, \\ z_{n+1} = \epsilon_n\xi f(z_n) + \sigma_n z_n + ((1-\sigma_n)I - \epsilon_n\mathcal{D})\mathcal{S}_n\bar{z}_n, \end{cases} \tag{3.54}$$

for all $n \in \mathbb{N}$, where $\sigma_n \in (0,1), \lambda_{i,n} \in (0, 2\alpha_i], i \in \{1, 2, \ldots, N\}$ satisfy $(H1)-(H2), \{\epsilon_n\} \subset [0,1], \{r_n\} \subset (r, \infty)$ with $r > 0, \gamma \subset (0, 1/L^2)$, L is the spectral radius of the operator $\mathcal{A}^*\mathcal{A}$ and $\mathcal{A}^*$ is the adjoint of $\mathcal{A}$ for each $i \in \{1, \ldots, N_1\}$ satisfying

- (C1) $\lim_{n\to\infty}\epsilon_n = 0$;
- (C2) $\sum_{n=1}^{\infty}\epsilon_n = \infty$;
- (C3) $\sum_{n=1}^{\infty}|\epsilon_{n+1} - \epsilon_n| < \infty$;
- (C4) $\sum_{n=1}^{\infty}|r_{n+1} - r_n| < \infty$;
- (C5) $\lim_{n\to\infty}\sigma_n = 0$.

Suppose that $\sum_{n=1}^{\infty} \sup\{\|\mathcal{S}_{n+1}z - \mathcal{S}_n z\| : z \in \mathcal{C}\} < \infty$*. Let* $\mathcal{S}$ *be a mapping of* $\mathcal{H}_1$ *into itself defined by* $Sz = \lim_{n\to\infty} \mathcal{S}_n z$*, for all* $z \in \mathcal{H}_1$ *and suppose that* $Fix(\mathcal{S}) = \cap_{n=1}^{\infty} Fix(\mathcal{S}_n)$*. Then, the sequence* $\{z_n\}$ *generated by (3.54) converges strongly to the point* $z = P_{\Omega}(I - \mathcal{D} + \xi f)(z)$*.*

Corollary 3.3. *Let* $\mathcal{H}_1, \mathcal{H}_2$ *be two real Hilbert spaces and* $\mathcal{C} \subset \mathcal{H}_1, \mathcal{Q} \subset \mathcal{H}_2$ *be nonempty closed convex subsets. Let* $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_2$ *be a bounded linear operator. Assume that* $\mathcal{F} : \mathcal{C} \times \mathcal{C} \to \mathcal{R}$ *is a bifunction that satisfies* $(A1) - (A4)$ *and let* $\mathcal{S}$ *be a nonexpansive mapping. Let* $\mathcal{B}_i : \mathcal{C} \to \mathcal{H}_1$ *be* α_i*-inverse strongly monotone mappings for each* $i = 1, 2, \dots, N_2$ *with* $N_2 \in \mathbb{N}$ *and let* $\mathcal{M}_i : \mathcal{H} \to 2^{\mathcal{H}}, i = 1, 2, \dots, N$ *be maximal monotone mappings such that* $\Omega := Fix(\mathcal{S}) \cap EP(\mathcal{F}) \cap (\cap_{n=1}^{N} I(\mathcal{B}_i, \mathcal{M}_i)) \neq \emptyset$*. Let* f *be a contraction of* $\mathcal{H}$ *into itself with constant* $\alpha \in (0, 1)$ *and let* $\mathcal{D}$ *be a strongly positive bounded linear operator on* $\mathcal{H}$ *with coefficient* $\bar{\xi} > 0$ *and* $0 < \xi < \frac{\bar{\xi}}{\alpha}$*. Let* $\{z_n\}, \{\bar{z}_n\}$ *and* $\{u_n\}$ *be sequences generated by* $x_1 \in H$ *and*

$$\begin{cases} u_n = \mathcal{T}_{r_n}^{\mathcal{F}} z_n, \\ \bar{z}_n = \mathcal{J}_{\mathcal{M}_N, \lambda_{N,n}}(I - \lambda_{N,n}\mathcal{B}_N) \dots \mathcal{J}_{\mathcal{M}_1, \lambda_{1,n}}(I - \lambda_{1,n}\mathcal{B}_1)u_n, \\ z_{n+1} = \epsilon_n \xi f(z_n) + \sigma_n z_n + ((1 - \sigma_n)I - \epsilon_n \mathcal{D})\mathcal{S}\bar{z}_n, \end{cases} \tag{3.55}$$

for all $n \in \mathbb{N}$*, where* $\sigma_n \in (0, 1), \lambda_{i,n} \in (0, 2\alpha_i], i \in \{1, 2, \dots, N\}$ *satisfy* $(H1) - (H2)$*,* $\{\epsilon_n\} \subset [0, 1], \{r_n\} \subset (r, \infty)$ *with* $r > 0$ *satisfy*

(C1) $\lim_{n\to\infty} \epsilon_n = 0$;

(C2) $\sum_{n=1}^{\infty} \epsilon_n = \infty$;

(C3) $\sum_{n=1}^{\infty} |\epsilon_{n+1} - \epsilon_n| < \infty$;

(C4) $\lim_{n\to\infty} \sigma_n = 0$.

Then, the sequence $\{z_n\}$ *generated by (3.55) converges strongly to the point* $z = P_{\Omega}(I - \mathcal{D} + \xi f)(z)$*.*

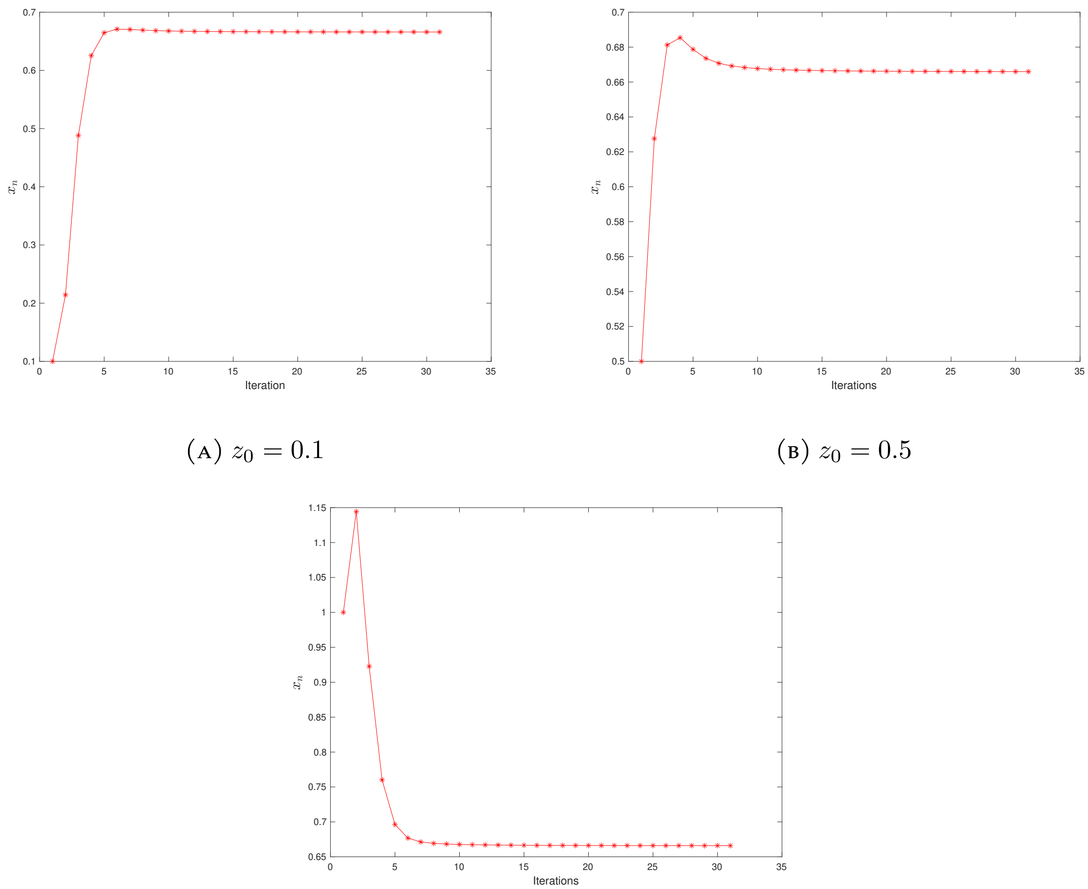
4. Numerical Illustration

In this section, the author present main result by considering the following numerical example.

Example 4.1. *Suppose* $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{R}$*, the set of all real numbers with inner product defined by* $\langle z, \bar{z}\rangle = z\bar{z}$*, for all* $z, \bar{z} \in \mathcal{R}$ *and induced norm* $|\cdot|$ *. Suppose that* $\mathcal{C} = [0, 1]$ *and* $\mathcal{Q} = [-3, 0]$*, let the bifunctions* $\mathcal{F} : \mathcal{C} \times \mathcal{C} \to \mathcal{R}$ *and* $\mathcal{F}_i : \mathcal{Q} \times \mathcal{Q} \to \mathcal{R}$ *be defined by* $\mathcal{F}(z, \bar{z}) = (z - 1)(\bar{z} - z)$ *for all* $z, \bar{z} \in \mathcal{C}$ *and* $\mathcal{F}_i(u, v) = (u + 6i)(v - u)$ *for all* $u, v \in \mathcal{Q}$ *respectively. Let the mappings* $\mathcal{B}_i : \mathcal{C} \to \mathcal{H}_1$ *be defined as* $\mathcal{B}_i(z) = 2iz$ *for each* $i = 1, 2, \cdots N_1$*. Suppose* $\mathcal{M}_i : \mathcal{R} \to 2^{\mathbb{H}}$ *be* $\mathcal{M}_i = \{2iz\}$ *for each* $i = 1, 2, \dots, N_2$*. We further supposed the mapping* $\mathcal{D}$ *to be identity mapping, the contraction* $f : \mathcal{C} \to \mathcal{C}$ *to be defined by* $f(z) = \frac{2}{3}z$*, for all* $x \in C$*,* $\mathcal{A}_i : \mathcal{H}_1 \to \mathcal{H}_2$ *be defined by* $\mathcal{A}_i(z) = -\frac{9}{4}iz$ *for each* $i = 1, 2, \dots, N_2$ *and for* $z \in \mathcal{C}$*, let* $\mathcal{S}_i(z) = \frac{z+2i}{1+3i}$ *for each* $i = 1, 2, \dots, N$*.*

It is easy to prove that the bifunctions $\mathcal{F}$ and $\mathcal{F}_i$ satisfy Assumption 2.8. The mapping $\mathcal{A}$ is a bounded linear operator on $\mathcal{R}$ with $\|\mathcal{A}\| = \|\mathcal{A}^*\| = \frac{9}{4}$. Further, $\mathcal{B}_i$ is $\frac{1}{2i}$- inverse strongly monotone mappings for each $i = 1, 2, \dots, N$. Moreover, it is easy to observe that $\mathcal{S}_i$ are nonexpansive mappings with $\text{Fix}(\mathcal{S}) = [0, \infty)$.

For the numerical illustration, setting $N = N_1 = N_2 = 2$ and choose the control parameters as $\epsilon_n = \frac{1}{n+1}, \sigma_n = \frac{1}{n^2}$, for $\lambda_{i,n} = \frac{1}{4}$ for each $i = 1, 2, \ldots, N, \xi = 1.4$ and $r_n = \frac{1}{4}$. For reference, the Matlab codes were written in personal computer.



(A) $z_0 = 0.1$

(B) $z_0 = 0.5$

(C) $z_0 = 1.0$

FIGURE 1. Convergence of z_n with different initial points.

Table 1. Convergence results of x_n

Iterations	$x_0 = 0.1$ x_n	Iterations	$x_0 = 0.5$ x_n	Iterations	$x_0 = 1.0$ x_n
1	0.1	1	0.5	1	1
2	0.2142	2	0.6276	2	1.1444
3	0.4881	3	0.6812	3	0.9227
4	0.6255	4	0.6854	4	0.7603
5	0.6647	5	0.6787	5	0.6963
6	0.6709	6	0.6736	6	0.6769
7	0.6703	7	0.6707	7	0.6713
8	0.669	8	0.6692	8	0.6693
9	0.6683	9	0.6683	9	0.6683
10	0.6677	10	0.6677	10	0.6677
11	0.6673	11	0.6673	11	0.6673
12	0.667	12	0.667	12	0.667
13	0.6668	13	0.6668	13	0.6668
14	0.6667	14	0.66667	14	0.6667
15	0.6667	15	0.6667	15	0.6667
16	0.6667	16	0.6667	16	0.6667
17	0.6667	17	0.6667	17	0.6667
18	0.6667	18	0.6667	18	0.6667
19	0.6667	19	0.6667	19	0.6667
20	0.6667	20	0.6667	20	0.6667

It can be observed from Figure 1 and Table 1 that, the iterates generated converge to the common solution of the considered problems.

5. Conclusion

In this paper, an iterative scheme for approximations of a common element of the set of solutions of finite family of variational inclusion problems consisting of set of finite family of inverse strongly monotone operators and finite family of multi-valued maximal monotone mappings, set of solutions of family of split equilibrium problems and the set of fixed points of a family of nonexpansive mappings in real Hilbert space. Under some suitable and easy to verify conditions, some strong convergence of the sequences generated by the proposed method to the common element of the solutions of the considered problems are established. A numerical example to illustrate the implementation of the proposed method indicates that, the proposed method is implementable and theorem extends and improves the corresponding results obtained in [3] and some existing results in the literature.

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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