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PITCHFORK EDGE DOMINATION IN GRAPHS

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Abstract. This study presents the pitchfork edge domination, a novel model of domination in graphs is introduced here. Let $G = (V, E)$ be a simple, finite and undirected graph without isolated edges. A set of edges D_e is said to be pitchfork edge dominating set if $r \leq |N(e) \bigcap (E - D_e)| \leq s$ for every $e \in D_e$, where r and s are non-negative integers. That means every edge $e \in D_e$ dominates at least r and at most s edges of $E - D_e$. The minimum cardinality for all pitchfork edge dominating sets in G is the pitchfork edge domination number $\gamma_{pfe}(G)$. Pitchfork edge domination at $r = 1$ and $s = 2$ is discussed in this paper. There are certain limitations on $\gamma_{pfe}(G)$ pertaining to the order, size, minimum degree and maximum degree of the graph and other properties are proved here. Pitchfork edge domination is applied for some well-known graphs.

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1. Introduction

Given a graph $G = (V, E)$ of size $m = |E|$ and order $n = |V|$ that has no isolated edges. An edge of degree 0 is regarded as an isolated edge. $Deg(e) = deg(u) + deg(v) - 2$ defines the degree of an edge $e = uv$ of a graph G as the number of edges that adjacent to it. The symbols $\Delta''(G)$ and $\delta''(G)$ indicate the maximum and minimum degrees of the graph G, respectively. The open neighborhood of an edge e in G is the set that contains all edges that are adjacent to it, and it is represented by the symbol $N(e)$. Moreover, the closed neighborhood of e in G is $N[e] = N(e) \cup \{e\}$. The subgraph $G[D_e]$ of a graph G induced by the edges of set D_e. The complement graph $\overline{G}$ is the graph in which two edges are adjacent if and only if they are not adjacent in a simple graph G. For specific terminology related to graph theory, see [15, 26, 28]. The study of dominating sets is a broad field in graph theory. See [16, 17] for a thorough overview of domination. A subset $D_e \subseteq E$ is edge dominating set, if every

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$e \in E$ is either belongs to D_e or adjacent with one or more edges from D_e. If there is no proper edge dominating subset in D_e, then it is considered as a minimal edge dominating set. The cardinality of the minimum edge dominating set is the edge domination number $\gamma_e(G)$. The terms domination number and dominating set were first used by Ore [26] in 1962. According to the relevance of domination in a variety of applications, several forms of domination have emerged based on their intended use. Some researchers focused on the vertices domination and provided several definitions and characteristics in their studies, such as [1,9,10,24,25,30]. While others concentrated on the edges domination, such as [13,14,29]. Domination theory has continued to evolve, with researchers exploring various types of domination from the prominent researchers who have contributed to this continuing discourse, V. R. Kulli and N. D. Soner presented complementary edge domination in graphs in 1997 [21]. In 2000, V. R. Kulli and B. Janakiram studied the nonsplit domination number of a graph [20]. Chin Lung Lu et al. [22] presented the concepts of perfect edge domination and efficient edge domination in graphs in 2002. However, disjoint dominating and total dominating sets in graphs were given by Michael A. Henning et al. [18] in 2010. In 2017, A. A. Omran and Y. Rajihy [25] investigated several characteristics of frame domination in graphs. The concepts of pitchfork domination and its inverse for corona and join operations in graphs were presented in 2019 by M. Al-Harere and M. Abdlhusein [6]. In [8,11] many properties were studied and the inverse pitchfork domination was found for certain complements of graphs and for some operations such as corona and join in 2020. M. A. Abdhusein introduced stability of inverse pitchfork domination [3] in 2021. Some modified types of pitchfork dominion and its inverse are given by M. A. Abdlhusein and others [2,4,5,7,12] in 2022. S. J. Radhi et al. [27] proposed the definition of arrow domination and its characteristics are established in 2021. A study on equality co-neighborhood domination in graphs was conducted in 2022 by A. A. Omran et al. [23]. In [19] new idea was given to construct new graphs from discrete topological space.

In this study, we work on the same definition of pitchfork vertex domination, but for edges, which is a novel model of domination in graphs. This kind of domination is depends on the number of dominated edges, that is useful for every kind of networks that needs these characteristics. Several bounds are given for the pitchfork edge domination number concerned to size, order, minimum degree, maximum degree of a graph and other attributes. Pitchfork edge domination is also determined for a known graphs. Some questions are discussed here: is there a pitchfork edge domination in every undirected, finite and simple graph G with no isolated edges? does the pitchfork edge domination appear on every graph that has the pitchfork vertex domination? and is the converse true or not?
After that, we will review a table showing the pitchfork vertex domination and pitchfork edge domination for some well-known graphs.

2. Pitchfork Edge Domination

This section presents the definition of pitchfork edge domination, a novel model of graph domination. Some characteristics for this kind of domination are presented.

Definition 2.1. *Let $G = (V, E)$ be a simple, finite and undirected graph with no isolated edges, a subset $D_e \subseteq E(G)$ if $r \leq |N(e) \bigcap (E - D_e)| \leq s \ \forall e \in D_e$, where $r, s \in Z^+ \cup \{0\}$ is called pitchfork edge dominating set that means each edge in it dominates at least r and at most s edges outside the dominating set and denoted by $PEDS$.*

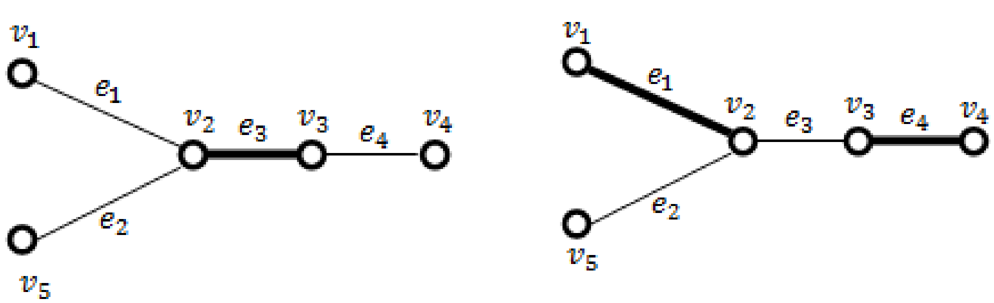
Definition 2.2. *The subset $D_e \subseteq E(G)$ is called minimal pitchfork edge dominating set if there is not a proper pitchfork edge dominating subset of D_e and denoted by $MPEDS$.*

Definition 2.3. *A smallest pitchfork edge dominating set of a graph G is said to be minimum pitchfork edge dominating set. Such set is referred as $\gamma_{pfe}-$set.*

Definition 2.4. *The minimum number of elements over all pitchfork edge dominating sets in a graph G is called pitchfork edge domination number and denoted by $\gamma_{pfe}(G)$.*

All results of the pitchfork edge domination in this paper are present for $r = 1$ and $s = 2$. We will give two examples to explain the difference between edge domination in general and pitchfork edge domination.

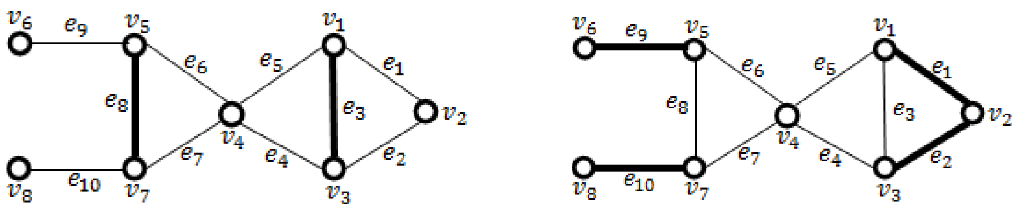
Example 2.5. *A graph G has five vertices and four edges. Let the set of vertices is $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and the set of edges is $E(G) = \{e_1, e_2, e_3, e_4\}$. Since in Fig. (1a), the edge e_3 dominates all other edges. So, $D_e = \{e_3\}$ is the minimum edge dominating set and $\gamma_e(G) = 1$. In Fig. (1b), the edge e_4 dominates e_3, while e_1 dominates e_2 and e_3. Thus, the minimum pitchfork edge dominating set $D_e = \{e_1, e_4\}$ because in it each edge dominates one or two edges. Then, $\gamma_{pfe}(G) = 2$*



(A) γ_e-set (B) $\gamma_{pfe}-$set

Figure 1. A pitchfork edge domination

Example 2.6. *Let G be a graph consists of eight vertices and ten edges, where the set of vertices is $V(G) = \{v_1, v_2, \dots, v_8\}$ and the set of edges is $E(G) = \{e_1, e_2, \dots, e_{10}\}$. The edge e_3 dominates e_1, e_2, e_4 and e_5 while e_8 dominates e_6, e_7, e_9 and e_{10}. Hence, the minimum edge dominating set in G is $\{e_3, e_8\}$. See Fig. (2a).*
In Fig. (2b), the edge e_1 dominates e_3 and e_5, the edge e_2 dominates e_3 and e_4, the edge e_9 dominates e_6 and e_8, the edge e_{10} dominates e_7 and e_8. Therefore, the minimum pitchfork edge dominating set is $D_e = \{e_1, e_2, e_9, e_{10}\}$ because in it every edge dominates exactly two edges.

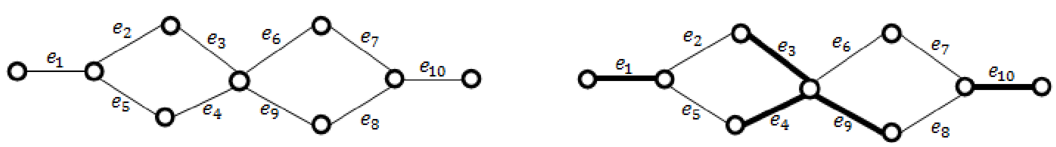


(A) γ_e−set (B) γ_{pfe}−set

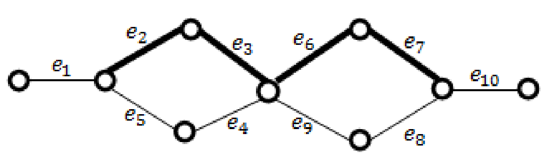
Figure 2. The pitchfork edge domination

The following example shows the difference between minimal and minimum pitchfork edge dominating sets.

Example 2.7. *In Fig. (3), the graph G has many minimal pitchfork edge dominating sets. Where: $D_e = \{e_1, e_3, e_4, e_9, e_{10}\}$ is minimal pitchfork edge dominating set of cardinality 5. $D'_e = \{e_2, e_3, e_6, e_7\}$ is minimal pitchfork edge dominating set of cardinality 4. Since $|D'_e| < |D_e|$, then D'_e is the minimum pitchfork edge dominating set. Hence, $\gamma_{pfe}(G) = |D'_e| = 4$.*



(A) (G) (B) (D_e)



(C) (D'_e)

Figure 3. The minimal and minimum pitchfork edge dominating sets

Proposition 2.8. *Suppose that $G(n,m)$ contains a pitchfork edge domination number $\gamma_{pfe}(G)$, then:*

(1) $m \geq 2$.
(2) *The order of G is* $n \geq 3$.
(3) $\Delta''(G) \geq 1$ *and* $\delta''(G) \geq 1$.
(4) $\gamma_e(G) \leq \gamma_{pfe}(G)$.
(5) $\gamma_{pfe}(G) \geq 1$.
(6) $N(e) \bigcap (E - D_e) \neq \emptyset \ \forall e \in D_e$.
(7) $N(e') \bigcap D_e \neq \emptyset \ \forall e' \in E - D_e$.
(8) $N[D_e] = E(G)$.
(9) $1 \leq |N(e) \cap (E - D_e)| \leq 2 \ \forall e \in D_e$.

Theorem 2.9. *Given a graph G and D_e be a pitchfork edge dominating set in it, D_e is the minimal pitchfork edge dominating set if one of the following conditions holds:*

(1) $|N(e) \cap (E - D_e)| = 2, \ \forall\ e \in D_e$.
(2) $|N(e') \cap D_e| = 1, \ \forall\ e' \in E - D_e$.
(3) *All edges of $G[D_e]$ are isolated edges.*

Proof. Considering D_e be a $PEDS$ in G, such that D_e is not the $MPEDS$, that mean there exist at least one edge say $e \in D_e$, such that the minimal pitchfork edge dominating set is $D_e - \{e\}$. Then, we address the above conditions as follows:

Case 1: There are two cases if the first condition is true:

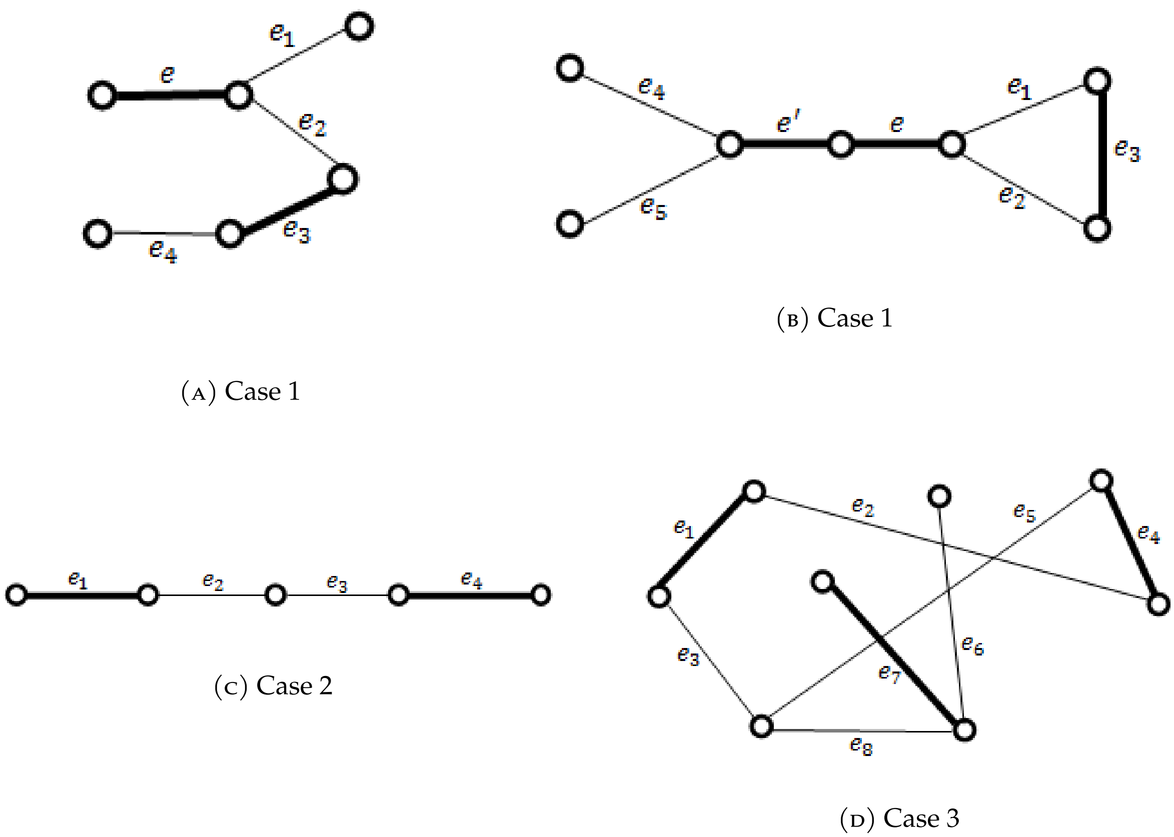
Subcase 1. If one or two edges e_1, e_2 from $E - D_e$ which are dominated by only the edge e. Then, there is no any edge in $D_e - \{e\}$ dominates e_1 or e_2. Thus, $D_e - \{e\}$ is not a $PEDS$ and this is a contradiction. See Fig. (4a) in Example 2.10.

Subcase 2. If $D_e - \{e\}$ has one or more edges that dominate the two edges e_1 and e_2 which are different from e. Suppose that there exist $e' \in D_e - \{e\}$ is adjacent to e. According to the first condition, each edge dominates exactly two edges, that mean e' dominates two edges and the edge e. Hence, e' dominates three edges and this is a contradiction. Therefore, $D_e - \{e\}$ is not pitchfork edge dominating set and this is contradiction. See Fig. (4b) in Example 2.10.

Case 2: Assume that the second condition is holds, such that $e' \in E - D_e$ is dominated by only one edge say $e \in D_e$. Then, there is no edge in $D_e - \{e\}$ dominates e'. As a result, $D_e - \{e\}$ is not a $PEDS$. See Fig. (4c) in Example 2.10.

Case 3: Assuming the third condition is true, meaning that e is not adjacent to any edge of D_e. As a result, no edge from $D_e - \{e\}$ dominates e. Therefore, the set $D_e - \{e\}$ is not pitchfork edge dominating set. Hence, $D_e - \{e\}$ is not a $PEDS$ in any of the above cases. Consequently, the minimal pitchfork edge dominating set is D_e. See Fig. (4d) in Example 2.10. □

Example 2.10. *Let G be a graph has $\gamma_{pfe}(G)$ we have $D_e = \{e, e_3\}$. The edge e dominates e_1 and e_2, the edge e_3 dominates e_2 and e_4. If we consider e outside the dominating set, then the edge e_1 does not has an edge that dominates it, and this is a contradiction. Also, the edge e does not has an edge that dominates it, because e_3 dominates only two edges, and this is contradiction. See Fig. (4a). In Fig. (4b), Let $D_e = \{e, e', e_3\}$ since all the edges of D_e dominate two edges of $E - D_e$. Where e dominates e_1 and e_2 and the edge e' dominates e_4 and e_5, while the edge e_3 dominates e_1 and e_2. If the edge e is removed from the dominating set D_e, then there exist $e_3 \in D_e - \{e\}$ dominates e_1 and e_2. Since there exist $e' \in D_e - \{e\}$ is adjacent to e, an edge e' dominates two edges and the edge e. Therefore, e' dominates three edges and this is contradiction. Thus, $D_e - \{e\}$ is not $PEDS$. In Fig. (4c), $D_e = \{e_1, e_4\}$, then in $D_e - \{e_1\}$ there is no edge dominates e_2. Also, in $D_e - \{e_4\}$ there is no edge dominates e_3. In Fig. (4d), let $D_e = \{e_1, e_4, e_7\}$ every edge of D_e is adjacent with only two edges from $E - D_e$. This means every two edges in D_e are not adjacent.*



(A) Case 1

(B) Case 1

(C) Case 2

(D) Case 3

FIGURE 4. Minimal pitchfork edge dominating set

Theorem 2.11. *Any graph $G = (n, m)$ has a pitchfork edge domination number $\gamma_{pfe}(G)$, then:*

$$\lceil \frac{m}{3} \rceil \leq \gamma_{pfe}(G) \leq m - 1$$

Proof. Assume that D_e be $\gamma_{pfe}-$set in G, then for the lower bound, in agreement with the definition of the pitchfork edge domination. Since each edge in the dominating set dominates one or two edges, if we assume that each edge of the dominating set dominates only one. This means $\lceil \frac{m}{2} \rceil \leq \gamma_{pfe}(G)$. But if we assume that each edge in the dominating set dominates exactly two edges, then $\lceil \frac{m}{3} \rceil \leq \gamma_{pfe}(G)$. Since one or two edges of $E - D_e$ are dominated by every edge $e \in D_e$. Therefore, the lower bound is $\lceil \frac{m}{3} \rceil \leq \gamma_{pfe}(G)$.

The upper bound is proven by using the fact that, the set $E - D_e$ can not be an empty set. So, it must contain at least one edge. Hence, $\gamma_{pfe}(G) \leq m - 1$. □

Proposition 2.12. *Suppose that $G(n, m)$ be any graph then:*

$$\gamma_e(G) \leq \gamma_{pfe}(G) \leq m - 1$$

Proof. The proof is directed from Theorem 2.11. □

The relationship between a graph's size and graph pitchfork edge domination number is found in the following theorem.

Theorem 2.13. *For any graph $G = (n, m)$ having $\gamma_{pfe}(G)$, then:*

$$2 \leq m \leq \binom{m}{2} + \gamma_{pfe}^2(G) - m\gamma_{pfe}(G)$$

Proof. Suppose that D_e be a pitchfork edge dominating set of a graph G, then:

Case 1: Assume that $G[D_e]$ and $G[E - D_e]$ are subgraphs contain only one edge such that G contains few edges in order to show the lower bound. According to the pitchfork edge domination definition, this edge of D_e dominates only one edge of $E - D_e$. Therefore, the total number of edges is equal to $m_1 = |D_e| = \gamma_{pfe}(G) = 1$ and $m_2 = |E - D_e| = 1$. Consequently, the lower bound is $2 \leq m$.

Case 2: To prove the upper bound, assume that $G[D_e]$ and $G[E - D_e]$ are two complete subgraphs such that G has a largest possible number of edges and let the number of edges of D_e and $E - D_e$ equals to m_1 and m_2 respectively. Then

$$\begin{aligned} m_1 &= \frac{|D_e||D_e - 1|}{2} = \frac{\gamma_{pfe}(\gamma_{pfe} - 1)}{2} \\ m_2 &= \frac{|E - D_e||E - D_e - 1|}{2} = \frac{(m - \gamma_{pfe})(m - \gamma_{pfe} - 1)}{2} \end{aligned}$$

According to the definition of pitchfork edge domination, suppose that each edge of D_e dominates two edges from $E - D_e$, then the graph G has the number of edges equal to

$$\begin{aligned} m &\leq m_1 + m_2 \\ &\leq \frac{1}{2}\left(\gamma_{pfe}^2 - \gamma_{pfe}\right) + \frac{1}{2}\left(m^2 - m\gamma_{pfe} - m - m\gamma_{pfe} + \gamma_{pfe}^2 + \gamma_{pfe}\right) \\ &\leq \gamma_{pfe}^2 - m\,\gamma_{pfe} + \frac{m^2 - m}{2} \end{aligned}$$

In general, this is the upper bound.

The lower bound is sharp for $G = P_3$ and the upper bound is sharp for $G = K_4$. □

3. Pitchfork Edge Domination of Some Graphs

The pitchfork edge domination is specify for several families of graphs. The pitchfork edge domination number and the minimum pitchfork edge dominating set are studied here.

Theorem 3.1. *For any path graph $P_n, (n \geq 3)$ we have:*

$$\gamma_{pfe}(P_n) = \begin{cases} \lceil \frac{n}{3} \rceil & if\ n \equiv 0, 2\,(mod\,3\,) \\ \lfloor \frac{n}{3} \rfloor & if\ n \equiv 1\,(mod\,3\,) \end{cases}$$

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices and $e_1, e_2, \ldots, e_{n-1}$ be the edges of P_n and let $D_e \subseteq E(P_n)$ defined as:

$$D_e = \begin{cases} \{e_{3i-1}, i = 1, 2, \ldots, \frac{n}{3}\} & if\ n \equiv 0\,(mod\,3\,) \\ \{e_{3i-1}, i = 1, 2, \ldots, \frac{n-1}{3}\} & if\ n \equiv 1\,(mod\,3\,) \\ \{e_{3i-2}, i = 1, 2, \ldots, \frac{n+1}{3}\} & if\ n \equiv 2\,(mod\,3\,) \end{cases}$$

To prove D_e is $PEDS$, we shall talk about three cases:

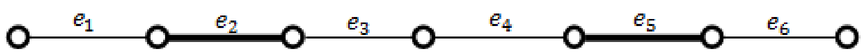
Case 1: If $n \equiv 0\,(mod\,3\,)$. Let $D_e = \{e_{3i-1}, i = 1, 2, \ldots, \frac{n}{3}\}$ every edge in D_e dominates two edges except the last edge where it dominates one edge. So, each edge in D_e dominates one or two edges. Thus, D_e is a pitchfork edge dominating set and $\gamma_{pfe} = \lceil \frac{n}{3} \rceil$.

Case 2: If $n \equiv 1\,(mod\,3\,)$. Let $D_e = \{e_{3i-1}, i = 1, 2, \ldots, \frac{n-1}{3}\}$, since every edge in D_e adjacent exactly two edges, then it dominates exactly two edges. Thus, D_e is the pitchfork edge dominating set and $\gamma_{pfe} = \lfloor \frac{n}{3} \rfloor$.

Case 3: If $n \equiv 2\,(mod\,3\,)$. Let $D_e = \{e_{3i-2}, i = 1, 2, \ldots, \frac{n+1}{3}\}$, the first and last edges of D_e dominates only one edge, while the rest of the edges of D_e dominate two edges. Thus, D_e is pitchfork edge dominating set and $\gamma_{pfe} = \lceil \frac{n}{3} \rceil$.

To prove that D_e is a minimum pitchfork edge dominating set in all the previous cases, assume that D'_e is a $PEDS$ such that $|D'_e| < |D_e|$. Then, there exist one or more edges of $E - D_e$ that are not dominated by any edge of D'_e. This contradicts with the concept of the pitchfork edge dominating set. Therefore, D_e is a $\gamma_{pfe}-$set. See Fig. (5).

(A) $n \equiv 0\,(mod\,3\,)$



(B) $n \equiv 1\,(mod\,3\,)$

(C) $n \equiv 2\,(mod\,3\,)$

FIGURE 5. The pitchfork edge domination of path graph

□

Theorem 3.2. *Given a cycle graph C_n, then:*

$$\gamma_{pfe}(C_n) = \lceil \frac{n}{3} \rceil$$

Proof. To prove D_e is the pitchfork edge dominating set in cycle graph. Let $e_1, e_2, \ldots, e_n$ be the edges of a cycle graph of order n and $D_e \subseteq E(C_n)$ such that

$$D_e = \begin{cases} \{e_{3i-2}, i = 1, 2, \ldots, \frac{n}{3}\} & if\ n \equiv 0\,(mod\,3\,) \\ \{e_{3i-2}, i = 1, 2, \ldots, \frac{n+2}{3}\} & if\ n \equiv 1\,(mod\,3\,) \\ \{e_{3i-2}, i = 1, 2, \ldots, \frac{n+1}{3}\} & if\ n \equiv 2\,(mod\,3\,) \end{cases}$$

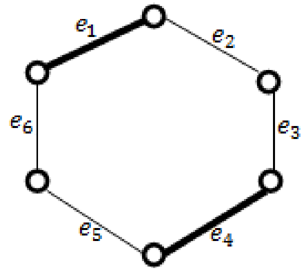
So, there are three cases to discuss as follows:

Case 1: If $n \equiv 0\,(mod\,3\,)$. Let $D_e = \{e_{3i-2}, i = 1, 2, \ldots, \frac{n}{3}\}$, then every $e \in D_e$ dominates exactly two edges. Then, D_e is a $PEDS$.

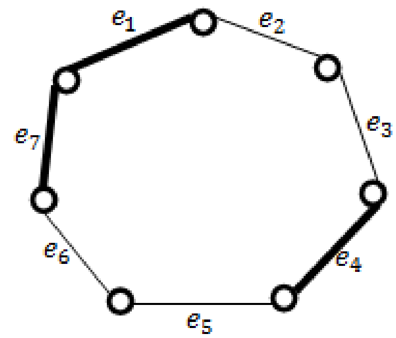
Case 2: If $n \equiv 1\,(mod\,3\,)$. Let $D_e = \{e_{3i-2}, i = 1, 2, \ldots, \frac{n+2}{3}\}$, then each edge of D_e dominates two edges expect the first and last edges which are dominate only one edge. Therefore, D_e is a pitchfork edge dominating set.

Case 3: If $n \equiv 2\,(mod\,3\,)$. Let $D_e = \{e_{3i-2}, i = 1, 2, \ldots, \frac{n+1}{3}\}$, then each edge of D_e dominates two edges. Hence, D_e is a $PEDS$. Thus, in all of the previous cases D_e is a pitchfork edge dominating set and $\gamma_{pfe}(C_n) = \lceil \frac{n}{3} \rceil$.

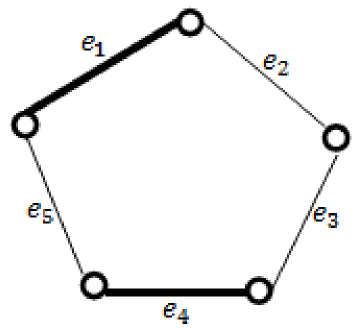
In all three of the above cases, the set D_e is a minimum pitchfork edge dominating set, and the proof of it is similar to the proof of Theorem 3.1.



(A) $n \equiv 0\,(mod\,3)$



(B) $n \equiv 1\,(mod\,3)$



(C) $n \equiv 2\,(mod\,3)$

FIGURE 6. The pitchfork edge domination of cycle graph

□

Theorem 3.3. *For a wheel graph W_n, then:*

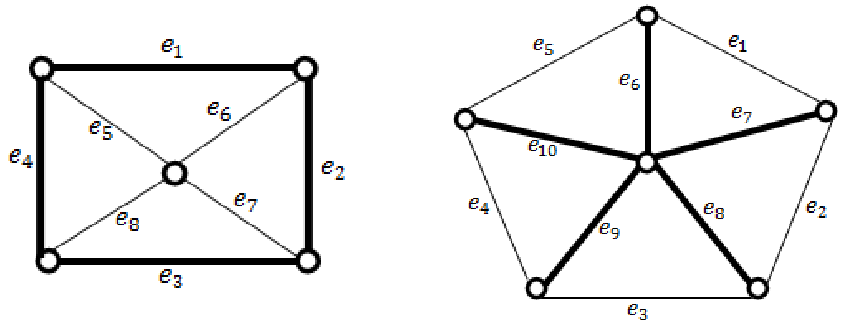
$$\gamma_{pfe}(W_n) = n$$

Proof. To prove the pitchfork edge dominating set. So, there are two possibilities cases for choosing the cardinality of D_e:

Case 1: Let D_e consists of all the edges that belong to the cycle C_n where the number of these edges are equal to n, then each edge of D_e dominates exactly two edges of $E - D_e$. Therefore, D_e is a pitchfork edge dominating set. Thus, $\gamma_{pfe}(W_n) = |D_e| = n$.

Case 2: Let D_e consists of all edges that join the vertex of K_1 with all vertices of C_n and the number of these edges are equal to n. Therefore, exactly two edges of $E - D_e$ are dominated by every edge of D_e. Hence, $\gamma_{pfe}(W_n) = |D_e| = n$.

Assume that D'_e is a pitchfork edge dominating set, such as $|D'_e| < |D_e|$. This indicates that some edges in D'_e dominate three edges of $E - D'_e$, which is contradictory. This proves that D_e in all above cases is the $MPEDS$.



(A) $\gamma_{pfe}(W_4) = 4$

(B) $\gamma_{pfe}(W_5) = 5$

FIGURE 7. The pitchfork edge domination of wheel graph

□

Theorem 3.4. *For a complete graph of order* $n, (n \geq 3)$ *and size* m *then:*

$$\gamma_{pfe}(K_n) = \begin{cases} 1 & \textit{if } n = 3 \\ 3 & \textit{if } n = 4 \\ m - \lfloor \frac{m}{2} \rfloor & \textit{if } n > 4 \end{cases}$$

Proof. To prove the pitchfork edge dominating set there are two cases.

Case 1: If $n = 3$, since K_3 has three edges and every edge of these edges is adjacent with the other two edges. We take one of these edges that it dominates the other two edges.

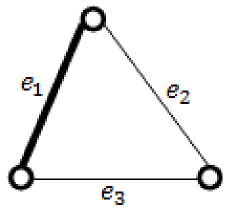
Case 2: Since K_4 has six edges and according to the definition of pitchfork edge domination, that every edge dominates at most two edges, let D_e is any three adjacent edges that have one common vertex. So, $\gamma_{pfe}(K_4) = 3$.

Case 3: If $n > 4$, let us label $\{e_i, i = 1, 2, \ldots, n\}$ be the set of the outer edges that represent the outer circle of K_n. Suppose that $D_e = E - \{e_i, i \text{ is even number}\}$. Then, there are two subcases as follows:

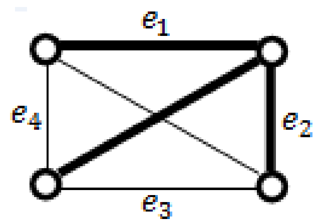
Subcase 1: If n is even, therefore exactly two edges are dominated by every edge of D_e. So, D_e is pitchfork edge dominating set.

Subcase 2: If n is odd, then every edge of D_e dominates two edges except e_1, e_n and all the edges which are adjacent with them by one common vertex. These edges dominate only one edge. Thus, D_e is pitchfork edge dominating set.

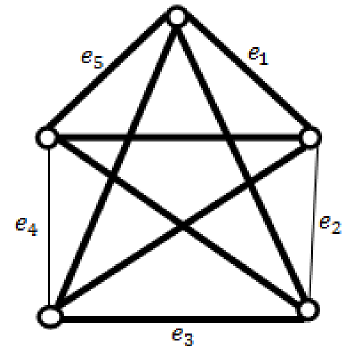
The proof that D_e is a $\gamma_{pfe}-$set in each of the given cases, assume that D'_e is pitchfork edge dominating set and $|D'_e| < |D_e|$, then there exist some edges of D'_e dominate three edges of $E - D'_e$. The definition of pitchfork edge domination will contradiction to this. Thus, D_e is $\gamma_{pfe}-$set. See Fig. (8).



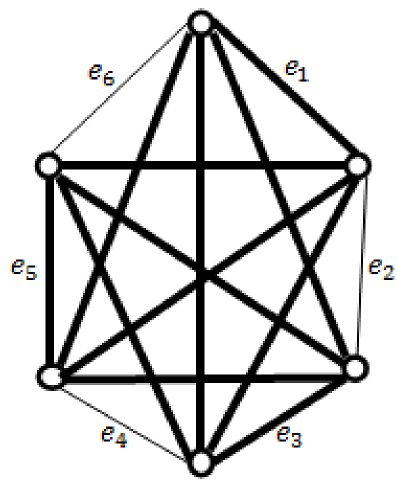
(A) $\gamma_{pfe}(K_3) = 1$



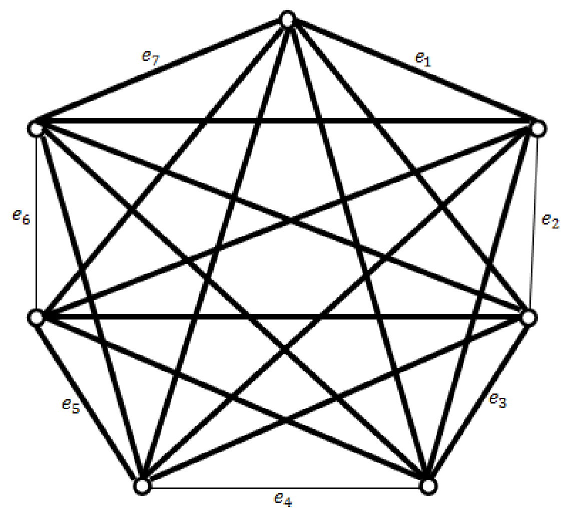
(B) $\gamma_{pfe}(K_4) = 3$



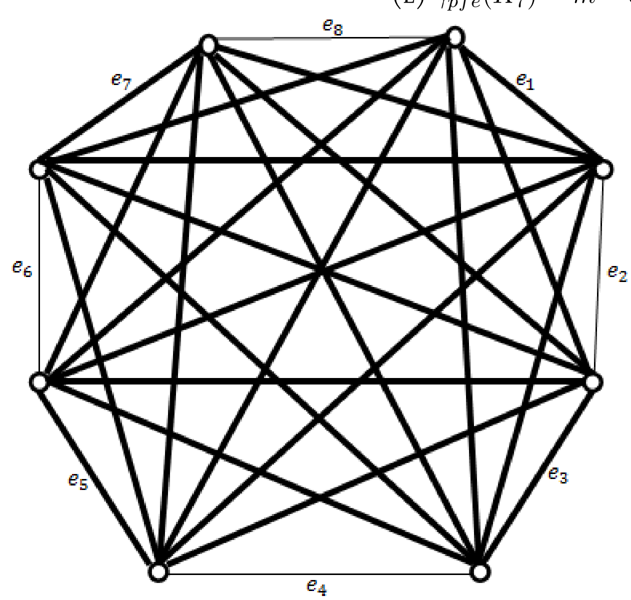
(C) $\gamma_{pfe}(K_5) = m - 2 = 8$



(D) $\gamma_{pfe}(K_6) = m - 3 = 12$



(E) $\gamma_{pfe}(K_7) = m - 3 = 18$



(F) $\gamma_{pfe}(K_8) = m - 4 = 24$

FIGURE 8. The pitchfork edge domination of complete graph

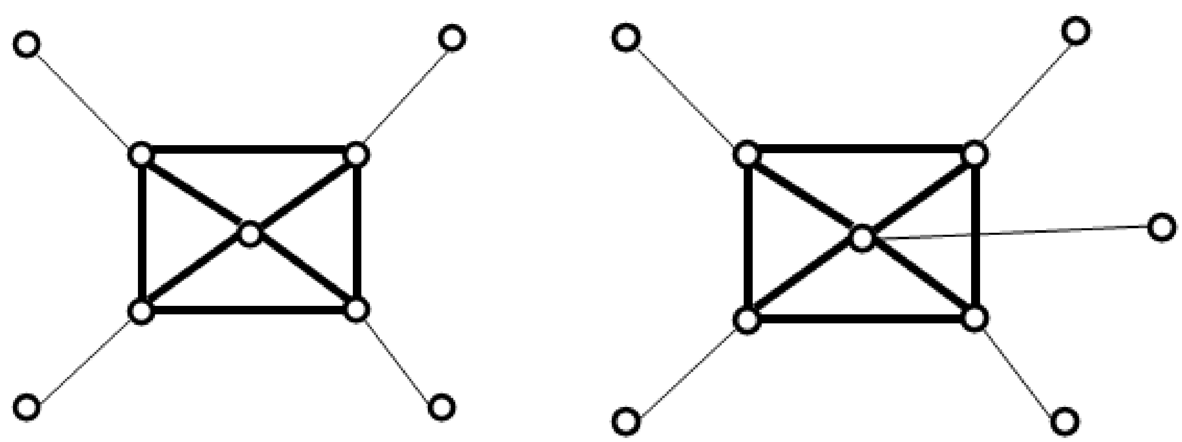
□

Proposition 3.5. *Suppose that H_n be a helm graph of order n and $\mathcal{H}_n$ be a big helm graph of order $n+1$, then:*

$$\gamma_{pfe}(H_n) = \gamma_{pfe}(\mathcal{H}_n) = 2n$$

Proof. Assume that D_e be a set of all edges of W_n. In the graph H_n every edge of C_n dominates exactly two edges of the single edges that joined with each vertex of C_n. While, the edges which are joining C_n to K_1, each edge of them dominates one edge of the single edges. In the graph $\mathcal{H}_n$, each edge of D_e dominates exactly two edges of the single edges that joined with each vertex of W_n. Therefore, D_e is a pitchfork edge dominating set.

To prove D_e is a minimum pitchfork edge dominating set, if any edge e is delete from D_e, then we get some edges in $D_e - \{e\}$ dominate three edges of $E - D_e$. Hence, D_e is the $\gamma_{pfe}-$set. See Fig. (9).



(A) H_n (B) $\mathcal{H}_n$

FIGURE 9. $\gamma_{pfe}-$sets of helm and big helm graphs

□

Theorem 3.6. *A bipartite complete graph has a pitchfork edge domination number, such that:*

$$\gamma_{pfe}(K_{n,m}) = \begin{cases} 1 & \text{if } n = 1 \wedge m = 2 \\ m-2 & \text{if } n = 1 \wedge m > 2 \\ m & \text{if } n = 2,3 \wedge m \geq 2 \\ m(n-2) & \text{if } n, m \geq 4 \end{cases}$$

Proof. Considering two disjoint sets of vertices of $K_{n,m}$, as A_1 and A_2 such that $|A_1| = n$ and $|A_2| = m$. The following four cases are obtained:

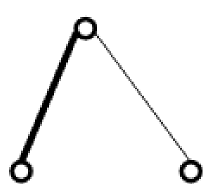
Case 1: If $n = 1 \wedge m = 2$, since $K_{1,2}$ has two edges. We take one of these edges to dominate the other edge.

Case 2: If $n = 1 \wedge m > 2$, since all vertices of A_2 are adjacent with one common vertex of A_1, then the number of the edges are equal to m. Let D_e represent the set of all edges except two edges. Then, exactly two edges are dominated by each edge of D_e. So, D_e is the $PEDS$ and $|D_e| = \gamma_{pfe}(K_{n,m}) = m - 2$.

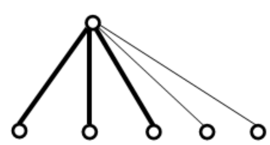
Case 3: If $n = 2, 3 \wedge m \geq 2$, let D_e be the set of all edges that joined one vertex of A_1 with all vertices of A_2 where these edges are equal to $|A_2| = m$. If $n = 2$, then each edge of D_e is dominates exactly one edge which is joined the other vertex of A_1 with the vertices of A_2. If $n = 3$, then each edge of D_e is dominates exactly two edges which are joined the two vertices of A_1 with the vertices of A_2. Therefore, D_e is the pitchfork edge dominating set and $\gamma_{pfe}(K_{n,m}) = |D_e| = m$.

Case 4: If $n, m \geq 4$, assume that D_e consists all the edges that joined $n - 2$ vertices from A_1 with all vertices of A_2, then each edge in the pitchfork edge dominating set D_e dominates exactly two edges from $E - D_e$. Thus, D_e is $\gamma_{pfe}-$set. Since each vertex of $n - 2$ vertices from A_1 has m edges where it is joined with all vertices of A_2. Hence, $|D_e| = m(n - 2) = \gamma_{pfe}(K_{n,m})$.

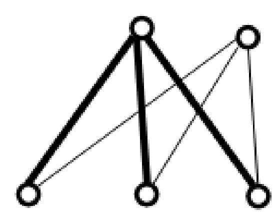
To show that D_e in all the previous cases is a $\gamma_{pfe}-$set. Assume that D'_e is a pitchfork edge dominating set such that $|D'_e| < |D_e|$. Then, either one or more edges of $E - D'_e$ are not dominated by D'_e, or there are some edges of D'_e dominate more than two edges of $E - D'_e$. This definition of pitchfork edge dominating set is disagreement with this. Thus, D_e is a minimum pitchfork edge dominating set. See Fig. (10).



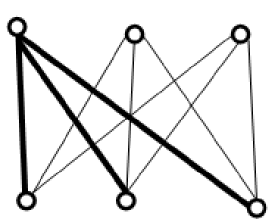
(A) $\gamma_{pfe}(K_{1,2}) = 1$



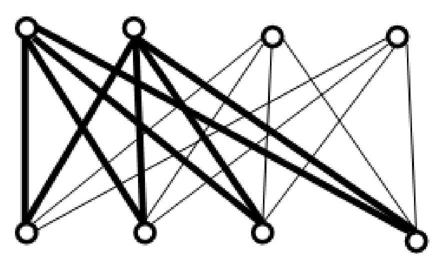
(B) $\gamma_{pfe}(K_{1,5}) = m-2 = 3$



(C) $\gamma_{pfe}(K_{2,3}) = m = 3$



(D) $\gamma_{pfe}(K_{3,3}) = m = 3$



(E) $\gamma_{pfe}(K_{4,4}) = m(n-2) = 8$

FIGURE 10. The pitchfork edge domination of a complete bipartite graph

□

Lastly, the following important question will be asked by everyone who has read pitchfork edge domination: does each graph G has pitchfork edge domination? does the pitchfork edge domination appear on every graph that has the pitchfork vertex domination? and is the converse true or not? the answer is yes for the first question where every finite, undirect and simple graph of order $n \geq 3$ without isolated edges has pitchfork edge domination. For the second question, the answer is no, where assume that T be a tree of order $n \geq 13$. From every four adjacent edges that have one common vertex, we choose two edges to form the set D_e. So, $D_e = \{e_3, e_4, e_5, e_6, e_9\}$, since all the edges of D_e dominates exactly two edges of $E - D_e$. Therefore, D_e is a $PEDS$. Thus, T having pitchfork edge domination but it has no pitchfork vertex domination. Since T having one vertex u adjacent with three support vertices w_1, w_2 and w_3. Since every support vertex adjacent with more than two pendant vertices, then the pendants belong to pitchfork vertex dominating set. If $u \in D$, then u dominates all support vertices w_1, w_2 and w_3. As a result, three vertices are dominated by u, which is contradictory with the pitchfork definition. If $u \notin D$, then there is no $v \in D$ such that v dominates u. Therefore, T has no pitchfork vertex dominating set. The converse has the same answer to the first question. See Fig. (11).

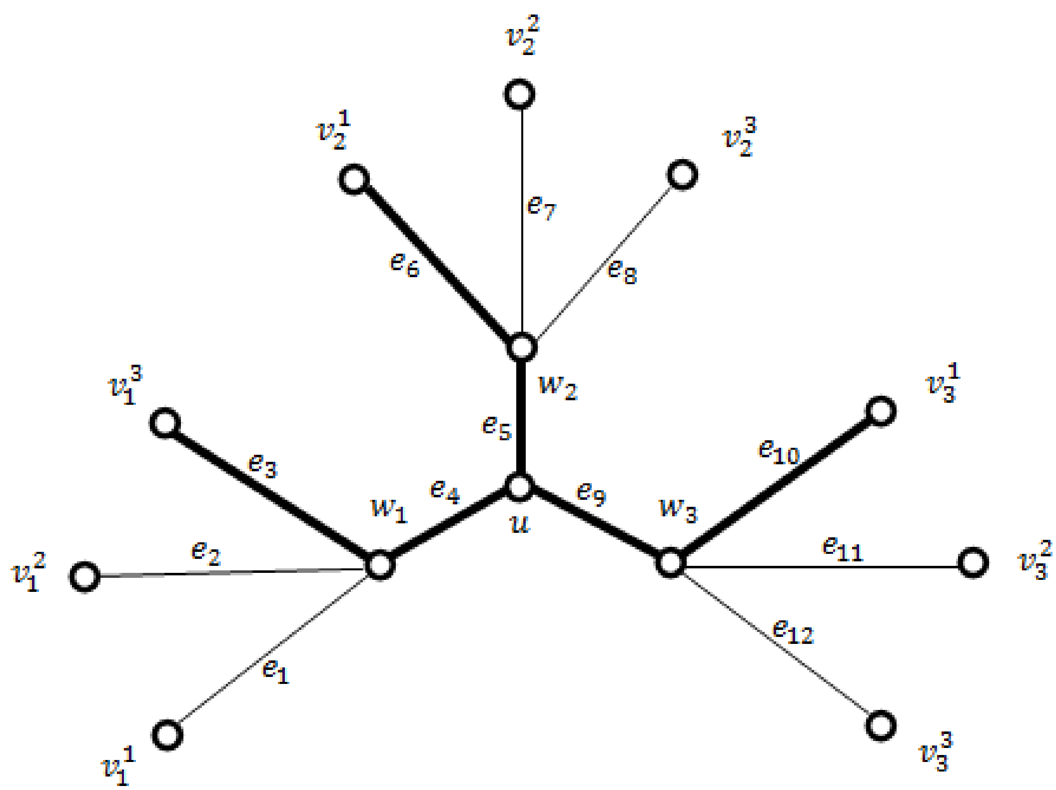


Figure 11. Tree graph

The following table reviews the comparable between a pitchfork vertex domination and the pitchfork edge domination for some graphs. Where $\gamma_{pf}(G)$ means the pitchfork vertex domination number.

Graph(G)	$\gamma_{pf}(G)$	$\gamma_{pfe}(G)$
Path graph	$\gamma_{pf}(P_n) = \lceil \frac{n}{3} \rceil$	$\gamma_{pfe}(P_n) = \begin{cases} \lceil \frac{n}{3} \rceil & if\ n \equiv 0, 2\,(mod\,3) \\ \lfloor \frac{n}{3} \rfloor & if\ n \equiv 1\,(mod\,3) \end{cases}$
Cycle graph	$\gamma_{pf}(C_n) = \lceil \frac{n}{3} \rceil$	$\gamma_{pfe}(C_n) = \lceil \frac{n}{3} \rceil$
Complete graph	$\gamma_{pf}(K_n) = n - 2, \quad \text{for } n \geq 3$	$\gamma_{pfe}(K_n) = \begin{cases} 1 & if n = 3 \\ 3 & if n = 4 \\ m - \lfloor \frac{m}{2} \rfloor & if n \neq 4 \end{cases}$
Bipartite complete	$\gamma_{pf}(K_{n,m}) = \begin{cases} m, \ if\, n = 2 \wedge m < 3\, or\, n = 1 \wedge m > 2 \\ m - 1, & if\ n = 2, m \geq 3 \\ n + m - 4, & if\ n, m > 2 \end{cases}$	$\gamma_{pfe}(K_{n,m}) = \begin{cases} 1 & if\ n = 1 \wedge m = 2 \\ m - 2 & if\ n = 1 \wedge m > 2 \\ m & if\ n = 2, 3 \wedge m \geq 2 \\ m(n-2) & if\ n, m \geq 4 \end{cases}$
Wheel graph	$\gamma_{pf}(W_n) = \begin{cases} 2\lceil \frac{n}{4} \rceil - 1, & if n \equiv 1\ (mod\,4) \\ 2\lceil \frac{n}{4} \rceil, & otherwise \end{cases}$	$\gamma_{pfe}(W_n) = n$
Helm graph	$\gamma_{pf}(H_n) = n, \quad \text{for } n \geq 3$	$\gamma_{pfe}(H_n) = 2n$
Big helm graph	$\gamma_{pf}(\mathcal{H}_n) = n + 1$	$\gamma_{pfe}(\mathcal{H}_n) = 2n$

Table 1. Pitchfork vertex domination and pitchfork edge domination

4. Conclusion

A novel type of domination known as "pitchfork edge domination" is introduced here. The relation between pitchfork edge domination number and the order, size, minimum degree and maximum degree are determined. In this paper the domination number was assessed for several standard graphs.

Authors' Contributions

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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