Asia Pac. J. Math. 2024 **11**:69

# BIPOLAR FUZZY BI-IDEALS OF Γ-SEMIRINGS

MADHULATHA PARVATHAM[1,2], YELLA BHARGAVI[1], AIYARED IAMPAN[3,*]

[1]Department of Engineering Mathematics, College of Engineering, Koneru Lakshmaiah Education Foundation, Vaddeswaram-522502, Andhra Pradesh, India

[2]Department of Basic Sciences and Humanities, Dhanekula Institute of Engineering and Technology, Vijayawada-521139, Andhra Pradesh, India

[3]Department of Mathematics, School of Science, University of Phayao, 19 Moo 2, Tambon Mae Ka, Amphur Mueang, Phayao 56000, Thailand

[*]Corresponding author: aiyared.ia@up.ac.th

Received May 15, 2024

Abstract. This paper explores several features and introduces the idea of bipolar fuzzy bi-ideals (BFBIs) of Γ-semirings (GSRs). We also find the results of the homomorphic pre-image and image of a BFBI of GSRs.

2020 Mathematics Subject Classification. 03E72; 16Y60; 16Y80.

Keywords and phrases: Γ-semiring; bi-ideal; bipolar fuzzy bi-ideal; homomorphic pre-image and image.

## 1. Introduction

Nobusawa [6] proposed the idea of Γ-rings, a generalization of rings. Vandiver [10] introduced and examined the idea of a semiring in detail. Rao [9] proposed Γ-semirings (GSRs) to generalise the ideas behind rings, Γ-rings, and semirings. In algebraic structures, ideals are crucial. Bi-ideals are introduced as a generalization of quasi-ideals in various algebraic systems. Many authors have expanded the notion of bi-ideals from semigroups to semirings by developing quasi-ideals; subsequently, Jagatap and Pawar [4] presented the concept of bi-ideals in GSRs.

In 1965, Zadeh [11] established the idea of fuzzy subsets of a set. Fuzzy sets have several extensions, including intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, neutrosophic sets, etc., which were developed. The idea of bipolar fuzzy sets, which is a significant extension of fuzzy sets whose membership degree interval is extended from the interval $[0, 1]$ to the interval $[-1, 1]$, was first suggested by Zhang [12] in 1994. Jun and Lee [5] have developed bipolar fuzzy sets (BFSs), a generalization

DOI: 10.28924/APJM/11-69

of fuzzy sets. Numerous scholars studied various algebraic structures, such as semigroups, groups, semirings, rings, etc., to build the BFS theory. In 2023, we continued the study of BFS theory by introducing bipolar fuzzy Γ-semirings [7] and bipolar fuzzy ideals [8] on GSRs. Currently, we are investigating the BFBIs of GSRs.

## 2. Preliminaries

In this section, we recall some basic concepts and definitions that we will need in the sequel.

**Definition 2.1.** [1] Let $D$ and $\Gamma$ be two additive commutative semigroups. Then $D$ is called a Γ-semiring (GSR) if there exists a mapping $D \times \Gamma \times D \to D$ image denoted by $\dot{c}\dot{\alpha}\dot{p}$ for $\dot{c}, \dot{p} \in D$ and $\dot{\alpha} \in \Gamma$, satisfying the following conditions:
(i) $\dot{c}\dot{\alpha}(\dot{p} + \dot{u}) = \dot{c}\dot{\alpha}\dot{p} + \dot{c}\dot{\alpha}\dot{u}$,
(ii) $(\dot{c} + \dot{p})\dot{\alpha}\dot{u} = \dot{c}\dot{\alpha}\dot{u} + \dot{p}\dot{\alpha}\dot{u}$,
(iii) $\dot{c}(\dot{\alpha} + \dot{\beta})\dot{u} = \dot{c}\dot{\alpha}\dot{u} + \dot{c}\dot{\beta}\dot{u}$,
(iv) $\dot{c}\dot{\alpha}(\dot{p}\dot{\beta}\dot{u}) = (\dot{c}\dot{\alpha}\dot{p})\dot{\beta}\dot{u}$, for all $\dot{c}, \dot{p}, \dot{u} \in D, \dot{\alpha}, \dot{\beta} \in \Gamma$.

**Definition 2.2.** [11] Let $\vee$ be any non-empty set. A mapping $\xi : \vee \to [0, 1]$ is called a fuzzy set of $\vee$.

**Definition 2.3.** [12] Let $\vee$ be the universe of discourse. A bipolar fuzzy set $\xi$ in $\vee$ is an object having the form $\xi := \{(\dot{v}, \xi^-(\dot{v}), \xi^+(\dot{v})) : \dot{v} \in \vee\}$, where $\xi^- : \vee \to [-1, 0]$ and $\xi^+ : \vee \to [0, 1]$ are mappings.

For the sake of simplicity, we shall use the symbol $\xi = \{\vee; \xi^-, \xi^+\}$ for the bipolar fuzzy set $\xi := \{(\dot{v}, \xi^-(\dot{v}), \xi^+(\dot{v})) : \dot{v} \in \vee\}$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar fuzzy sets.

**Definition 2.4.** [12] Let $\xi = \{\vee; \xi^-, \xi^+\}$ be a bipolar fuzzy set and $s \times t \in [-1, 0] \times [0, 1]$, the sets $\xi^N_s = \{\dot{v} \in \vee : \xi^-(\dot{v}) \leq s\}$ and $\xi^P_t = \{\dot{v} \in \vee : \xi^+(\dot{v}) \geq t\}$ are called negative $s$-cut and positive $t$-cut, respectively. For $s \times t \in [-1, 0] \times [0, 1]$, the set $\xi_{(s,t)} = \xi^N_s \cap \xi^P_t$ is called the $(s, t)$-set of $\xi = \{\vee; \xi^-, \xi^+\}$.

**Definition 2.5.** [12] Let $\xi = \{\vee; \xi^-, \xi^+\}$ and $\eta = \{\vee; \eta^-, \eta^+\}$ be two bipolar fuzzy sets of a universe of discourse $\vee$. The intersection of $\xi$ and $\eta$ is defined as

$$(\xi^- \cap \eta^-)(\dot{v}) = \min\{\xi^-(\dot{v}), \eta^-(\dot{v})\} \text{ and } (\xi^+ \cap \eta^+)(\dot{v}) = \min\{\xi^+(\dot{v}), \eta^+(\dot{v})\}, \text{ for all } \dot{v} \in \vee.$$

The union of $\xi$ and $\eta$ is defined as

$$(\xi^- \cup \eta^-)(\dot{v}) = \max\{\xi^-(\dot{v}), \eta^-(\dot{v})\} \text{ and } (\xi^+ \cup \eta^+)(\dot{v}) = \max\{\xi^+(\dot{v}), \eta^+(\dot{v})\}, \text{ for all } \dot{v} \in \vee.$$

A bipolar fuzzy set $\xi$ is contained in another bipolar fuzzy set $\eta$, written with $\xi \subseteq \eta$ if

$$\xi^-(\dot{v}) \geq \eta^-(\dot{v}) \text{ and } \xi^+(\dot{v}) \leq \eta^+(\dot{v}), \text{ for all } \dot{v} \in \vee.$$

**Definition 2.6.** [3] Let $J : K \to L$ be a homomorphism from a set $K$ onto a set $L$ and let $\xi = \{K; \xi^-, \xi^+\}$ be a bipolar fuzzy set of $K$ and $\eta = \{L; \eta^-, \eta^+\}$ be a bipolar fuzzy set of $L$, then the homomorphic image $J(\xi)$ of $\xi$ is $J(\xi) = \{L; (J(\xi))^-, (J(\xi))^+\}$ defined as for all $\dot{v} \in L$,

$$(J(\xi))^-(\dot{v}) = \begin{cases} \inf\{\xi^-(\dot{u}) : \dot{u} \in J^{-1}(\dot{v})\}, \text{ if } J^{-1}(\dot{v}) \neq \emptyset \\ 0, \text{ otherwise} \end{cases}$$

and

$$(J(\xi))^+(\dot{v}) = \begin{cases} \sup\{\xi^+(\dot{u}) : \dot{u} \in J^{-1}(\dot{v})\}, \text{ if } J^{-1}(\dot{v}) \neq \emptyset \\ 0, \text{ otherwise.} \end{cases}$$

The pre-image $J^{-1}(\eta)$ of $\eta$ under $J$ is a bipolar fuzzy set defined as $(J^{-1}(\eta))^-(\dot{u}) = \eta^-(J(\dot{u}))$ and $(J^{-1}(\eta))^+(\dot{u}) = \eta^+(J(\dot{u}))$, for all $\dot{u} \in K$.

**Definition 2.7.** [1] Let $\vee$ be a subset of a $\Gamma$-semiring $D$. The characteristic function of $\vee$ taking values in $[0, 1]$ is a fuzzy set $\delta_\vee$ given by

$$\delta_\vee(\dot{v}) = \begin{cases} 1, \text{ if } \dot{v} \in \vee \\ 0, \text{ otherwise.} \end{cases}$$

Then $\delta_\vee$ is a fuzzy characteristic function of $\vee$.

**Definition 2.8.** [7] Let $\vee$ be a subset of a $\Gamma$-semiring $D$. The bipolar fuzzy characteristic function $\delta_\vee$ of $\vee$ is given by

$$\delta_\vee^+(\dot{v}) = \begin{cases} 1, \text{ if } \dot{v} \in \vee \\ 0, \text{ otherwise} \end{cases} \text{ and } \delta_\vee^-(\dot{v}) = \begin{cases} -1, \text{ if } \dot{v} \in \vee \\ 0, \text{ otherwise.} \end{cases}$$

Then $\delta_\vee$ is a bipolar fuzzy characteristic function of $\vee$.

**Definition 2.9.** [1] A bipolar fuzzy set $\xi = \{D; \xi^-, \xi^+\}$ in $D$ is called a bipolar fuzzy $\Gamma$-semiring of $D$ if it satisfies the following properties: for all $\dot{c}, \dot{p} \in D$ and $\ddot{\gamma} \in \Gamma$,

(i) $\xi^-(\dot{c} + \dot{p}) \leq \max\{\xi^-(\dot{c}), \xi^-(\dot{p})\}$,

(ii) $\xi^-(\dot{c}\ddot{\gamma}\dot{p}) \leq \max\{\xi^-(\dot{c}), \xi^-(\dot{p})\}$,

(iii) $\xi^+(\dot{c} + \dot{p}) \geq \min\{\xi^+(\dot{c}), \xi^+(\dot{p})\}$,

(iv) $\xi^+(\dot{c}\ddot{\gamma}\dot{p}) \geq \min\{\xi^+(\dot{c}), \xi^+(\dot{p})\}$.

**Definition 2.10.** [2] A non-empty subset $\vee$ of a $\Gamma$-semiring $D$ is said to be a bi-ideal of $D$ if $\vee$ is a sub-$\Gamma$-semiring of $D$ and $\vee\Gamma D\Gamma\vee \subseteq \vee$.

**Definition 2.11.** [2] Let $\xi$ be a fuzzy set in a $\Gamma$-semiring $D$. Then $\xi$ is called a fuzzy bi-ideal of $D$ if for any $\dot{c}, \dot{p}, \dot{u} \in D$ and $\dot{\alpha}, \dot{\beta} \in \Gamma$,

(i) $\xi(\dot{c} + \dot{p}) \geq \min\{\xi(\dot{c}), \xi(\dot{p})\}$,

(ii) $\xi(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}) \geq \min\{\xi(\dot{c}), \xi(\dot{u})\}$.

## 3. Main Results

This session introduces and studies BFBIs and discusses their properties. After this, we will replace $D$ as a $\Gamma$-semiring.

**Definition 3.1.** A BFS $\xi = (D; \xi^+, \xi^-)$ of $D$ is called a bipolar fuzzy bi-ideal (BFBI) of $D$ if for any $\dot{c}, \dot{p}, \dot{u} \in D$ and $\dot{\alpha}, \dot{\beta} \in \Gamma$,
(i) $\xi^-(\dot{c} + \dot{p}) \leq \max\{\xi^-(\dot{c}), \xi^-(\dot{p})\}$,
(ii) $\xi^-(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}) \leq \max\{\xi^-(\dot{c}), \xi^-(\dot{u})\}$,
(iii) $\xi^+(\dot{c} + \dot{p}) \geq \min\{\xi^+(\dot{c}), \xi^+(\dot{p})\}$,
(iv) $\xi^+(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}) \geq \min\{\xi^+(\dot{c}), \xi^+(\dot{u})\}$.

**Theorem 3.2.** *A BFS $\xi = (D; \xi^-, \xi^+)$ of $D$ is a BFBI of $D$ if and only if for all $s \times t \in [-1, 0] \times [0, 1], \emptyset \neq \xi_s^N$ and $\emptyset \neq \xi_t^P$ are bi-ideals of $D$.*

*Proof.* Let $\xi = (D; \xi^-, \xi^+)$ be a BFBI of $D$. Let $s \times t \in [-1, 0] \times [0, 1]$ be such that $\xi_s^N \neq \emptyset$ and $\xi_t^P \neq \emptyset$. Let $\dot{d}, \dot{g}, \dot{j} \in \xi_s^N$, $\dot{c}, \dot{p}, \dot{u} \in \xi_t^P$, and $\dot{\alpha}, \dot{\beta} \in \Gamma$. Then $\xi^-(\dot{d}) \leq s, \xi^-(\dot{g}) \leq s, \xi^-(\dot{j}) \leq s$ and $\xi^+(\dot{c}) \geq t, \xi^+(\dot{p}) \geq t, \xi^+(\dot{u}) \geq t$. Since $\xi = (D; \xi^-, \xi^+)$ is a BFBI of $D$, we have
(i) $\xi^-(\dot{d} + \dot{g}) \leq \max\{\xi^-(\dot{d}), \xi^-(\dot{g})\} \leq s$,
(ii) $\xi^-(\dot{d}\dot{\alpha}\dot{g}\dot{\beta}\dot{j}) \leq \max\{\xi^-(\dot{d}), \xi^-(\dot{j})\} \leq s$,
(iii) $\xi^+(\dot{c} + \dot{p}) \geq \min\{\xi^+(\dot{c}), \xi^+(\dot{p})\} \geq t$,
(iv) $\xi^+(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}) \geq \min\{\xi^+(\dot{c}), \xi^+(\dot{u})\} \geq t$.
Then $(\dot{d} + \dot{g}) \in \xi_s^N, \dot{d}\dot{\alpha}\dot{g}\dot{\beta}\dot{j} \in \xi_s^N$ and $\dot{c} + \dot{p} \in \xi_t^P, \dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u} \in \xi_t^P$. Thus, $\xi_s^N$ and $\xi_t^P$ are bi-ideals of $D$.

Conversely, suppose that for all $s \times t \in [-1, 0] \times [0, 1], \emptyset \neq \xi_s^N$ and $\emptyset \neq \xi_t^P$ are bi-ideals of $D$. Let $\dot{d}, \dot{g}, \dot{j} \in \xi_s^N$, $\dot{c}, \dot{p}, \dot{u} \in \xi_t^P$, and $\dot{\alpha}, \dot{\beta} \in \Gamma$. Then $\dot{d} + \dot{g} \in \xi_s^N, \dot{d}\dot{\alpha}\dot{g}\dot{\beta}\dot{j} \in \xi_s^N$ and $\dot{c} + \dot{p} \in \xi_t^P, \dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u} \in \xi_t^P$. Choose $s = \max\{\xi^-(\dot{d}), \xi^-(\dot{g}), \xi^-(\dot{j})\}$. Let $\xi^-(\dot{d}) \leq s_1, \xi^-(\dot{g}) \leq s_2, \xi^-(\dot{j}) \leq s_3$, and $s_1 \leq s_2 \leq s_3 \leq s$.
(i) $\xi^-(\dot{d} + \dot{g}) \leq \max\{\xi^-(\dot{d}), \xi^-(\dot{g})\} \leq \max\{s_1, s_2\} = s_2 \leq s$,
(ii) $\xi^-(\dot{d}\dot{\alpha}\dot{g}\dot{\beta}\dot{j}) \leq \max\{\xi^-(\dot{d}), \xi^-(\dot{j})\} \leq \max\{s_1, s_3\} = s_3 \leq s$.
Choose $t = \min\{\xi^+(\dot{c}), \xi^+(\dot{p}), \xi^+(\dot{u})\}$. Let $\xi^+(\dot{c}) \geq t_1, \xi^+(\dot{p}) \geq t_2, \xi^+(\dot{u}) \geq t_3$, and $t_1 \geq t_2 \geq t_3 \geq t$.
(iii) $\xi^+(\dot{c} + \dot{p}) \geq \min\{\xi^+(\dot{c}), \xi^+(\dot{p})\} \geq \min\{t_1, t_2\} = t_2 \geq t$,
(iv) $\xi^+(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}) \geq \min\{\xi^+(\dot{c}), \xi^+(\dot{u})\} \geq \min\{t_1, t_3\} = t_3 \geq t$.
Hence, $\xi = (D; \xi^-, \xi^+)$ is a BFBI of $D$. □

**Theorem 3.3.** *If $\xi = (D; \xi^-, \xi^+)$ and $\eta = (D; \eta^-, \eta^+)$ are two BFBIs of $D$, then $\xi \cap \eta$ is a BFBI of $D$.*

*Proof.* Assume that $\xi = (D; \xi^-, \xi^+)$ and $\eta = (D; \eta^-, \eta^+)$ are BFBIs of $D$. Let $\dot{c}, \dot{p}, \dot{u} \in D$ and $\dot{\alpha}, \dot{\beta} \in \Gamma$. Then

$$(\xi^- \cap \eta^-)(\dot{c} + \dot{p}) = \min\{\xi^-(\dot{c} + \dot{p}), \eta^-(\dot{c} + \dot{p})\}$$

$$\leq \min\{\max\{\xi^-(\dot{c}), \xi^-(\dot{p})\}, \max\{\eta^-(\dot{c}), \eta^-(\dot{p})\}\}$$
$$\leq \min\{\max\{\xi^-(\dot{c}), \eta^-(\dot{c})\}, \max\{\xi^-(\dot{p}), \eta^-(\dot{p})\}\}$$
$$\leq \max\{\min\{\xi^-(\dot{c}), \eta^-(\dot{c})\}, \min\{\xi^-(\dot{p}), \eta^-(\dot{p})\}\}$$
$$= \max\{(\xi^- \cap \eta^-)(\dot{c}), (\xi^- \cap \eta^-)(\dot{p})\},$$

$$\begin{aligned}(\xi^- \cap \eta^-)(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}) &= \min\{\xi^-(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}), \eta^-(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u})\}\\ &\leq \min\{\max\{\xi^-(\dot{c}), \xi^-(\dot{u})\}, \max\{\eta^-(\dot{c}), \eta^-(\dot{u})\}\}\\ &\leq \min\{\max\{\xi^-(\dot{c}), \eta^-(\dot{c})\}, \max\{\xi^-(\dot{u}), \eta^-(\dot{u})\}\}\\ &\leq \max\{\min\{\xi^-(\dot{c}), \eta^-(\dot{c})\}, \min\{\xi^-(\dot{u}), \eta^-(\dot{u})\}\}\\ &= \max\{(\xi^- \cap \eta^-)(\dot{c}), (\xi^- \cap \eta^-)(\dot{u})\},\end{aligned}$$

$$\begin{aligned}(\xi^+ \cap \eta^+)(\dot{c}+\dot{p}) &= \min\{\xi^+(\dot{c}+\dot{p}), \eta^+(\dot{c}+\dot{p})\}\\ &\geq \min\{\min\{\xi^+(\dot{c}), \xi^+(\dot{p})\}, \min\{\eta^+(\dot{c}), \eta^+(\dot{p})\}\}\\ &= \min\{\min\{\xi^+(\dot{c}), \eta^+(\dot{c})\}, \min\{\xi^+(\dot{p}), \eta^+(\dot{p})\}\}\\ &= \min\{(\xi^+ \cap \eta^+)(\dot{c}), (\xi^+ \cap \eta^+)(\dot{p})\},\end{aligned}$$

$$\begin{aligned}(\xi^+ \cap \eta^+)(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}) &= \min\{\xi^+(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}), \eta^+(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u})\}\\ &\geq \min\{\min\{\xi^+(\dot{c}), \xi^+(\dot{u})\}, \min\{\eta^+(\dot{c}), \eta^+(\dot{u})\}\}\\ &= \min\{\min\{\xi^+(\dot{c}), \eta^+(\dot{c})\}, \min\{\xi^+(\dot{u}), \eta^+(\dot{u})\}\}\\ &= \min\{(\xi^+ \cap \eta^+)(\dot{c}), (\xi^+ \cap \eta^+)(\dot{u})\}.\end{aligned}$$

Hence, $\xi \cap \eta$ is a BFBI of $D$. □

**Theorem 3.4.** *If $\xi = (D; \xi^-, \xi^+)$ and $\eta = (D; \eta^-, \eta^+)$ are two BFBIs of $D$, then $\xi \cup \eta$ is a BFBI of $D$ when $\xi \subseteq \eta$ or $\eta \subseteq \xi$.*

*Proof.* Suppose $\xi \subseteq \eta$. Let $\dot{c}, \dot{p}, \dot{u} \in D$ and $\dot{\alpha}, \dot{\beta} \in \Gamma$. Then

$$\begin{aligned}(\xi^- \cup \eta^-)(\dot{c}+\dot{p}) &= \max\{\xi^-(\dot{c}+\dot{p}), \eta^-(\dot{c}+\dot{p})\}\\ &= \xi^-(\dot{c}+\dot{p})\\ &\leq \max\{\xi^-(\dot{c}), \xi^-(\dot{p})\}\\ &= \max\{\max\{\xi^-(\dot{c}), \eta^-(\dot{c})\}, \max\{\xi^-(\dot{p}), \eta^-(\dot{p})\}\}\\ &= \max\{(\xi^- \cup \eta^-)(\dot{c}), (\xi^- \cup \eta^-)(\dot{p})\},\end{aligned}$$

$$\begin{aligned}(\xi^- \cup \eta^-)(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}) &= \max\{\xi^-(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}), \eta^-(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u})\}\\ &= \xi^-(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u})\\ &\leq \max\{\xi^-(\dot{c}), \xi^-(\dot{u})\}\\ &\leq \max\{\max\{\xi^-(\dot{c}), \eta^-(\dot{c})\}, \max\{\xi^-(\dot{u}), \eta^-(\dot{u})\}\}\\ &= \max\{(\xi^- \cup \eta^-)(\dot{c}), (\xi^- \cup \eta^-)(\dot{u})\},\end{aligned}$$

$$\begin{aligned}(\xi^+ \cup \eta^+)(\dot{c}+\dot{p}) &= \max\{\xi^+(\dot{c}+\dot{p}), \eta^+(\dot{c}+\dot{p})\}\\ &= \eta^+(\dot{c}+\dot{p})\\ &\geq \min\{\eta^+(\dot{c}), \eta^+(\dot{p})\}\\ &= \min\{\max\{\xi^+(\dot{c}), \eta^+(\dot{c})\}, \max\{\xi^+(\dot{p}), \eta^+(\dot{p})\}\}\\ &= \min\{(\xi^+ \cup \eta^+)(\dot{c}), (\xi^+ \cup \eta^+)(\dot{p})\},\end{aligned}$$

$$\begin{aligned}(\xi^+ \cup \eta^+)(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}) &= \max\{\xi^+(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}), \eta^+(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u})\}\\ &= \eta^+(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u})\\ &\geq \min\{\eta^+(\dot{c}), \eta^+(\dot{u})\}\\ &= \min\{\max\{\xi^+(\dot{c}), \eta^+(\dot{c})\}, \max\{\xi^+(\dot{u}), \eta^+(\dot{u})\}\}\\ &= \min\{(\xi^+ \cup \eta^+)(\dot{c}), (\xi^+ \cup \eta^+)(\dot{u})\}.\end{aligned}$$

Hence, $\xi \cup \eta$ is a BFBI of $D$. Similarly, if $\eta \subseteq \xi$, then $\xi \cup \eta$ is a BFBI of $D$. □

**Theorem 3.5.** *Let $g$ be a homomorphism from a $\Gamma$-semiring $C$ onto a $\Gamma$-semiring $D$. If $\eta$ is a BFBI of $D$, then the homomorphic pre-image $g^{-1}(\eta)$ of $\eta$ is a BFBI of $C$.*

*Proof.* Assume that $\eta$ is a BFBI of $D$. Let $\dot{c}, \dot{p}, \dot{u} \in C$ and $\dot{\alpha}, \dot{\beta} \in \Gamma$. Then

$$\begin{aligned}(g^{-1}(\eta)^-)(\dot{c}+\dot{p}) &= \eta^-(g(\dot{c}+\dot{p}))\\ &= \eta^-(g(\dot{c})+g(\dot{p}))\\ &\leq \max\left\{\eta^-(g(\dot{c})), \eta^-(g(\dot{p}))\right\}\\ &= \max\left\{(g^{-1}(\eta)^-)(\dot{c}), (g^{-1}(\eta)^-)(\dot{p})\right\},\end{aligned}$$

$$\begin{aligned}(g^{-1}(\eta)^-)(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}) &= \eta^-(g(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}))\\ &= \eta^-(g(\dot{c})\dot{\alpha}g(\dot{p})\dot{\beta}g(\dot{u}))\\ &\leq \max\left\{\eta^-(g(\dot{c})), \eta^-(g(\dot{u}))\right\}\\ &= \max\left\{(g^{-1}(\eta)^-)(\dot{c}), (g^{-1}(\eta)^-)(\dot{u})\right\},\end{aligned}$$

$$
\begin{aligned}
(g^{-1}(\eta)^{+})(\dot{c}+\dot{p}) &= \eta^{+}(g(\dot{c}+\dot{p})) \\
&= \eta^{+}(g(\dot{c})+g(\dot{p})) \\
&\geq \min\left\{\eta^{+}(g(\dot{c})), \eta^{+}(g(\dot{p}))\right\} \\
&= \min\left\{(g^{-1}(\eta)^{+})(\dot{c}), (g^{-1}(\eta)^{+})(\dot{p})\right\},
\end{aligned}
$$

$$
\begin{aligned}
(g^{-1}(\eta)^{+})(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}) &= \eta^{+}(g(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u})) \\
&= \eta^{+}(g(\dot{c})\dot{\alpha}g(\dot{p})\dot{\beta}g(\dot{u})) \\
&\geq \min\left\{\eta^{+}(g(\dot{c})), \eta^{+}(g(\dot{u}))\right\} \\
&= \min\left\{(g^{-1}(\eta)^{+})(\dot{c}), (g^{-1}(\eta)^{+})(\dot{u})\right\}.
\end{aligned}
$$

Hence, $g^{-1}(\eta)$ is a BFBI of $C$. □

**Theorem 3.6.** *Let $g$ be a homomorphism from a $\Gamma$-semiring $C$ onto a $\Gamma$-semiring $D$. If $\xi$ is a BFBI of $C$, then the homomorphic image $g(\xi)$ of $\xi$ is a BFBI of $D$.*

*Proof.* Assume that $\xi$ is a BFBI of $C$. Let $\dot{c}, \dot{p}, \dot{u} \in D$ and $\dot{\alpha}, \dot{\beta} \in \Gamma$. Now, $g^{-1}(\dot{c})$, $g^{-1}(\dot{p})$, and $g^{-1}(\dot{u})$ are non-empty. Since $g$ is a homomorphism and there exist $\dot{d}, \dot{g}, \dot{j} \in C$ such that $g(\dot{d}) = \dot{c}$, $g(\dot{g}) = \dot{p}$, and $g(\dot{j}) = \dot{u}$, we have that $\dot{d}+\dot{g} \in g^{-1}(\dot{c}+\dot{p})$ and $\dot{d}\dot{\alpha}\dot{g}\dot{\beta}\dot{j} \in g^{-1}(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u})$. Then

$$
\begin{aligned}
(g(\xi))^{-}(\dot{c}+\dot{p}) &= \inf\{\xi^{-}(\dot{u}) : \dot{u} \in g^{-1}(\dot{c}+\dot{p})\} \\
&= \inf\{\xi^{-}(\dot{d}+\dot{g}) : \dot{d} \in g^{-1}(n), \dot{g} \in g^{-1}(\dot{p})\} \\
&\leq \inf\{\max\{\xi^{-}(\dot{d}), \xi^{-}(\dot{g})\} : \dot{d} \in g^{-1}(\dot{c}), \dot{g} \in g^{-1}(\dot{p})\} \\
&\leq \max\{\inf\{\xi^{-}(\dot{d}) : \dot{d} \in g^{-1}(\dot{c})\}, \inf\{\xi^{-}(\dot{g}) : \dot{g} \in g^{-1}(\dot{p})\}\} \\
&= \max\{(g(\xi))^{-}(\dot{c}), (g(\xi))^{-}(\dot{p})\},
\end{aligned}
$$

$$
\begin{aligned}
(g(\xi))^{-}(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}) &= \inf\{\xi^{-}(\dot{u}) : \dot{u} \in g^{-1}(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u})\} \\
&= \inf\{\xi^{-}(\dot{d}\dot{\alpha}\dot{g}\dot{\beta}\dot{j}) : \dot{d} \in g^{-1}(\dot{c}), \dot{g} \in g^{-1}(\dot{p}), \dot{j} \in g^{-1}(\dot{u})\} \\
&\leq \inf\{\max\{\xi^{-}(\dot{d}), \xi^{-}(\dot{j})\} : \dot{d} \in g^{-1}(\dot{c}), \dot{j} \in g^{-1}(\dot{u})\} \\
&\leq \max\{\inf\{\xi^{-}(\dot{d}) : \dot{d} \in g^{-1}(\dot{c})\}, \inf\{\xi^{-}(\dot{j}) : \dot{g} \in g^{-1}(\dot{u})\}\} \\
&= \max\{(g(\xi))^{-}(\dot{c}), (g(\xi))^{-}(\dot{u})\},
\end{aligned}
$$

$$
\begin{aligned}
(g(\xi))^{+}(\dot{c}+\dot{p}) &= \sup\{\xi^{+}(\dot{u}) : \dot{u} \in g^{-1}(\dot{c}+\dot{p})\} \\
&= \sup\{\xi^{+}(\dot{d}+\dot{g}) : \dot{d} \in g^{-1}(n), \dot{g} \in g^{-1}(\dot{p})\} \\
&\geq \sup\{\min\{\xi^{+}(\dot{d}), \xi^{+}(\dot{g})\} : \dot{d} \in g^{-1}(\dot{c}), \dot{g} \in g^{-1}(\dot{p})\}
\end{aligned}
$$

$$\geq \min\{\sup\{\xi^{+}(\dot{d}) : \dot{d} \in g^{-1}(\dot{c})\}, \sup\{\xi^{+}(\dot{g}) : \dot{g} \in g^{-1}(\dot{p})\}\}$$
$$= \min\{(g(\xi))^{+}(\dot{c}), (g(\xi))^{+}(\dot{p})\},$$

$$\begin{aligned}(g(\xi))^{+}(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}) &= \sup\{\xi^{+}(\dot{u}) : \dot{u} \in g^{-1}(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u})\}\\ &= \sup\{\xi^{+}(\dot{d}\dot{\alpha}\dot{g}\dot{\beta}\dot{j}) : \dot{d} \in g^{-1}(\dot{c}), \dot{g} \in g^{-1}(\dot{p}), \dot{j} \in g^{-1}(\dot{u})\}\\ &\geq \sup\{\min\{\xi^{+}(\dot{d}), \xi^{+}(\dot{j})\} : \dot{d} \in g^{-1}(\dot{c}), \dot{j} \in g^{-1}(\dot{u})\}\\ &\geq \min\{\sup\{\xi^{+}(\dot{d}) : \dot{d} \in g^{-1}(\dot{c})\}, \sup\{\xi^{+}(\dot{j}) : \dot{g} \in g^{-1}(\dot{u})\}\}\\ &= \min\{(g(\xi))^{+}(\dot{c}), (g(\xi))^{+}(\dot{u})\}.\end{aligned}$$

Hence, $g(\xi)$ is a BFBI of a $D$. □

**Theorem 3.7.** *Let $B$ be a non-empty subset of $D$. Then the bipolar fuzzy characteristic function $\delta_B$ of $B$ is a BFBI of $D$ if and only if $B$ is a bi-ideal of $D$.*

*Proof.* Assume that $\delta_B$ is a BFBI of $D$. Let $\dot{c}, \dot{p}, \dot{u} \in D$ and $\dot{\alpha}, \dot{\beta} \in \Gamma$. Then
(i) $\dot{c}, \dot{p} \in B \Rightarrow \delta_B^{-}(\dot{c} + \dot{p}) \leq \max\{\delta_B^{-}(\dot{c}), \delta_B^{-}(\dot{p})\} = \max\{-1, -1\} = -1 \Rightarrow \dot{c} + \dot{p} \in B$.
(ii) $\dot{c}, \dot{u} \in B \Rightarrow \delta_B^{-}(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}) \leq \max\{\delta_B^{-}(\dot{c}), \delta_B^{-}(\dot{u})\} = \max\{-1, -1\} = -1 \Rightarrow \dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u} \in B$.
Hence, $B$ is a bi-ideal of $D$.

Conversely, assume that $B$ is a bi-ideal of $D$.

Case 1: If $\dot{c}, \dot{p}, \dot{u} \in B$ and $\dot{\alpha}, \dot{\beta} \in \Gamma$. Then $\dot{c} + \dot{p} \in B$ and $\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u} \in B$. Thus
(i)${\delta_B}^{-}(\dot{c} + \dot{p}) = -1 = \max\{{\delta_B}^{-}(\dot{c}), {\delta_B}^{-}(\dot{p})\}$,
(ii) ${\delta_B}^{-}(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}) = -1 = \max\{{\delta_B}^{-}(\dot{c}), {\delta_B}^{-}(\dot{u})\}$,
(iii) ${\delta_B}^{+}(\dot{c} + \dot{p}) = 1 = \min\{{\delta_B}^{+}(\dot{c}), {\delta_B}^{+}(\dot{p})\}$,
(iv) ${\delta_B}^{+}(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}) = 1 = \min\{{\delta_B}^{+}(\dot{c}), {\delta_B}^{+}(\dot{u})\}$.

Case 2: If $\dot{c}, \dot{p}, \dot{u} \notin B$ and $\dot{\alpha}, \dot{\beta} \in \Gamma$. Then $\delta_B^{+}(\dot{c}) = \delta_B^{+}(\dot{p}) = \delta_B^{+}(\dot{u}) = 0$ and $\delta_B^{-}(\dot{c}) = \delta_B^{-}(\dot{p}) = \delta_B^{-}(\dot{u}) = 0$. Thus
(i) ${\delta_B}^{-}(\dot{c} + \dot{p}) \leq 0 = \max\{{\delta_B}^{-}(\dot{c}), {\delta_B}^{-}(\dot{p})\}$,
(ii) ${\delta_B}^{-}(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}) \leq 0 = \max\{{\delta_B}^{-}(\dot{c}), {\delta_B}^{-}(\dot{u})\}$,
(iii) ${\delta_B}^{+}(\dot{c} + \dot{p}) \geq 0 = \min\{{\delta_B}^{+}(\dot{c}), {\delta_B}^{+}(\dot{p})\}$,
(iv) ${\delta_B}^{+}(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}) \geq 0 = \min\{{\delta_B}^{+}(\dot{c}), {\delta_B}^{+}(\dot{u})\}$.

Case 3: If $\dot{c}, \dot{p} \in B, \dot{u} \notin B$, and $\dot{\alpha}, \dot{\beta} \in \Gamma$. Then $\delta_B^{+}(\dot{c}) = \delta_B^{+}(\dot{p}) = 1, \delta_B^{+}(\dot{u}) = 0$ and $\delta_B^{-}(\dot{c}) = \delta_B^{-}(\dot{p}) = -1, \delta_B^{-}(\dot{u}) = 0$. Thus
(i) ${\delta_B}^{-}(\dot{c} + \dot{p}) = -1 = \max\{{\delta_B}^{-}(\dot{c}), {\delta_B}^{-}(\dot{p})\}$,
(ii) ${\delta_B}^{-}(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}) \leq 0 = \max\{{\delta_B}^{-}(\dot{c}), {\delta_B}^{-}(\dot{u})\}$,
(iii) ${\delta_B}^{+}(\dot{c} + \dot{p}) = 1 = \min\{{\delta_B}^{+}(\dot{c}), {\delta_B}^{+}(\dot{p})\}$,
(iv) ${\delta_B}^{+}(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}) \geq 0 = \min\{{\delta_B}^{+}(\dot{c}), {\delta_B}^{+}(\dot{u})\}$.

Case 4: If $\dot{c}, \dot{u} \in B, \dot{p} \notin B$, and $\dot{\alpha}, \dot{\beta} \in \Gamma$. Then $\delta_B^+(\dot{c}) = \delta_B^+(\dot{u}) = 1, \delta_B^+(\dot{p}) = 0$ and $\delta_B^-(\dot{c}) = \delta_B^-(\dot{u}) = -1, \delta_B^-(\dot{p}) = 0$. Thus

(i) $\delta_B{}^-(\dot{c}+\dot{p}) \leq 0 = \max\{\delta_B{}^-(\dot{c}), \delta_B{}^-(\dot{p})\}$,

(ii) $\delta_B{}^-(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}) \leq 0 = \max\{\delta_B{}^-(\dot{c}), \delta_B{}^-(\dot{u})\}$,

(iii) $\delta_B{}^+(\dot{c}+\dot{p}) \geq 0 = \min\{\delta_B{}^+(\dot{c}), \delta_B{}^+(\dot{p})\}$,

(iv) $\delta_B{}^+(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}) \geq 0 = \min\{\delta_B{}^+(\dot{c}), \delta_B{}^+(\dot{u})\}$.

Case 5: If $\dot{p}, \dot{u} \in B, \dot{c} \notin B$, and $\dot{\alpha}, \dot{\beta} \in \Gamma$. Then $\delta_B^+(\dot{p}) = \delta_B^+(\dot{u}) = 1, \delta_B^+(\dot{c}) = 0$ and $\delta_B^-(\dot{p}) = \delta_B^-(\dot{u}) = -1, \delta_B^-(\dot{c}) = 0$. Thus

(i) $\delta_B{}^-(\dot{c}+\dot{p}) \leq 0 = \max\{\delta_B{}^-(\dot{c}), \delta_B{}^-(\dot{p})\}$,

(ii) $\delta_B{}^-(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}) \leq 0 = \max\{\delta_B{}^-(\dot{c}), \delta_B{}^-(\dot{u})\}$,

(iii) $\delta_B{}^+(\dot{c}+\dot{p}) \geq 0 = \min\{\delta_B{}^+(\dot{c}), \delta_B{}^+(\dot{p})\}$,

(iv) $\delta_B{}^+(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}) \geq 0 = \min\{\delta_B{}^+(\dot{c}), \delta_B{}^+(\dot{u})\}$.

Case 6: If $\dot{c} \in B, \dot{p}, \dot{u} \notin B$, and $\dot{\alpha}, \dot{\beta} \in \Gamma$. Then $\delta_B^+(\dot{c}) = 1, \delta_B^+(\dot{p}) = \delta_B^+(\dot{u}) = 0$ and $\delta_B^-(\dot{c}) = -1, \delta_B^-(\dot{p}) = \delta_B^-(\dot{u}) = 0$. Thus

(i) $\delta_B{}^-(\dot{c}+\dot{p}) \leq 0 = \max\{\delta_B{}^-(\dot{c}), \delta_B{}^-(\dot{p})\}$,

(ii) $\delta_B{}^-(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}) \leq 0 = \max\{\delta_B{}^-(\dot{c}), \delta_B{}^-(\dot{u})\}$,

(iii) $\delta_B{}^+(\dot{c}+\dot{p}) \geq 0 = \min\{\delta_B{}^+(\dot{c}), \delta_B{}^+(\dot{p})\}$,

(iv)$\delta_B{}^+(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}) \geq 0 = \min\{\delta_B{}^+(\dot{c}), \delta_B{}^+(\dot{u})\}$.

Case 7: If $\dot{p} \in B, \dot{c}, \dot{u} \notin B$, and $\dot{\alpha}, \dot{\beta} \in \Gamma$. Then $\delta_B^+(\dot{p}) = 1, \delta_B^+(\dot{c}) = \delta_B^+(\dot{u}) = 0$ and $\delta_B^-(\dot{p}) = -1, \delta_B^-(\dot{c}) = \delta_B^-(\dot{u}) = 0$. Thus

(i) $\delta_B{}^-(\dot{c}+\dot{p}) \leq 0 = \max\{\delta_B{}^-(\dot{c}), \delta_B{}^-(\dot{p})\}$,

(ii) $\delta_B{}^-(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}) \leq 0 = \max\{\delta_B{}^-(\dot{c}), \delta_B{}^-(\dot{u})\}$,

(iii) $\delta_B{}^+(\dot{c}+\dot{p}) \geq 0 = \min\{\delta_B{}^+(\dot{c}), \delta_B{}^+(\dot{p})\}$,

(iv) $\delta_B{}^+(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}) \geq 0 = \min\{\delta_B{}^+(\dot{c}), \delta_B{}^+(\dot{u})\}$.

Case 8: If $\dot{u} \in B, \dot{c}, \dot{p} \notin B$, and $\dot{\alpha}, \dot{\beta} \in \Gamma$. Then $\delta_B^+(\dot{u}) = 1, \delta_B^+(\dot{c}) = \delta_B^+(\dot{p}) = 0$ and $\delta_B^-(\dot{u}) = -1, \delta_B^-(\dot{c}) = \delta_B^-(\dot{p}) = 0$. Thus

(i) $\delta_B{}^-(\dot{c}+\dot{p}) = -1 = \max\{\delta_B{}^-(\dot{c}), \delta_B{}^-(\dot{p})\}$,

(ii) $\delta_B{}^-(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}) \leq 0 = \max\{\delta_B{}^-(\dot{c}), \delta_B{}^-(\dot{u})\}$,

(iii) $\delta_B{}^+(\dot{c}+\dot{p}) = 1 = \min\{\delta_B{}^+(\dot{c}), \delta_B{}^+(\dot{p})\}$,

(iv) $\delta_B{}^+(\dot{c}\dot{\alpha}\dot{p}\dot{\beta}\dot{u}) \geq 0 = \min\{\delta_B{}^+(\dot{c}), \delta_B{}^+(\dot{u})\}$.

Therefore, $\delta_B$ is a BFBI of $D$. □

## 4. Conclusion

In this paper, we examine the notion of BFBIs and their one-to-one correspondence. Furthermore, we proved that a BFBI is a homomorphic image and its pre-image.

### Acknowledgment

This research was supported by the University of Phayao and the Thailand Science Research and Innovation Fund (Fundamental Fund 2024).

### Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

### References

[1] Y. Bhargavi, T. Eswarlal, Fuzzy Γ-semirings, Int. J. Pure Appl. Math. 98 (2015), 339–349. https://doi.org/10.12732/ijpam.v98i3.6.

[2] Y. Bhargavi, T. Eswarlal, Vague bi-ideals and vague quasi ideals of a Γ-semiring, Int. J. Sci. Res. 4 (2015), 2694–2699.

[3] H. Hedayati, K.P. Shum, An introduction to Γ-semirings, Int. J. Algebra, 5 (2011), 709–726.

[4] R.D. Jagatap, Y.S. Pawar, Bi-ideals in Γ-semirings, Bull. Int. Math. Virtual Inst. 6 (2016), 169–179.

[5] K.J. Lee, Bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI-algebras, Bull. Malays. Math. Sci. Soc. 32 (2009), 361–373. http://eudml.org/doc/45571.

[6] N. Nobusawa, On a generalization of the ring theory, Osaka J. Math. 1 (1964), 81–89. https://doi.org/10.18910/12354.

[7] M. Parvatham, Y. Bhargavi, Bipolar fuzzy Γ-semirings, AIP Conf. Proc. 2707 (2023), 020006. https://doi.org/10.1063/5.0143364.

[8] M. Parvatham, Y. Bhargavi, A. Iampan, Bipolar fuzzy ideals of Γ-semirings, Asia Pac. J. Math. 10 (2023), 38. https://doi.org/10.28924/APJM/10-38.

[9] M.M.K. Rao, Γ-semiring, Southeast Asian Bull. Math. 19 (1995), 49–54.

[10] H.S. Vandiver, Note on a simple type of algebra in which the cancellation law of addition does not hold, Bull. Amer. Math. Soc. 40 (1934), 914–920.

[11] L.A. Zadeh, Fuzzy sets, Inf. Control. 8 (1965), 338–353. https://doi.org/10.1016/s0019-9958(65)90241-x.

[12] W.R. Zhang, Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis, in: Proceedings of the First International Joint Conference of The North American Fuzzy Information Processing Society IEEE, San Antonio, TX, USA, 1994: pp. 305–309. https://doi.org/10.1109/IJCF.1994.375115.