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ALMOST (τ_1, τ_2)-CONTINUITY AND WEAK (τ_1, τ_2)-CONTINUITY

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Abstract. This paper is concerned with the notions of almost (τ_1, τ_2)-continuous multifunctions and weakly (τ_1, τ_2)-continuous multifunctions. Moreover, the relationships between almost (τ_1, τ_2)-continuous multifunctions and weakly (τ_1, τ_2)-continuous multifunctions are established.

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1. Introduction

The branch of mathematics called topology is concerned with all questions directly or indirectly related to continuity. Stronger and weaker forms of open sets play an important role in the researches of generalization of continuity. Viriyapong and Boonpok [52] investigated some characterizations of (Λ, sp)-continuous functions by utilizing the notions of (Λ, sp)-open sets and (Λ, sp)-closed sets due to Boonpok and Khampakdee [16]. Furthermore, several characterizations of strongly $\theta(\Lambda, p)$-continuous functions, $\theta(\Lambda, p)$-continuous functions, $\theta(\star)$-precontinuous functions, $\star$-continuous functions, θ-$\mathscr{I}$-continuous functions and (τ_1, τ_2)-continuous functions were presented in [48], [41], [13], [12], [22] and [3], respectively. Singal and Singal [43] introduced and studied the notion of almost continuous functions as a generalization of continuity. Srisarakham and Boonpok [46] introduced and investigated the concept of almost (Λ, p)-continuous functions. Moreover, several characterizations of almost strongly

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$\theta(\Lambda, p)$-continuous functions, almost (g, m)-continuous functions and almost (τ_1, τ_2)-continuous functions were investigated in [5], [26] and [6], respectively. Popa [39] introduced and studied the notions of upper and lower almost continuous multifunctions. Popa et al. [37] introduced and investigated the concepts of upper and lower almost precontinuous multifunctions. Laprom et al. [31] introduced and studied the notion of almost $\beta(\tau_1, \tau_2)$-continuous multifunctions. In particular, some characterizations of almost $\beta(\star)$-continuous multifunctions, α-$\star$-continuous multifunctions, almost α-$\star$-continuous multifunctions, $s\beta(\star)$-continuous multifunctions, almost $s\beta(\star)$-continuous multifunctions, $\theta(\star)$-quasi continuous multifunctions, $\alpha(\Lambda, sp)$-continuous multifunctions, almost $\alpha(\Lambda, sp)$-continuous multifunctions, almost $\beta(\Lambda, sp)$-continuous multifunctions and almost (τ_1, τ_2)-continuous multifunctions were established in [19], [2], [7], [9], [8], [15], [29], [17], [49] and [30], respectively.

The concept of weakly continuous functions was introduced by Levine [32]. Duangphui et al. [28] introduced and studied the notion of weakly $(\mu, \mu')^{(m,n)}$-continuous functions. Furthermore, some characterizations of pairwise weakly M-continuous functions, weakly (Λ, b)-continuous functions and weakly (τ_1, τ_2)-continuous functions were presented in [25], [11] and [4], respectively. Popa [40] and Smithson [44] independently introduced the notion of weakly continuous multifunctions. The present authors introduced and studied other weak forms of continuous multifunctions: weakly quasicontinuous multifunctions [35], almost weakly continuous multifunctions [34], weakly α-continuous multifunctions [38], weakly β-continuous multifunctions [36]. These multifunctions have similar characterizations. The analogy in their definitions and results suggests the need of formulating a unified theory. Noiri and Popa [33] introduced and studied the notions of upper and lower weakly m-continuous multifunctions as a multifunction from a set satisfying certain minimal condition into a topological space. Srisarakham et al. [45] introduced and studied the concept of weakly (τ_1, τ_2)-continuous multifunctions. Moreover, several characterizations of weakly $(\tau_1, \tau_2)\alpha$-continuous multifunctions, weakly $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions, almost weakly (τ_1, τ_2)-continuous multifunctions, weakly $\star$-continuous multifunctions, weakly α-$\star$-continuous multifunctions, weakly $\imath^\star$-continuous multifunctions, weakly quasi (Λ, sp)-continuous multifunctions, weakly (Λ, sp)-continuous multifunctions and weakly (τ_1, τ_2)-continuous multifunctions were studied in [53], [20], [18], [23], [10], [21], [51], [14] and [47], respectively. In this paper, we investigate the relationships between almost (τ_1, τ_2)-continuous multifunctions and weakly (τ_1, τ_2)-continuous multifunctions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space

(X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [24] if $A = \tau_1$-Cl(τ_2-Cl(A)). The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [24] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [24] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [24] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) *$A \subseteq \tau_1\tau_2$-Cl(A) and $\tau_1\tau_2$-Cl($\tau_1\tau_2$-Cl(A)) $= \tau_1\tau_2$-Cl(A).*
(2) *If $A \subseteq B$, then $\tau_1\tau_2$-Cl(A) $\subseteq \tau_1\tau_2$-Cl(B).*
(3) *$\tau_1\tau_2$-Cl(A) is $\tau_1\tau_2$-closed.*
(4) *A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2$-Cl(A).*
(5) *$\tau_1\tau_2$-Cl($X - A$) $= X - \tau_1\tau_2$-Int(A).*

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$-*open* [53] (resp. $(\tau_1, \tau_2)s$-*open* [20], $(\tau_1, \tau_2)p$-*open* [20], $(\tau_1, \tau_2)\beta$-*open* [20]) if $A = \tau_1\tau_2$-Int($\tau_1\tau_2$-Cl(A)) (resp. $A \subseteq \tau_1\tau_2$-Cl($\tau_1\tau_2$-Int(A)), $A \subseteq \tau_1\tau_2$-Int($\tau_1\tau_2$-Cl(A)), $A \subseteq \tau_1\tau_2$-Cl($\tau_1\tau_2$-Int($\tau_1\tau_2$-Cl(A)))). The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is said to be $(\tau_1, \tau_2)r$-*closed* (resp. $(\tau_1, \tau_2)s$-*closed*, $(\tau_1, \tau_2)p$-*closed*, $(\tau_1, \tau_2)\beta$-*closed*). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\alpha(\tau_1, \tau_2)$-*open* [50] if $A \subseteq \tau_1\tau_2$-Int($\tau_1\tau_2$-Cl($\tau_1\tau_2$-Int(A))). The complement of an $\alpha(\tau_1, \tau_2)$-open set is called $\alpha(\tau_1, \tau_2)$-*closed*.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*paracompact* [24] if every cover of A by $\tau_1\tau_2$-open sets of X is refined by a cover of A which consists of $\tau_1\tau_2$-open sets of X and is $\tau_1\tau_2$-locally finite in X. A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*regular* [24] if for each $x \in A$ and each $\tau_1\tau_2$-open set U of X containing x, there exists a $\tau_1\tau_2$-open set V of X such that $x \in V \subseteq \tau_1\tau_2$-Cl($V$) $\subseteq U$.

Lemma 2. [24] *If A is a $\tau_1\tau_2$-regular $\tau_1\tau_2$-paracompact set of a bitopological space (X, τ_1, τ_2) and U is a $\tau_1\tau_2$-open neighbourhood of A, then there exists a $\tau_1\tau_2$-open set V of X such that $A \subseteq V \subseteq \tau_1\tau_2$-Cl($V$) $\subseteq U$.*

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [1] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and

$$F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}.$$

In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *upper* (τ_1, τ_2)-*continuous* [42] (resp. *upper almost* (τ_1, τ_2)-*continuous* [30],*upper weakly* (τ_1, τ_2)-*continuous* [47]) at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq V$ (resp. $F(U) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$, $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$). A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *upper* (τ_1, τ_2)*-continuous* (resp. *upper almost* (τ_1, τ_2)*-continuous*, *upper weakly* (τ_1, τ_2)*-continuous*) if F has this property at each point of X. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *lower* (τ_1, τ_2)*-continuous* [42] (resp. *lower almost* (τ_1, τ_2)*-continuous* [30], *lower weakly* (τ_1, τ_2)*-continuous* [47]) at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(z) \cap V \neq \emptyset$ (resp. $\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)) \cap F(z) \neq \emptyset$, $\sigma_1\sigma_2\text{-Cl}(V) \cap F(z) \neq \emptyset$) for each $z \in U$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *lower* (τ_1, τ_2)*-continuous* (resp. *lower almost* (τ_1, τ_2)*-continuous*, *lower weakly* (τ_1, τ_2)*-continuous*) if F has this property at each point of X.

3. Almost (τ_1, τ_2)-continuity and weak (τ_1, τ_2)-continuity

In this section, we discuss the relationships between almost (τ_1, τ_2)-continuous multifunctions and weakly (τ_1, τ_2)-continuous multifunctions.

Definition 1. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be almost (τ_1, τ_2)-open if $F(U) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(F(U)))$ for every $\tau_1\tau_2$-open set U of X.*

Theorem 1. *If a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is upper weakly (τ_1, τ_2)-continuous and almost (τ_1, τ_2)-open, then F is upper almost (τ_1, τ_2)-continuous.*

Proof. Let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. Then, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. Since F is almost (τ_1, τ_2)-open, $F(U) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(F(U))) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$. Thus, F is upper almost (τ_1, τ_2)-continuous. □

Theorem 2. *Let $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a multifunction such that $F(x)$ is $\sigma_1\sigma_2$-open in Y for each $x \in X$. Then, the following properties are equivalent:*

(1) *F is lower (τ_1, τ_2)-continuous;*
(2) *F is lower almost (τ_1, τ_2)-continuous;*
(3) *F is lower weakly (τ_1, τ_2)-continuous.*

Proof. (1) ⇒ (2) and (2) ⇒ (3): The proofs of these implications are obvious.

(3) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \cap V \neq \emptyset$. There exists a $\tau_1\tau_2$-open set U of X containing x such that $\sigma_1\sigma_2\text{-Cl}(V) \cap F(z) \neq \emptyset$ for each $z \in U$. Since $F(z)$ is $\sigma_1\sigma_2$-open, $V \cap F(z) \neq \emptyset$ and hence F is lower (τ_1, τ_2)-continuous. □

Definition 2. *A subset A of a bitopological space (X, τ_1, τ_2) is said to be almost (τ_1, τ_2)-regular if for each $x \in A$ and each $\tau_1\tau_2$-open set U containing x, there exists a $\tau_1\tau_2$-open set V of X such that $x \in V \subseteq \tau_1\tau_2\text{-Cl}(V) \subseteq U$.*

Lemma 3. *If A is an almost (τ_1, τ_2)-regular $\tau_1\tau_2$-paracompact set of a bitopological space (X, τ_1, τ_2) and U is a $(\tau_1, \tau_2)r$-open neighbourhood of A, then there exists a $\tau_1\tau_2$-open set V of X such that $A \subseteq V \subseteq \tau_1\tau_2$-Cl$(V) \subseteq U$.*

Lemma 4. [30] *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is upper almost (τ_1, τ_2)-continuous;*
(2) *$F^+(V) \subseteq \tau_1\tau_2$-Int$(F^+(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))))$ for every $\sigma_1\sigma_2$-open set V of Y;*
(3) *$\tau_1\tau_2$-Cl$(F^-(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(K)))) \subseteq F^-(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;*
(4) *$\tau_1\tau_2$-Cl$(F^-(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(B))))) \subseteq F^-(\sigma_1\sigma_2$-Cl$(B))$ for every subset B of Y;*
(5) *$F^+(\sigma_1\sigma_2$-Int$(B)) \subseteq \tau_1\tau_2$-Int$(F^+(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(B)))))$ for every subset B of Y;*
(6) *$F^+(V)$ is $\tau_1\tau_2$-open in X for every $(\sigma_1, \sigma_2)r$-open set V of Y;*
(7) *$F^-(K)$ is $\tau_1\tau_2$-closed in X for every $(\sigma_1, \sigma_2)r$-closed set K of Y.*

Theorem 3. *If $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is upper weakly (τ_1, τ_2)-continuous and $F(x)$ is an almost (σ_1, σ_2)-regular $\sigma_1\sigma_2$-paracompact set of Y for each point $x \in X$, then F is upper almost (τ_1, τ_2)-continuous.*

Proof. Let V be any $(\sigma_1, \sigma_2)r$-open set of Y and $x \in F^+(V)$. Then, $F(x) \subseteq V$. Since $F(x)$ is an almost (σ_1, σ_2)-regular $\tau_1\tau_2$-paracompact, by Lemma 3 there exists a $\sigma_1\sigma_2$-open set W of Y such that $F(x) \subseteq W \subseteq \sigma_1\sigma_2$-Cl$(W) \subseteq V$. Since F is upper weakly (τ_1, τ_2)-continuous, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2$-Cl$(W) \subseteq V$. Thus, $x \in U \subseteq F^+(V)$ and hence $F^+(V)$ is $\tau_1\tau_2$-open in X. It follows from Lemma 4 that F is upper almost (τ_1, τ_2)-continuous. □

Definition 3. [27] *A bitopological space (X, τ_1, τ_2) is said to be almost (τ_1, τ_2)-regular if for each $(\tau_1, \tau_2)r$-closed set F and each $x \notin F$, there exist disjoint $\tau_1\tau_2$-open sets U and V such that $x \in U$ and $F \subseteq V$.*

Corollary 1. *If $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is upper weakly (τ_1, τ_2)-continuous, where (Y, σ_1, σ_2) is almost (σ_1, σ_2)-regular and $F(x)$ is $\sigma_1\sigma_2$-paracompact for each point $x \in X$, then F is upper almost (τ_1, τ_2)-continuous.*

Lemma 5. *If A is an almost (τ_1, τ_2)-regular $\tau_1\tau_2$-paracompact set of a bitopological space (X, τ_1, τ_2) and U is a $(\tau_1, \tau_2)r$-open set such that $A \cap U \neq \emptyset$, then there exists a $\tau_1\tau_2$-open set V of X such that $A \cap V \neq \emptyset$ and $\tau_1\tau_2$-Cl$(V) \subseteq U$.*

Lemma 6. [30] *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is lower almost (τ_1, τ_2)-continuous;*
(2) *$F^-(V) \subseteq \tau_1\tau_2$-Int$(F^-(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))))$ for every $\sigma_1\sigma_2$-open set V of Y;*
(3) *$\tau_1\tau_2$-Cl$(F^+(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(K)))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;*
(4) *$\tau_1\tau_2$-Cl$(F^+(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(B))))) \subseteq F^+(\sigma_1\sigma_2$-Cl$(B))$ for every subset B of Y;*
(5) *$F^-(\sigma_1\sigma_2$-Int$(B)) \subseteq \tau_1\tau_2$-Int$(F^-(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(B)))))$ for every subset B of Y;*
(6) *$F^-(V)$ is $\tau_1\tau_2$-open in X for every $(\sigma_1, \sigma_2)r$-open set V of Y;*
(7) *$F^+(K)$ is $\tau_1\tau_2$-closed in X for every $(\sigma_1, \sigma_2)r$-closed set K of Y.*

Theorem 4. *If $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower weakly (τ_1, τ_2)-continuous and $F(x)$ is an almost (σ_1, σ_2)-regular set of Y for each point $x \in X$, then F is lower almost (τ_1, τ_2)-continuous.*

Proof. Let V be any $(\sigma_1, \sigma_2)r$-open set of Y and $x \in F^-(V)$. Then, we have $F(x) \cap V \neq \emptyset$. Since $F(x)$ is almost (σ_1, σ_2)-regular, by Lemma 5 there exists a $\sigma_1\sigma_2$-open set W of Y such that $F(x) \cap W \neq \emptyset$ and $\sigma_1\sigma_2$-Cl$(W) \subseteq V$. Since F is lower weakly (τ_1, τ_2)-continuous, there exists a $\tau_1\tau_2$-open set U of X containing x such that

$$\sigma_1\sigma_2\text{-Cl}(W) \cap F(z) \neq \emptyset;$$

hence $F(z) \cap V \neq \emptyset$ for each $z \in U$. Thus, $x \in U \subseteq F^-(V)$ and hence $F^-(V)$ is $\tau_1\tau_2$-open in X. It follows from Lemma 6 that F is lower almost (τ_1, τ_2)-continuous. □

Corollary 2. *If $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower weakly (τ_1, τ_2)-continuous and (Y, σ_1, σ_2) is almost (σ_1, σ_2)-regular, then F is lower almost (τ_1, τ_2)-continuous.*

Definition 4. *A subset A of a bitopological space (X, τ_1, τ_2) is said to be semi-(τ_1, τ_2)-regular if for each $x \in A$ and each $\tau_1\tau_2$-open set U of X containing x, there exists a $(\tau_1, \tau_2)r$-open set V of X such that $x \in V \subseteq U$.*

Lemma 7. *If A is a semi-(τ_1, τ_2)-regular set of a bitopological space (X, τ_1, τ_2), then for every $\tau_1\tau_2$-open set U such that $A \cap U \neq \emptyset$, there exists a $(\tau_1, \tau_2)r$-open set W such that $A \cap W \neq \emptyset$ and $W \subseteq U$.*

Lemma 8. [30] *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is lower almost (τ_1, τ_2)-continuous at $x \in X$;*

(2) *$x \in \tau_1\tau_2$-Int$(F^-(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))))$ for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$;*

(3) *$x \in \tau_1\tau_2$-Int$(F^-((\sigma_1, \sigma_2)$-sCl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$;*

(4) *$x \in \tau_1\tau_2$-Int$(F^-(V))$ for every $(\sigma_1, \sigma_2)r$-open set V of Y such that $F(x) \cap V \neq \emptyset$;*

(5) *for each $(\sigma_1, \sigma_2)r$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $U \subseteq F^-(V)$.*

Lemma 9. [42] *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is lower (τ_1, τ_2)-continuous;*

(2) *$F^-(V)$ is $\tau_1\tau_2$-open in X for every $\sigma_1\sigma_2$-open set V of Y;*

(3) *$F^+(K)$ is $\tau_1\tau_2$-closed in X for every $\sigma_1\sigma_2$-closed set K of Y;*

(4) *$\tau_1\tau_2$-Cl$(F^+(B)) \subseteq F^+(\sigma_1\sigma_2$-Cl$(B))$ for every subset B of Y;*

(5) *$F^-(\sigma_1\sigma_2$-Int$(B)) \subseteq \tau_1\tau_2$-Int$(F^-(B))$ for every subset B of Y.*

Theorem 5. *If $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower almost (τ_1, τ_2)-continuous and $F(x)$ is semi-(σ_1, σ_2)-regular in Y for each $x \in X$, then F is lower (τ_1, τ_2)-continuous.*

Proof. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y such that

$$F(x) \cap V \neq \emptyset.$$

Since $F(x)$ is semi-(σ_1, σ_2)-regular, by Lemma 7 there exists a $(\sigma_1, \sigma_2)r$-open set W such that $F(x) \cap W \neq \emptyset$ and $W \subseteq V$. Since F is lower almost (τ_1, τ_2)-continuous, by Lemma 8 there exists a $\tau_1\tau_2$-open set U of X containing x such that $x \in U \subseteq F^-(W) \subseteq F^-(V)$. Thus, $F^-(V) \subseteq \tau_1\tau_2\text{-Int}(F^-(V))$ and hence $F^-(V)$ is $\tau_1\tau_2$-open in X. It follows from Lemma 9 that F is lower (τ_1, τ_2)-continuous. □

Corollary 3. *Let (Y, σ_1, σ_2) be a semi-(σ_1, σ_2)-regular space. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower almost (τ_1, τ_2)-continuous if and only if F is lower (τ_1, τ_2)-continuous.*

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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