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FIXED POINT RESULTS OF TRI-SIMULATION FUNCTION FOR S-METRIC SPACE INVOLVING APPLICATIONS OF INTEGRAL EQUATIONS

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Abstract. This paper introduces novel findings in the field of fixed-point theory for S-metric spaces that make use of the tri-simulation function. In addition, we offer real-world implementations of integral equations to improve outcomes. Our findings represent advancements and improvements over earlier scholars' writings.

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Key words and phrases. S-metric space; fixed point; α-permissible mapping; simulation function; α-orbital permissible mapping.

1. Introduction

A subfield of mathematics known as fixed-point results focuses on the study of specific points at which functions become fixed. A fixed point of a function is a point at which the output of the function is identical to the input. This is a straightforward definition of the term. The public first heard about fixed-point theory in the latter half of the 19th and early 20 th centuries. It is several mathematicians contributed to the generalization of the concept of metric spaces when they first introduced it in 1992 [1]. Dhage, British Columbia, developed a generalized metric space and mapping with a fixed point. In 1950 E. Michael, a pioneer in the field of quasi-metric spaces, relaxed the triangle inequality constraint that is present in metric spaces, which made it possible to construct distance measures that were more flexible. In 1960, Lotfi A. Zadeh presented fuzzy metric spaces, which are an extension of the concept of a metric space that allows for degrees of membership rather than rigid distances [13]. Around the same time, others developed fuzzy metric spaces. George A. Hunt, Bruce J. Pettis, and others developed probabilistic metric spaces in the middle of the 20th century. These spaces include probability measures in the framework of metric spaces in order to model uncertainty. Felix Hausdorff first introduced

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S-metric spaces as a generalization of metric spaces in the early 20 th century. They are distinguish by the substitution of an S-metric for the metric function, which enables them to satisfy a modified set of axioms. S. Sedghi and N. Shobe presented the idea of S-metric spaces in 2007 [9]. S-metric spaces are a generalization of metric spaces. Sedghi's research laid the foundation for subsequent investigations in this field. In 2012 [10], S. Sedghi and N. Shobe extended the theory of S-metric spaces by investigating a variety of features and applications. Their combined efforts led to a deeper understanding of this mathematical framework and its potential applications in various sectors. Generally, a wide range of disciplines can use the fixed-point theory as a powerful tool to prove the existence and stability of solutions. We have established generalizations and provided a fixed-point theorem for self-mapping on complete S-metric spaces. These generalizations have found applications in a variety of branches of mathematics and computer science, as well as in engineering and physics.

2. Preliminaries

This section will delve into a variety of definitions and outcomes previously explored by other authors. This section will benefit the reader because it will help them understand the primary findings of this research.

Definition 2.1. [11] *Assume that the set $X \neq \emptyset$ and $S : X^3 \to [0, \infty)$ to be a function that meets all of the criteria for any $\mu, v, \omega, a \in X$.*

1- $S(\mu, v, \omega) \geq 0$

2- $S(\mu, v, \omega) = 0$ *if and only if* $\mu = v = \omega$*;*

3- $S(\mu, v, \omega) \leq S(\mu, \mu, a) + S(v, v, a) + S(\omega, \omega, a)$.

Then, S is called an S-metric function on X and the pair (X, S) is called on S-metric space.

Lemma 2.1. [11] *A S-metric space is considered symmetric if $S(\mu, \mu, v) = S(v, v, \mu)$.*

Definition 2.2. [11] *Consider the (X, S) be S-metric space, A sequence $\{\mu_n\}$ of X is defined as follows;*

1. *If for any $\mu \in A$, there is $r > 0$ in which $B_s(\mu, r) \subset A$, then the set A is said an open set of X;*
2. *A set* A *of X is called to be S-bounded if there is $r > 0$ in which $S(\mu, \mu, v) < r$ for any $\mu, v \in A$;*
3. *A sequence $\{\mu_n\}$ in X converges to X if and only if $S(\mu_n, \mu_n, \mu) \to 0$ as $n \to \infty$, for any $\varepsilon > 0$, there is $n_0 \in N$ in which* $\mathrm{S}(\mu_n, \mu_n, \mu) < \varepsilon$ *for all $n \geq n_0$ and we denote this by $\lim_{n\to\infty} \mu_n = \mu$;*
4. *A sequence $\{\mu_n\}$ in X is called a Cauchy sequence if $\forall \varepsilon > 0$, there is $n_0 \in N$ in which $S(\mu_n, \mu_n \mu_m) < \varepsilon$ for any $n, m \geq n_0$;*
5. *If every Cauchy sequence is convergent in S-metric space, then it is termed the complete space.*

Definition 2.3. [5] *Consider that $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be mapping then ζ called a simulation function if while fulfilling the following requirements;*

1. $\zeta(0,0)=0$;
2. $\zeta(x,y)\leq 0$, *for all* $x,y>0$;
3. *If* $\{x_n\},\{y_n\}$ *are sequences in* $(0,\infty)$ *in which* $\lim_{n\to\infty} x_n=\lim_{n\to\infty} y_n>0$ *then* $\lim_{n\to\infty}\sup\zeta\left(x_n,y_n\right)<0$.

Definition 2.4. [2] *Consider that* $T:\mathbb{R}_+^3\to\mathbb{R}$ *then* T *said to be a tri-simulation function if which fulfills a requirements listed below:*
$T_1) T(z,y,x)<x-yz \quad \forall x,y>0, z\geq 0$;
T_2) *If* $\{x_n\},\{y_n\}$ *and* $\{z_n\}$ *are sequence in* $(0,\infty)$ *in which* $y_n<x_n$ *for any* $n\in N$
$\lim_{n\to\infty} z_n\geq 1$ *and* $\lim_{n\to\infty} y_n=\lim_{n\to\infty} x_n>0$ *then* $\lim_{n\to\infty}\sup T\left(x_n,y_n,z_n\right)<0$.

Example 2.1. $M=\mathbb{R}$ *and the function* $S:M^3\to[0,\infty)$ *be defined as :* $S(a,b,c)=|a-c|+|a+c-2b|$ $\forall a,b,c\in\mathbb{R}$, *then the pair* (M,S) *is an* S*-metric space.*

Example 2.2. *Let* $X=\mathbb{R}^n$ *and* $\|.\| benormon X$ *then* $S(a,b,c)=\|a-c\|+\|b-c\|$ *is an* S*-metric on* X. *In the case when* $X=\mathbb{R}\cdot S(a,b,c)=|a-c|+|b-c|$ *is called the an usual* S*-metric on* X.

Definition 2.5. [5] *Consider that metric space* (M,d), *a mapping* $T:M\to M$. *Then,* T *is an* Z*-contractive mapping in relates* $\zeta\in Z$ *if*

$$\zeta(d(T\mu,Tv),d(\mu,v))\geq 0,\forall\mu,v\in M.$$

Theorem 2.1. [8] *Consider a space* (M,d) *is metric space and the mapping* $T:M\to M$, *take into consideration the next set of circumstances:*

1. $\alpha(\mu,v)d(T\mu,Tv)\leq\psi(d(\mu,v))$ *for any* $\mu,v\in M$, *such that* $\psi:\mathbb{R}_+\to\mathbb{R}_+$*is non-decreasing function in which* $\sum_{n=1}^{\infty}\psi^n(t)<\infty,\forall t>0$ *and* $\alpha:M\times M\to\mathbb{R}_+$;
2. *There is* $\mu_0\in M$ *in which* $\alpha\left(\mu_0,T\mu_0\right)\geq 1$;
3. T *is* α*-admissible, That is,* $\alpha(\mu,v)\geq 1$ *then* $\alpha(T\mu,Tv)\geq 1,\forall\mu,v\in M$.

Consequently, there is a fixed point for T.

Definition 2.6. [2] *Consider that a metric space* (M,d) *and the mapping* $f:M\to M$ *is said to be a* αZ*-contraction in relates* $\mathrm{T}\in z$ *if* $\forall\mu,v\in M$

$$T(\alpha(\mu,v),d(f\mu,fv),d(\mu,v))\geq 0,\ \textit{where}\ \alpha:M\times M\to\mathbb{R}_+.$$

Definition 2.7. [2] *Consider that (M,d) is a metric space and the mapping* $f:M\to M$ *and* $\alpha:M\times M\to\mathbb{R}_+$*is called* α*-permissible if* $\forall m\leq n\leq 1$ *and* $\mu,v\in M$.

$$\alpha(\mu,v)\geq 1,\ \textit{then}\ \alpha\left(f^n\mu,f^mv\right)\geq 1.$$

Definition 2.8. [2] *Consider that a metric space* (M, d) *and* $f : M \to M$ *and* $\alpha : M \times M \to \mathbb{R}_+$*is said to be* α*-orbital permissible if*

$$\forall m \leq n \leq 1, \mu \in M, \quad \alpha(\mu, f\mu) \geq 1 \text{ then } \alpha\left(f^n\mu, f^m\mu\right) \geq 1.$$

Definition 2.9. [12] *Consider that* M *is a nonempty set,* $f : M \to M$ *is a mapping. Then,* $\alpha : M \times M \to \mathbb{R}_+$*is called* α *-admissible if for any* $\mu, v \in M$ *such that*

$$\alpha(\mu, v) \geq 1 \text{ then } \alpha(f\mu, fv) \geq 1.$$

Theorem 2.2. [7] *On a complete metric space, any* Z *- contraction has a single fixed point.*

Theorem 2.3. [4] *Consider that* (X, d) *be a metric space and let* $T : X \to X$ *be an* αZ*-contraction in relates* ζ*. Consider that.*

i. T *is* α*-admissible ;*
ii. *there is* $\mu_0 \in X$ *in which* $\alpha\left(\mu_0, T\mu_0\right) \geq 1$*;*
iii. T *is continuous.*

Then there is $\mu_0 \in X$ *in which* $T\mu = \mu$*.*

3. Main Results

Here under the context of S-metric space, we introduce new definition about be αZ - contraction, α - admissible, triangular α-admissible and α-permissible. Moreover, the uniqueness of the fixed point is studies.

Definition 3.1. *Consider that* M *is self mapping on S-metric space (X,S) then* M *is called to be* αZ *- contraction in relates* $T \in Z$ *if* $\forall \mu, v \in X$

$$T\left(\alpha(\mu, \mu, v), S\left(M_\mu, M_\mu, M_v\right), S(\mu, \mu, v)\right) \geq 0 \tag{3.1}$$

where $\alpha : X \times X \to [0, \infty)$*.*

Remark 3.1. *If* M *is an* αZ*-contraction for some* $T \in Z$ *then, inview of the condition* (T_1) *we have*

$$\alpha(\mu, \mu, v) \cdot S\left(M_\mu, M_\mu, M_v\right) < S(\mu, \mu, v) \qquad \forall \mu, v \in X.$$

Definition 3.2. *Let* $M \neq \emptyset$ *on S-metric space* (X, S) *and* $M : X \to X$ *be a mapping and it is called* α *-admissible if Where* $\alpha : X \times X \to [0, \infty), \forall \mu.v \in X$*.*

$$\alpha(\mu, \mu, v) \geq 1 \text{ then } \alpha(M_\mu, M_\mu, M_v) \geq 1.$$

Definition 3.3. *Let* $M \neq \emptyset$ *on S-metric space and* $M : X \to X$ *a mapping is said to be triangular* α*-admissible if,* $\alpha : \mathrm{X} \times \mathrm{X} \to [0, \infty), \forall \mu, \mathrm{v}, \omega \in \mathrm{X}$*.*

1- M *is* α*-admissible;*

2- $\alpha(\mu,\mu,v)\geq 1$ *and* $\alpha(v,v,\omega)\geq 1$ *then* $\alpha(\mu,\mu,\omega)\geq 1$.

Definition 3.4. *Let $M\neq\emptyset$ on S-metric space and $M: X\times X$ a mapping is said to be α-permissible if*

$$\forall m\geq n\geq 1,\mu,v\in X,\quad \alpha(\mu,\mu,v)\geq 1 \text{ then } \alpha\left(M_\mu^n,M_\mu^n,M_v^m\right)\geq 1.$$

Theorem 3.1. *Let (X,S) be complete S-metric space and $M: X\to X$ be an αZ-contaction under some conditions of tri-simulation function M, suppose that*

1- *M^{-1} is α-permissible;*

2- *there is $\mu_0\in X$ in which $\alpha\left(\mu_0,\mu_0,M_{\mu_0}^{-1}\right)\geq 1$.*

3- *M is continuous function.*

Then, M has fixed point.

Proof. Consider that $\mu_0\in X$ in which $\alpha\left(\mu_0,\mu_0,M_{\mu_0}^{-1}\right)\geq 1$ by $M\mu_{n-1}=\mu_n \quad n\in N$, then $\alpha\left(\mu_0,\mu_0,\mu_1\right)\geq 1$, Since M is α-permissible mapping

$$\alpha\left(\mu_n,\mu_n,\mu_{n+1}\right)\geq 1,\forall n\in N. \tag{3.2}$$

Now $\mu=\mu_n, v=\mu_{n+1}$ in equation (3.1) we get

$$T\left(\alpha\left(\mu_n,\mu_n,\mu_{n+1}\right),S\left(M\mu_n,M\mu_n,M\mu_{n+1}\right),S\left(\mu_n,\mu_n,\mu_{n+1}\right)\geq 0.\right.$$

By using condition (T_1)

$$0<\left(S\left(\mu_{n-1},\mu_{n-1},\mu_n\right)-\alpha\left(\mu_n,\mu_n,\mu_{n+1}\right)S\left(\mu_n,\mu_n,\mu_{n+1}\right)\right). \tag{3.3}$$

Then,

$$S\left(\mu_n,\mu_n,\mu_{n+1}\right)\leq\alpha\left(\mu_n,\mu_n,\mu_{n+1}\right)S\left(\mu_n,\mu_n,\mu_{n+1}\right)<S\left(\mu_{n-1},\mu_{n-1},\mu_n\right).$$

This implies

The sequence $S\left(\mu_n,\mu_n,\mu_{n+1}\right)$ is a non-negative real decreasing. So, $S\left(\mu_n,\mu_n,\mu_{n+1}\right)$ is converge to Point say r. Consequently, if $r\neq 0$, then letting $n\longrightarrow\infty$ on both side of equation (3.3) and using equation (3.2) we get $\lim_{n\to\infty}\alpha\left(\mu_n,\mu_n,\mu_{n+1}\right)=1$ by (T_2)

$$0\leqslant\lim_{n\to\infty}\sup T\left(\alpha\left(\mu_n,\mu_n,\mu_{n+1}\right),S\left(\mu_n,\mu_n,\mu_{n+1}\right),S\left(\mu_{n-1},\mu_{n-1},\mu_n\right)<0\right.$$

Which is contradiction therefore $r=0$

So,

$$S\left(\mu_n,\mu_n,\mu_{n+1}\right)=0. \tag{3.4}$$

We will show that $\{\mu_n\}$ is bounded sequence. Assume that $\{\mu_n\}$ is not bounded, then there is a sub sequence $\{\mu_{n_k}\}$ in which $n_1=1$. $\quad\forall k\in N, n_{k+1}$ is the minimum integer in which

$$S\left(\mu_{n_k},\mu_{n_k},\mu_{n_{k+1}}\right)>1. \tag{3.5}$$

And for $n_k \leq m \leq n_{k+1}-1$ we have $\mathrm{S}\left(\mu_{n_k}, \mu_{n_k}, \mu_m\right) \leq 1$.

Utilizing the triangular inequality, (3.5) we get

$$\begin{aligned}
1 &< S\left(\mu_{n_k}, \mu_{n_k}, \mu_{n_{k+1}}\right) \\
&\leq S\left(\mu_{n_k}, \mu_{n_k}, \mu_{n_{k+1}-1}\right)+S\left(\mu_{n_k}, \mu_{n_k}, \mu_{n_{k+1}-1}\right)+S\left(\mu_{n_{k+1}}, \mu_{n_{k+1}}, \mu_{n_{k+1}-1}\right) \\
&\leq 2S\left(\mu_{n_k}, \mu_{n_k}, \mu_{n_{k+1}-1}\right)+S\left(\mu_{n_{k+1}}, \mu_{n_{k+1}}, \mu_{n_{k+1}-1}\right) \\
&\leq 2S\left(\mu_{n_k}, \mu_{n_k}, \mu_{n_{k+1}-1}\right)+1.
\end{aligned}$$

Letting $k \rightarrow \infty$ we get

$$\lim_{n \rightarrow \infty} S\left(\mu_{n_k}, \mu_{n_k}, \mu_{n_{k+1}}\right)=1. \tag{3.6}$$

By remark (3.1) taking $\mu=\mu_{n_{k-1}}, v=\mu_{n_{k-1}-1}$ we have

$$\alpha\left(\mu_{n_k}, \mu_{n_k}, \mu_{n_{k+1}}\right) S\left(\mu_{n_k}, \mu_{n_k}, \mu_{n_{k+1}}\right)<S\left(\mu_{n_{k-1}}, \mu_{n_{k-1}}, \mu_{n_{k+1}-1}\right).$$

Using equation (3.2) and (3.5) we get

$$\begin{aligned}
1 &< \alpha\left(\mu_{n_k}, \mu_{n_k}, \mu_{n_{k+1}}\right) S\left(\mu_{n_k}, \mu_{n_k}, \mu_{n_{k+1}}\right)) \\
&< \mathrm{S}\left(\mu_{n_{k-1}}, \mu_{n_{k-1}}, \mu_{n_{k+1}-1}\right) \\
&\leq 2\, \mathrm{S}\left(\mu_{n_{k-1}}, \mu_{n_{k-1}}, \mu_{n_k}\right)+\mathrm{S}\left(\mu_{n_{k+1}-1}, \mu_{n_{k+1}-1}, \mu_{n_k}\right) \\
&\leq 2S\left(\mu_{n_{k-1}}, \mu_{n_{k-1}}, \mu_{n_k}\right)+1.
\end{aligned}$$

Taking $k \rightarrow \infty$ we get

$$\lim_{k \rightarrow \infty} S\left(\mu_{n_{k-1}}, \mu_{n_{k-1}}, \mu_{n_{k+1}-1}\right). \tag{3.7}$$

By using (T_2) equation (3.6) and (3.7) we have

$$0 \leq \lim_{k \rightarrow \infty} \sup T\left(\alpha\left(\mu_{n_k}, \mu_{n_k}, \mu_{n_{k+1}}\right), S\left(\mu_{n_k}, \mu_{n_k}, \mu_{n_{k+1}}\right), S\left(\mu_{n_{k-1}}, \mu_{n_{k-1}}, \mu_{n_{k+1}-1}\right)\right)<0.$$

This is contradiction.

Consequently $\{\mu_n\}$ is bounded. Let $\lambda_n=\sup\left\{S\left(\mu_i, \mu_i, \mu_j\right) i, j \geq n\right\}$, where $\{\lambda_n\}$ is decreasing sequence and $\{\mu_n\}$ is bounded sequence.

Thus $\lambda_n<\infty \quad \forall n \in N$, there is $\lambda \geq 0$ in which $\lim_{n \rightarrow \infty} \lambda_n=\lambda$.

If, by definition of $\lambda_n, \forall k \in N$, there is n_k, m_k in which $m_k>n_k \geq k$

$$\lambda_k-\frac{1}{k}<S\left(\mu_{m_k}, \mu_{m_k}, \mu_{n_k}\right) \leq \lambda_k.$$

Therefore, we get, taking $k \rightarrow \infty$

$$\lim_{k \rightarrow \infty} S\left(\mu_{m_k}, \mu_{m_k}, \mu_{n_k}\right)=\lambda. \tag{3.8}$$

From equation (3.1) and (3.2) we get

$$\mathrm{S}\left(\mu_{m_k},\mu_{m_k},\mu_{n_k}\right)\leq \mathrm{S}\left(\mu_{m_k},\mu_{m_k},\mu_{n_k}\right)$$
$$\leq 2S\left(\mu_{m_{k-1}},\mu_{m_{k-1}},\mu_{m_k}\right)+\mathrm{S}\left(\mu_{n_{k-1}},\mu_{n_{k-1}},\mu_{m_k}\right).$$

Taking $k\to\infty$ we get

$$\lim_{k\to\infty} S\left(\mu_{m_{k-1}},\mu_{m_{k-1}},\mu_{n_{k-1}}\right)=\lambda. \tag{3.9}$$

Using the αZ-contraction and equation (3.8) and (3.9) we get

$$0\leq \lim_{k\to\infty}\sup T\left(\alpha\left(\mu_{m_k},\mu_{m_k},\mu_{n_k}\right),S\left(\mu_{m_k},\mu_{m_k},\mu_{n_k}\right)\mathrm{S}\left(\mu_{m_{k-1}},\mu_{m_{k-1}},\mu_{n_{k-1}}\right)\right)<0.$$

This indicates that a contradiction, consider that $\lambda=0$, and $\{\mu_n\}$ is Cauchy sequence and (X,S) is complete S-metric space, there is $a\in X$.
Letting $n\to\infty$, we obtaing $\lim_{n\to\infty}\mu_n=\mathrm{a}$. By continuity of T, implies that $\lim_{n\to\infty}\mu_{n-1}=\lim_{n\to\infty}T\mu_n$, then $Ta=a$ so, a is fixed point □

Corollary 3.1. *Assume that Theorem (3.1) is hypothesis is true as well as that if we remove condition (3) and put this condition in its place:*
Consider that $\{\mu_n\}$ a sequence in X, in which $\alpha\left(\mu_n,\mu_n,\mu_{n+1}\right)\geq 1 \quad ,\forall n\in N$, and $\{\mu_n\}\to a\in X$, as $n\to\infty$. Then there is a Subsequence $\left\{\mu_{n_{(k)}}\right\}$ of $\{\mu_n\}$ in which $\alpha\left(\mu_{n_{(k)}},\mu_{n_{(k)}},a\right)\geq 1$. Consequently, there is fixed point for T.

Proof. From theorem (3.1). We obtain that $\{\mu_n\}$ is convergent sequence. There is a subsequence based on the newly specified criteria described above. $\left\{\mu_{n_{(k)}}\right\}$ of $\{\mu_n\}$ in which $\alpha\left(\mu_{n_{(k)}},\mu_{n_{(k)}},a\right)\geq 1$, for any k, from equation (3.1)

$$0\leq T\left(\alpha\left(\mu_{n_{(k)}},\mu_{n_{(k)}},a\right),S\left(\mu_{n_{(k)}},\mu_{n_{(k)}},a\right),S\left(\mu_{n_{(k)-1}},\mu_{n_{(k)-1}},Ta\right)\right).$$

Then,

$$\alpha\left(\mu_{n_{(k)}},\mu_{n_{(k)}},a\right).S\left(\mu_{n_{(k)}},\mu_{n_{(k)}},a\right)\leq S\left(\mu_{n_{(k)-1}},\mu_{n_{(k)-1}},Ta\right)$$

Then so

$$S\left(\mu_{n_{(k)}},\mu_{n_{(k)}},a\right)\leq \alpha\left(\mu_{n_{(k)})},u_{n_{(k)}},a\right)S\left(\mu_{n_{(k)}},\mu_{n_{(k)}}a\right)$$
$$\leq S\left(\mu_{n_{(k)-1}},\mu_{n_{(k)-1}},Ta\right).$$

Taking $k\to\infty$ we get $S(a,a,Ta)=0$. So, $Ta=a$, this concludes the proof □

Corollary 3.2. *For the fixed point of* T *that theorem (3.1) obtains to remain unique, one of the following requirements must be satisfied.*

1. $\alpha(\mu,\mu,v)\geq 1,\forall\mu,v\in$ *fix* $(T)=(x\in X:T_x=X)$;

2. *T is α-permissible and $\forall \mu, v \in X$ there is $a \in X$,In which $\alpha(\mu, \mu, a) \geq 1$ and $\alpha(v, v, a) \geq 1$.*

Proof. Consider that μ and v are two district fixed points of T. Consider the condition (1) is true, then

$$
\begin{aligned}
0 &\leq T(\alpha(\mu, \mu, v), S(\mu, \mu, v), S(T\mu, T\mu, Tv)) \\
&= T(\alpha(\mu, \mu, v), S(\mu, \mu, v), S(\mu, \mu, v)) \\
&< S(\mu, \mu, v) - \alpha(\mu, \mu, v).S(\mu, \mu, v).
\end{aligned}
$$

Which is contradiction so that $\mu = v$. Alternately if requirement two holds, one can get $\alpha \in X$, in which $\alpha(\mu, \mu, a) \geq 1$ and $\alpha(v, v, a) \geq 1$. If one of two fixed points (say μ) is same as a then one can prove that $a = v$. This brings about a contradiction. As a result, we assume that μ and v are separate points because the function α permissible. It means that $\alpha\left(\mu, \mu, a_n\right) \geq 1$ and $\alpha\left(v, v, a_n\right) \geq 1 \quad \forall n \geq 1$. Here, we will to show that $\lim_{n\to\infty} a_n = \mu$. If $a_n = \mu$ for some $m \in N$ and immediately following that is the assertion. Else, assume that $S\left(\mu, \mu, a_n\right) > 0, \forall n \in \mathbb{N}$.

$$
\begin{aligned}
0 &\leq T\left(\alpha\left(\mu_n, \mu_n, a_n\right), S\left(\mu_n, \mu_1, a_n\right) S\left(T\mu_{n1}, T\mu_n, Ta_n\right)\right) \\
&\leq S\left(\mu_{n-1}, \mu_{n-1}, a_{n-1}\right) - \alpha\left(\mu, \mu, a_n\right).S\left(\mu, \mu, a_n\right).
\end{aligned}
$$

So $\{S\left(\mu, \mu, a_n\right)\}$ is decreasing sequence of non-negative real so convergent to $r \geq 0$. If $r \neq 0$ then by (T_2) we have

$$
0 \leq \lim_{n\to\infty} \sup T\left(\alpha\left(\mu, \mu, a_n\right), S\left(\mu, \mu, a_{n-1}\right), S\left(\mu, \mu, a_n\right)\right) < 0.
$$

This is contradiction.

We can show that $\lim_{n\to\infty} a_n = v$ Now, the uniqueness of the limit point i.e, $\mu = v$ □

4. Applications of Integral Equations

Fixed-point theorems acquire their most intriguing applications when the underlying metric spaces is a function space. It's theorems for integral equations in S-metric spaces extend classical results to more general settings. These theorems provide powerful tools for proving the existence and uniqueness of solutions to integral equations, significantly broadening the scope of applicable mathematical analysis. the existence and uniqueness of solutions to the Fredholm and Volterra equations are the topics of discussion. In the same metric space in this chapter.

4.1. Integral equation of Fredholm.

An application of Corollary 3..1 shown in this section. Considering the set $X = C([0, 1], \mathbb{R})$ and the following Fredholm type integral equation:

$$
f(t) = \varphi(t) + \lambda \int_0^1 K(t, s, f(s))ds, t \in [0, 1]. \tag{4.1}
$$

Where, $\varphi : [0, 1] \to \mathbb{R}$, and $K : [0, 1]^2 \times \mathbb{R} \to \mathbb{R}$ are continuous function on $[0, 1]$,

Define $S : X^3 \to \mathbb{R}^+, \mathrm{S}(\alpha, \beta, \mu) = \sup_{0\leq t\leq 1} \mid \max\{\alpha(t).\beta(t)\} - \mu(t)) \mid^2$.
So, (X, S) is complete S-metric space. And explain the transformation $T : X \to X$ as follows:

$$Tf(t) = \varphi(t) + \lambda \int_0^1 K(t, s, f(s))d(s), t \in [0, 1]. \tag{4.2}$$

Now, the Fredholm equation becomes $f(t) = Tf(t)$. So, suppose that the next inequality is satisfied:

$$|K(t, s, f(s)) - K(t, s, Tf(s))| \leq \frac{1}{2}|f(s) - Tf(s)|.$$

For $s, t \in [0, 1]$ and $f \in X$, then the solution of the Fredholm integral equation

$$S(Tf(t), T(f(t))), T(Tf(t)) = |Tf(t) - T(Tf(t))|^2$$
$$\leq \left(\int_0^1 |K(t, s, f(s)) - K(t, s, Tf(s))|\right)^2 ds$$
$$\leq \frac{1}{4}S(f, f, Tf).$$

Consequently, the Fixed point $f \in X$ has been proven to exist and unique as Corollary 3..1 is applicable to T.

4.2. **Integral equation of Volterra.** An application of Corollary 3..1, we consider the $X = C([0, 1], \mathbb{R})$ and following Volterra integral equation:

$$f(t) = \varphi(t) + \lambda \int_0^t K(t, s, f(s))ds. \tag{4.3}$$

Where, $t \in I = [0, 1]$ and $\varphi : [0, 1] \to \mathbb{R}$ and $K : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ are continouous functions. So, (X, S) is a complete S-metric space. Explain the transformation $T : X \to X$ as follows .

$$Tf(t) = \varphi(t) + \lambda \int_0^t K(t, s, f(s))ds. \tag{4.4}$$

Where, $t \in I = [0, 1]$. Assume that the following conditions are satisfied:

1. K : $[0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}, \varphi : [0, 1] \to \mathbb{R}, f : \mathbb{R} \to \mathbb{R}$ are continuous
2. K : $[0, 1] \times [0, 1] \to \mathbb{R}$ where $|K(t, s, x) - K(t, s, y)| \leq \theta|x - y|$
3. $S(x, x, y) = \sup_{t\in[0,1]} |x(t) - y(t)|$
4. $\sup_{t\in[0,1]} \int_0^t \theta ds < \beta, \quad \beta \in (\theta, 1)$

Then the integral equation has fixed point

Proof. Let $X = C([0, 1], \mathbb{R})$

$$|Tf(t) - Tg(t)| = \mid \lambda \left(\int_0^t k\left(t, s, f(s)\right) ds - \int_0^t k(t, s, g(s))ds\right) \mid$$
$$\leqslant |\lambda| \mid \int_0^t \left(k\left(t, s, f(s)\right) - k\left(t, s, g(s)\right) ds \mid\right.$$
$$\leqslant |\lambda| \int_0^t \theta|f(s) - g(s)|ds$$

$$\leqslant |\lambda| \int_0^t \theta ds |f(s) - g(s)|$$
$$\leqslant |\lambda| \, \beta \, S(f, f, g)$$

This implies that

$$|Tf(t) - Tg(t)| \leqslant | \, \lambda \, | \;\; \beta \, S(f, f, g)$$

Consequently, the fixed point $f \in X$ has been proven to unique as corollary 3..1 □

5. Conclusions

This study presents new discoveries in fixed-point theory for S-metric spaces that make use of the tri-simulation function. Our paper presents these discoveries. In addition, the existence of solutions to the Fredholm and Volterra equations, as well as the uniqueness of those solutions, are the subjects of discussion in S-metric spaces, respectively. When compared to the writings of earlier experts. Our findings represent developments and improvements for the field (see [6] and [3]).

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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