Asia Pac. J. Math. 2024 **11**:70

ON FUZZY SOFT $-(\alpha, \beta) - n-$ NORMAL OPERATOR

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Received May 21, 2024

Abstract. This work defines an operator T operating on a Hilbert space as $fs - (\alpha, \beta) - n-$ Normal operator $0 \leq \alpha \leq 1 \leq \beta$ if $\alpha^2 T^*T^n \leq T^nT^* \leq \beta^2 T^*T^n$ for real number, α, β and give some properties and application of $fs - (\alpha, \beta) - n-$ Normal operator. We set the commutativity condition in order to prove that S_1S_2 and $(S+T)$ are a $fs - (\alpha, \beta) - n-$ Normal operator. We also added a condition for direct sum and direct multiplication to his proof $fs - (\alpha, \beta) - n-$ Normal operator.

So, We established several disparities between the $fs-$ Normal operator with its numerical radius of $fs - (\alpha, \beta) - n-$ Normal operator in Hilbert spaces.

2020 Mathematics Subject Classification. 47S40.

Key words and phrases. fuzzy soft Normal operator; fuzzy soft n-normal operator; $(\alpha, \beta)-$ normal operator; $(\alpha, \beta) - n-$ normal operator; fuzzy soft $-(\alpha, \beta) - n-$ normal operator.

1. Introduction

The fuzzy set theory, first developed by Zadeh [22], has been widely proposed in several prior research. Let X be represented by a fuzzy set whose function from X to $[0, 1]$ is well-defined. The resulting theory was named the theory of fuzzy sets. It quickly gained recognition as an extremely helpful method for solving situations involving ambiguity. Additionally, Molodsov created new sets in Form [17] that were known as "soft sets."

Subsequently, a number of researchers also presented new, refined ideas derived from soft sets. Since then, it has been used to tackle difficult problems in disciplines including computer science, engineering, and medicine. A soft set is a parameterized collection of a universal set. After that, the concept of a soft set was expanded to encompass a number of functional analytic mathematical concepts, giving rise to concepts like soft points. Furthermore, the study examined their characteristics, including soft points, of which instances can be found in [1], [20], [3], and [21] for soft-normed space, the soft inner

DOI: 10.28924/APJM/11-70

product space, and soft Hilbert spaces, respectively. Many searchers, like Maji [18], combined fuzzy and soft sets to define fuzzy soft sets. Terms such as fuzzy soft normed space in [12] and fuzzy soft point in [2] are provided by the study.
The fuzzy soft inner product space and fuzzy soft Hilbert space [13] were recently provided by Faried et al. There is also the fuzzy soft linear operator available [4]. In [5], the characteristics of the fuzzy soft self adjoint operator were examined.
S. Dawood and A. Qassim examined and talked about a novel kind of normal operator called the fuzzy soft normal operator. They also gave certain characteristics along with the operator's description. Furthermore, there have been provided theorems on the fuzzy soft normal operator. Additionally, in [4], they introduced the connection between this operator and other sorts. Some fuzzy, soft notions were introduced by S. Dawood and H. Khalid. The study examined a broader type of hyponormal worker known as the soft fuzzy soft $\mathcal{M}$ of hyponormal worker in [5]. They also discussed several important elements of this operator and a number of hypotheses pertaining to it. The hyponormal operator was characterized in fuzzy soft Hilbert space.

Therefore, Fuzzy soft n normal operators hence pique the interest of many research projects [14]. invested in a fuzzy soft ($n-\widetilde{N}$)- quasi normal operator, a type of soft quasi-normal operator, in [15]. Some generalizations of fuzzy soft $\left(\mathrm{k}^{\wedge *}-\hat{\mathrm{A}}\right)$-quasinormal operators in fuzzy soft Hilbert space were researched by Salim Dawood [10]. Further research on the (α, β)-Normal operator in Hilbert spaces is also done in [11]. Additionally, research In [19], m-Quasi-Totally - (α, β) Normal operators are discussed. Additionally, establish distinct inequalities in Hilbert spaces between the worker norm and the numerical radius of the $-(\alpha, \beta)$-Normal Ordinary operator [6].

Numerous research studies are fascinated by fuzzy soft n normal operators in [14]. Invested in a Salim Dawood studied Some generalizations of fuzzy soft $\left(\mathrm{k}^{\wedge *}-\hat{\mathrm{A}}\right)$-quasi-normal operators in fuzzy soft Hilbert space [10]. And they study more on (α, β)-Normal operator in Hilbert spaces in [11]. Also studied On m-Quasi-Totyally $-(\alpha, \beta)$-Normal operators in [19]. And Establish different inequities between worker norm and numerical radius of $-(\alpha, \beta)$-Normal Ordinary operator in Hilbert spaces in [6].

In this paper we define $fs-(\alpha, \beta)-n-$ Normal operator and give some properties and application of $fs-(\alpha, \beta)-n-$ Normal operator. We give condition commuting to prove product two operators and sum two operators that $fs-(\alpha, \beta)-n-$ Normal operators, and also we prove direct sum and direct multiplication.

Establish numerous disparities between the $fs-$ Normal operator with its numerical radius of $fs-(\alpha, \beta)-n-$ Normal operator on Hilbert spaces. For this objective, we use certain traditional disparities between Vectors of disease in a within product space.

2. Main Result

We define $fs-(\alpha,\beta)-n-Norml$ operator, and we have presented some theories to prove it $fs-(\alpha,\beta)-n-$ Normal operator on Hilbert spaces

Definition 2.1. *Let T be $fs-n-$ Normal operator Then T is called as $fs-(\alpha,\beta)-n-$ Normal operator if for real numbers, α,β with $0\le\alpha\le 1\le\beta$*

Then

$$\alpha^2 T^*T^n \le T^nT^* \le \beta^2 T^*T^n \tag{2.1}$$

Theorem 2.1. *If S_1, S_2 are double commuting $fs-(\alpha,\beta)-n-$ Normal operator , then S_1S_2 is an $fs-(\alpha,\beta)-n-$ Normal operator .*

Proof.

$$\begin{aligned}\alpha^2\left(S_1S_2\right)^*\left(S_1S_2\right)^n = \alpha^2\left(S_2^*S_1^*S_1^nS_2^n\right) &\le \frac{1}{\alpha^2}\left(S_1^nS_2^nS_2^*S_1^*\right)\\ &\le \frac{1}{\alpha^2}\left(S_1S_2\right)^n\left(S_1S_2\right)^*\end{aligned}$$

Then

$$\alpha^2\left(S_1S_2\right)^*\left(S_1S_2\right)^n \le \frac{1}{\alpha^2}\left(S_1S_2\right)^n\left(S_1S_2\right)^*$$

Thus

$$\alpha^4\left(S_1S_2\right)^*\left(S_1S_2\right)^n \le \left(S_1S_2\right)^n\left(S_1S_2\right)^* \tag{2.2}$$

$$\begin{aligned}\beta^2\left(S_1S_2\right)^*\left(S_1S_2\right)^n = \beta^2\left(S_2^*S_1^*S_1^nS_2^n\right) &\ge \frac{1}{\beta^2}\left(S_1^nS_2^nS_2^*S_1^*\right)\\ &\ge \frac{1}{\beta^2}\left(S_1S_2\right)^n\left(S_1S_2\right)^*\end{aligned}$$

Moreover

$$\beta^2\left(S_1S_2\right)^*\left(S_1S_2\right)^n \ge \frac{1}{\beta^2}\left(S_1S_2\right)^n\left(S_1S_2\right)^*$$

Therefore

$$\left(S_1S_2\right)^n\left(S_1S_2\right)^* \le \beta^2\left(S_1S_2\right)^*\left(S_1S_2\right)^n \tag{2.3}$$

From (2.2) (2.3) we get

$$\alpha^2\left(S_1S_2\right)^*\left(S_1S_2\right)^n \le \left(S_1S_2\right)^n\left(S_1S_2\right)^* \le \beta^2\left(S_1S_2\right)^*\left(S_1S_2\right)^n \tag{2.4}$$

So S_1S_2 is $fs-(\alpha,\beta)-n-$ Normal operator □

Proposition 2.2. *Let $\mathbb{T}, S$ is be a commuting $fs-(\alpha,\beta)-n-$ Normal operator such that $(S+T)^*$ commuting with $\sum_{k=1}^{n-1}\binom{n}{k}S^{n-k}T^k$ Then $(S+T)$ is an $fs-(\alpha,\beta)-n-$ Normal operator.*

Proof.

$$\alpha^2(S+T)^*(S+T)^n = \alpha^2\left((S^*+T^*)\sum_{k=0}^{n}\binom{n}{k}S^{n-k}T^k\right)$$
$$= \alpha^2\left((S^*S^n + (S+T)^*)\sum_{k=1}^{n-1}\binom{n}{k}S^{n-k}T^k + S^*T^n + \mathbb{T}^*S^n + \mathbb{T}^*\mathbb{T}^n\right)$$

□

And since $(S+T)^*$ is commuting with $\sum_{k=1}^{n-1}\binom{n}{k}S^{n-k}T^k$

$$\alpha^2(S+T)^*(S+T)^n \leq (S^n+T^n)(S+T)^* + \sum_{k=1}^{n-1}\binom{n}{k}S^{n-k}\mathbb{T}^k(S+T)^*$$
$$\leq \left(\sum_{k=0}^{n}\binom{n}{k}S^{n-k}T^k\right)(S+T)^*$$

Then

$$\alpha^2(S+T)^*(S+T)^n \leq (S+T)^n(S+T)^* \tag{2.5}$$
$$\beta^2\left((S+T)^*(S+T)^n\right) = \beta^2\left((S^*+T^*)\sum_{k=0}^{n}\binom{n}{k}S^{n-k}T^k\right)$$
$$= \beta^2\left(S^*S^n + (S+T)^*\right)\sum_{k=1}^{n-1}\binom{n}{k}S^{n-k}T^k + S^*T^n + T^*S^n + T^*T^n\Big)$$

And since $(S+T)^*$ is commuting with $\sum_{k=1}^{n-1}\binom{n}{k}S^{n-k}T^k$

$$\beta^2(S+T)^*(S+T)^n \geq \left(S^nS^* + \sum_{k=1}^{n-1}\binom{n}{k}S^{n-k}T^k(S+T)^* + T^nS^* + S^nT^* + T^nT^*\right)$$
$$\geq (S^n+T^n)(S+T)^* + \sum_{k=1}^{n-1}\binom{n}{k}S^{n-k}T^k(S+T)^*\Big)$$
$$\geq \left(\sum_{k=0}^{n}\binom{n}{k}S^{n-k}T^k\right)(S+T)^*$$

Then

$$\beta^2(S+T)^*(S+T)^n \geq (S+T)^n(S+T)^* \tag{2.6}$$

So

From (2.5),(2.6) we get

$$\alpha^2(S+T)^*(S+T)^n \leq (S+T)^n(S+T)^* \leq \beta^2(S+T)^*(S+T)^n$$

Thus, $(S+T)$ is an $fs-(\alpha,\beta)-n-$ Normal operator

Theorem 2.3. *Let* $T_1 \ldots\ldots T_m$ *to commute* $fs-(\alpha,\beta)-n-$ *Normal operator in* $\mathrm{B(H)}$ *then* $(T_1 \oplus \ldots\ldots \oplus T_m)$ *are* $fs-(\alpha,\beta)-n-$ *Normal operator.*

Proof.

$$\alpha^2 (T_1 \oplus \ldots \oplus T_m)^* (T_1 \oplus \ldots \oplus T_m)^n$$
$$= \alpha^2 (T_1^* T_1^n \oplus \ldots \oplus T_m^* T_m^n)$$
$$\leq (T_1 \oplus \ldots \oplus T_m)^n (T_1 \oplus \ldots \oplus T_m)^*$$

Then

$$\alpha^2 (T_1 \oplus \ldots \oplus T_m)^* (T_1 \oplus \ldots \oplus T_m)^n \leq (T_1 \oplus \ldots \oplus T_m)^n (T_1 \oplus \ldots \oplus T_m)^* \tag{2.7}$$

$$\beta^2 (T_1 \oplus \ldots \oplus T_m)^* (T_1 \oplus \ldots \oplus T_m)^n = \beta^2 (T_1^* T_1^n \oplus \ldots \oplus T_m^* T_m^n)$$
$$\geq (T_1 \oplus \ldots \oplus T_m)^n (T_1 \oplus \ldots \oplus T_m)^*$$

Thus

$$\beta^2 (T_1 \oplus \ldots \oplus T_m)^* (T_1 \oplus \ldots \oplus T_m)^n \geq (T_1 \oplus \ldots \oplus T_m)^n (T_1 \oplus \ldots \oplus T_m)^* \tag{2.8}$$

From (2.7) (2.8) we get

$$\alpha^2 (T_1 \oplus \ldots \oplus T_m)^* (T_1 \oplus \ldots \oplus T_m)^n$$
$$\leq (T_1 \oplus \ldots \oplus T_m)^n (T_1 \oplus \ldots \oplus T_m)^*$$
$$\leq \beta^2 (T_1 \oplus \ldots \oplus T_m)^* (T_1 \oplus \ldots \oplus T_m)^n$$

Hence $(T_1 \oplus \ldots \oplus T_m)$ is $fs - (\alpha, \beta) - n-$ Normal operator □

Theorem 2.4. *Let $T_1 \ldots\ldots T_m$ to commute $fs-(\alpha, \beta)-n-$ Normal operator in* $\mathrm{B(H)}$ *then $(T_1 \otimes \ldots\ldots\ldots \otimes T_m)$ are $fs - (\alpha, \beta) - n-$ Normal operator.*

Proof.

$$\alpha^2 (T_1 \otimes \ldots \otimes T_m)^* (T_1 \otimes \ldots \otimes T_m)^n = \alpha^2 (T_1^* T_1^n \otimes \ldots \otimes T_m^* T_m^n)$$
$$\leq (T_1^n T_1^* \otimes \ldots \otimes T_m^n T_m^*) \leq (T_1 \otimes \ldots \otimes T_m)^n (T \otimes \ldots \otimes T_m)^*$$

Then

$$\alpha^2 (T_1 \otimes \ldots \otimes T_m)^* (T_1 \otimes \ldots \otimes T_m)^n \leq (T_1 \otimes \ldots \otimes T_m)^n (T_1 \otimes \ldots \otimes T_m)^* \tag{2.9}$$

$$\beta^2 (T_1 \otimes \ldots \otimes T_m)^* (T_1 \otimes \ldots \otimes T_m)^n$$
$$= \beta^2 (T_1^* T_1^n \otimes \ldots \otimes T_m^* T_m^n) \geq (T_1^n T_1^* \otimes \ldots \otimes T_m^n T_m^*)$$
$$\geq (T_1 \otimes \ldots \otimes T_m)^n (T_1 \otimes \ldots \otimes T_m)^*$$

Then

$$\beta^2 (T_1 \otimes \ldots \otimes T_m)^* (T_1 \otimes \ldots \otimes T_m)^n \geq (T_1 \otimes \ldots \otimes T_m)^n (T_1 \otimes \ldots \otimes T_m)^* \tag{2.10}$$

From (2.9) (2.10) we get

$$\alpha^2 (T_1 \otimes \ldots \otimes T_m)^* (T_1 \otimes \ldots \otimes T_m)^n$$
$$\leq (T_1 \oplus \ldots \otimes T_m)^n (T_1 \otimes \ldots \otimes T_m)^*$$
$$\leq \beta^2 (T_1 \otimes \ldots \otimes T_m)^* (T_1 \otimes \ldots \otimes T_m)^n$$

Then $(T_1 \otimes \ldots . \otimes T_m)$ is $fs - (\alpha, \beta) - n-$ Normal operator □

Theorem 2.5. *Let T be $fs - (\alpha, \beta) - n-$ Normal operator such that T^{2m} is $fs - (\alpha, \beta) - n-$ Normal operator for every $n \in N$. Then, we have $\beta (T^* T^n) \leq r (T^* T^n) \leq (T^* T^n)$.*

Proof. By the definition of $fs - (\alpha, \beta) - n-$ Normal operator, we have

IF $m = 1$, then

$$\left(T^{*2m} T^{n^{2m}}\right) = (T^* T^* T^n T^n) \geq \frac{1}{\beta^2} (T^* T^n T^* T^n) \geq \frac{1}{\beta^2} (T^* T^n)^2$$

IF $\mathrm{m} = 2$, then

$$\left(T^{*^4} T^{n^4}\right) = (T^* T^* T^* T^* T^n T^n T^n T^n) \geq \frac{1}{\beta^2} (T^* T^* T^* T^n T^* T^n T^n T^n) \tag{2.11}$$
$$\geq \frac{1}{\beta^4} (T^* T^* T^n T^* T^* T^n T^n T^n) \geq \frac{1}{\beta^6} (T^* T^n T^* T^* T^* T^n T^n T^n) \tag{2.12}$$
$$\geq \frac{1}{\beta^8} (T^* T^n T^* T^* T^n T^* T^n T^n) \geq \frac{1}{\beta^{10}} (T^* T^n T^* T^n T^* T^* T^n T^n) \tag{2.13}$$
$$\geq \frac{1}{\beta^{12}} (T^* T^n T^* T^n T^* T^n T^* T^n) \tag{2.14}$$

Then

$$\left(T^{*^4} T^{n^4}\right) \geq \frac{1}{\beta^{12}} (T^* T^n)^4$$

Using induction on n, we infer

$$\left(T^{*^{2m}}, T^{n^{2m}}\right) \geq \frac{1}{\beta^{2m(2m-1)}} (T^* T^n)^{2m}$$

From which, we get

$$r (T^* T^n) = r (T^*) r (T^n) = \lim_{m \to \infty} \left[(T^*)^{2m} (T^n)^{2m}\right]^{\frac{1}{2m}}$$
$$\geq \lim_{m \to \infty} \left[\left(T^{*^{2m}} T^{n^{2m}}\right)\right]^{\frac{1}{2m}} \geq \lim_{m \to \infty} \left[\frac{1}{\beta^{2m(2m-1)}} (T^* T^n)^{2m}\right]^{\frac{1}{2m}}$$
$$\geq \frac{1}{\beta^{-1}} \lim_{m \to \infty} \left[\frac{1}{\beta^{2m-1}} (T^* T^n)^{2m}\right]^{\frac{1}{2m}} \geq \beta (T^* T^n)$$

Therefore, we get

$$\beta (T^* T^n) \leq r (T^* T^n) \leq (T^* T^n)$$

□

Theorem 2.6. *Take $T \in \mathrm{B(H)}$ be an $fs-(\alpha,\beta)-n-$ Normal operator, if $0 \le p \le 1$ or $p \ge 2$, Then we've*

$$\begin{aligned}&\left((T^*T^n+T^nT^*)^2+(T^*T^n-T^nT^*)^2\right)^p\\&\ge (T^*T^n)^{2p}\,\varphi(\alpha,p) \quad \textit{Where } \varphi(\alpha,p)=2^p\left(1+\alpha^{2p}\right)^2+\left(2^p-2^2\right)\alpha^{2p}\Big)\end{aligned} \tag{2.15}$$

Proof. We apply the following discrepancy [18, Theorem 8, page 551]

$$\left((a+b)^2+(a-b)^2\right)^p \ge 2^p\left((a)^p+(b)^p\right)^2+\left(2^p-2^2\right)\left((a)^p(b)^p\right) \tag{2.16}$$

In a Hilbert space, a and b are two vectors and $0 \le p \le 1$ or $p \ge 2$ Now, let's place $a = T^*T^nx$ and $b = T^nT^*x$ in (2.16)

Then we get

$$\begin{aligned}&\left((T^*T^nx+T^nT^*x)^2+(T^*T^nx-T^nT^*x)^2\right)^P\\&\ge 2^p\left((T^*T^nx)^p+(T^nT^*x)^p\right)^2+\left(2^p-2^2\right)(T^*T^nx)^p\,(T^nT^*x)^p\Big)\\&\ge 2^p\left((T^*T^nx)^{2p}\left(1+\alpha^{2p}\right)^2+\left(2^p-2^2\right)\alpha^{2p}\left((T^*T^nx)\right)^{2p}\right)\\&=2^p\left((T^*T^nx)^{2p}\left(\left(1+\alpha^{2p}\right)^2+\left(2^p-2^2\right)\alpha^{2p}\right)\right)\\&=(T^*T^n\chi)^{2p}\,\varphi(\alpha,p)\end{aligned} \tag{2.17}$$

Now ,

$$\left((T^*T^n+T^nT^*)^2+(T^*T^n-T^nT^*)^2\right)^p \ge \left((T^*T^n)\right)^{2p}\varphi(\alpha,p)$$

where $\varphi(\alpha,p)=2^p\left(1+\alpha^{2p}\right)^2+\left(2^p-2^2\right)\alpha^{2p}\Big)$ □

Theorem 2.7. *Assume that $T \in \mathrm{B(H)}$ be an $fs-(\alpha,\beta)-n-$ Normal operator, if $\mathcal{N}(T^n)=0$*
Additionally, Regarding any $x \in H$ with
We've got $\left(\frac{T^*T^nx}{(T^nT^*x)}-\frac{T^nT^*x}{(T^*T^nx)}\right) \le \rho$

Then, we have $\alpha^2\left((T^*T^n)\right)^2 \le \omega\left((T^nT^*)^*(T^nT^*)\right)+\frac{1}{2}\rho^2\beta^2(T^*T^nx)^2$

Proof. We apply as follows inverse of the Schwarz inequality:

$$(0 \le ab)-|\langle a,b\rangle| \le \left(ab-\mathrm{Re}\langle a,b\rangle \le \frac{1}{2}\rho^2(ab)\right) \tag{2.18}$$

Which is true for $a,b \in H/0$ and $\rho > 0$, with $\left(\frac{a}{(b)}-\frac{b}{(a)}\right) \le \rho$ (notice [9]).
We utilize $a = T^*T^nx$ and $b = T^nT^*x$ in (2.18) to get

$$\Big((T^*T^nx)\left((T^nT^*x)\right) \le |\langle T^*T^nx, T^nT^*x\rangle| + \frac{1}{2}\rho^2\left((T^*T^nx)(T^nT^*x)\right) \tag{2.19}$$

Thus, we obtain

$$\alpha^2\left((T^*T^nx)\right)^2 \le |\langle T^*T^nx, T^nT^*x\rangle| + \frac{1}{2}\rho^2\beta^2\left((T^*T^nx)\right)^2 \tag{2.20}$$

$$\le |\langle (T^nT^*)^*\left(T^nT^*\right)x, x\rangle| + \frac{1}{2}\rho^2\beta^2\left((T^*T^nx)\right)^2 \tag{2.21}$$

Now,

$$\alpha^2\left((T^*T^n)\right)^2 \le \omega\left((T^nT^*)^*\left(T^nT^*\right)\right) + \frac{1}{2}\rho^2\beta^2\left((T^*T^n)\right)^2$$

□

Theorem 2.8. *Assume that T is $fs-(\alpha,\beta)-n-Normal$ operator, then for every real s with $0 \le s \le 1$ we have*

$$\left[\left(\left(\frac{1-s}{\beta^2}\right)+s\right)\left((1-s)+\frac{s}{\beta^2}\right)\left((T^nT^*)\right)^4\right] \le$$
$$\left[(1-s)\left((T^*T^n)\right)^2 + s\beta^2\left((T^*T^n)\right)^2\right]$$
$$\left((T^nT^*-T^*T^n)\right)^2 + \omega\left((T^nT^*)^*\left(T^*T^n\right)\right)^2$$

Proof. Through [7], Theorem 2.5 (Refer to also [20], Theorem 2.3) We've

$$\left[(1-s)(a)^2+s(b)^2\right]\left[(1-s)(b)^2+s(a)^2\right] - |\langle a,b\rangle|^2$$
$$\le \left[(1-s)(a)^2+s(b)^2\right]\left[(1-s)(b)^2+s(a)^2\right] \tag{2.22}$$

Where $0 \le s \le 1, m \in R$ and $a, b \in H$. By taking $m=1, a = T^*T^nx$ and $b = T^nT^*x$ in (2.22), we have

$$\left[(1-s)\left((T^*T^nx)\right)^2 + s\left((T^nT^*x)\right)^2\right]\left[(1-s)\left((T^nT^*x)\right)^2 + s\left((T^*T^nx)\right)^2\right]|\langle T^*T^nx, T^nT^*x\rangle|^2$$
$$\le \left[(1-s)\left((T^*T^nx)\right)^2 + s\left((T^nT^*x)\right)^2\right]\left[(1-s)\left((T^nT^*x-T^*T^nx)\right)^2 +\right.$$
$$\left. s\left((T^nT^*x-T^*T^nx)\right)^2\right] \tag{2.23}$$

Thus, we have

$$\left[\left(\frac{1-s}{\beta^2}\right)\left((T^nT^*x)\right)^2 + s\left((T^nT^*x)\right)^2\right]\left[(1-s)\left((T^nT^*x)\right)^2 + \frac{s}{\beta^2}\left((T^nT^*x)\right)^2\right] -$$
$$|\langle (T^nT^*)^*T^*T^nx, x)|^2$$
$$\le \left[(1-s)\left((T^*T^nx)\right)^2 + s\left((T^nT^*x)\right)^2\right]$$
$$\left[(1-s)\left((T^nT^*x-T^*T^nx)\right)^2 + s\left((T^nT^*x-T^*T^nx)\right)^2\right]$$
$$\le \left[(1-s)\left((T^*T^nx)\right)^2 + s\beta^2\left((T^*T^nx)\right)^2\right]$$
$$\left((T^nT^*x-T^*T^nx)\right)^2 \tag{2.24}$$

Then

$$\left[\left(\left(\frac{1-s}{\beta^2}\right)+s\right)\left((1-s)+\frac{s}{\beta^2}\right)((T^nT^*x))^4\right]$$
$$\leq\left[(1-s)((T^*T^nx))^2+s\beta^2((T^*T^nx))^2\right]$$
$$((T^nT^*-T^*T^nx))^2+|\langle(T^nT^*)^*T^*T^nx,x\rangle|^2$$
$$\left[\left(\left(\frac{1-s}{\beta^2}\right)+s\right)\left((1-s)+\frac{s}{\beta^2}\right)(T^nT^*)^4\right]$$
$$\leq\left[(1-s)(T^*T^n)^2+s\beta^2(T^*T^n)^2\right]$$
$$(T^nT^*-T^*T^n)^2+\omega((T^nT^*)^*(T^*T^n))^2$$

□

3. Conclusion

We define the $fs-(\alpha,\beta)-n-$ Normal operator, give some of its characteristics, explain how to use it to prove certain theorems and properties, and show conditions under which the $fs-(\alpha,\beta)-n-$ Normal operator can be proven to have certain qualities.Furthermore, we establish a number of inequalities between the $fs-$ Normal operator and its numerical radius in Hilbert spaces, which is $fs-(\alpha,\beta)-n-$ Normal operator.

Authors' Contributions

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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