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POST-QUANTUM GENERALIZATION OF THE OPIAL'S TYPE INEQUALITIES

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Abstract. In this work, we obtain a (p, q)-analogue of Generalized Opial's integral inequalities. The main tool used in the study to obtain results was (p, q)-calculus, specifically, the concepts of (p, q)-differentiability, (p, q)-integrability and continuity of functions, convexity properties of functions, and among others. The analytical tools applied to prove the theorems established include the (p, q)-Hölder's and (p, q)-Cauchy-Schwarz's integral inequalities.

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1. Introduction

Opial established an inequality involving integral of a function and its derivative in [18] as

$$\int_0^h |f(t)f'(t)|dt \leq \frac{h}{4}\int_0^h (f'(t))^2 dt, \tag{1.1}$$

where $f \in C^1[0, h]$, such that $f(0) = f(h) = 0$, $f'(t) > 0$ and $t \in [0, h]$. The coefficient $h/4$ is the best constant possible.

This inequality, due to its significance, experienced a lot of extensions and generalizations over time in the classical field. See [5], [6], [8] and [26], among others.

The generalizations of the classical Opial's Inequality established in 1960 is presented in [22] as

i) If $f(a) = f(b) = 0$, then

$$\int_a^b \left|f(x)f'(x)\right| dx \leq \frac{3(b-a)}{16}\int_a^b \left|f'(x)\right|^2 dx. \tag{1.2}$$

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ii) If $f(a) = 0$ or $f(b) = 0$, then

$$\int_a^b \left|f(x)f'(x)\right| dx \leq \frac{3(b-a)}{8} \int_a^b \left|f'(x)\right|^2 dx. \tag{1.3}$$

where $f \in [a,b] \to \mathbf{R}$ is an absolutely continuous function such that $f > 0$. (p,q)-Calculus is a generalization of q-calculus. There has been a lot of development in the study of (p,q)-calculus. Recently, Sadjang [20] investigated on fundamental concepts of (p,q)-calculus. In [11], (p,q)-derivatives and (p,q)-integrals and their properties are also presented.

In [10], a (p,q)-analogue of a generalized Opial type inequality is established as

$$\int_0^b |\omega(px)||D_{p,q}\omega(x)|d_{p,q}x \leq \frac{b}{4}\int_0^b |(D_{p,q}\omega(x)|^2 d_{p,q}x. \tag{1.4}$$

where $\omega \in C[0,b]$ with $\omega(0) = \omega(b) = 0$ and $0 < q < p \leq 1$.

See also [1], [2], [3], [10], [15] and [23] for more analogues of the Opial's type inequalities.

The Opial inequality plays essential role in establishing the existence and uniqueness of initial and boundary values problems for both ordinary and partial differential equations [3] and [10].

The main purpose of this work is to establish (p,q)-analogue of the generalization of the Opial integral inequality in [22].

2. Methodology

The basic concepts of (p,q)-calculus employed in this work are presented in this section. The definitions provided can also be seen in [7], [11], [12], [13], [15], [16], [19] and [20].

Definition 2.1. [11] Let ϕ be a (p,q)-differentiable function, the (p,q)-derivative is defined as

$$D_{p,q}\phi(x) = \frac{\phi(px) - \phi(qx)}{(p-q)x}, \quad x \neq 0. \tag{2.1}$$

Definition 2.2. Let $\alpha > 0$, the (p,q)-bracket is defined as

$$[\alpha]_{p,q} = p^{\alpha-1} + p^{\alpha-2}q + \cdots + pq^{\alpha-2} + q^{\alpha-1} = \begin{cases} \frac{p^\alpha - q^\alpha}{p-q}, & (p \neq q \neq 1), \\ \frac{1-q^\alpha}{1-q}, & (p = 1), \\ \alpha, & (p = q = 1), \end{cases} \tag{2.2}$$

for $\quad 0 < q < p \leq 1 \quad \alpha \in \mathbf{R}$.

The (p,q)-Derivative of sum or difference of ϕ and ξ is defined as

$$D_{p,q}(\alpha\phi(x) \pm \beta\xi(x)) = \alpha D_{p,q}\phi(x) \pm \beta D_{p,q}\xi(x). \tag{2.3}$$

The (p,q)-Derivative of product of ϕ and ξ is defined as

$$\begin{aligned} D_{p,q}(\phi(x)\xi(x)) &= \xi(px)D_{p,q}\phi(x) + \phi(qx)D_{p,q}\xi(x) \\ &= \phi(px)D_{p,q}\xi(x) + \xi(qx)D_{p,q}\phi(x). \end{aligned} \tag{2.4}$$

The (p,q)-Derivative of a quotient of ϕ and ξ is defined as

$$\begin{aligned} D_{p,q}\left(\frac{\phi(x)}{\xi(x)}\right) &= \frac{\xi(px)D_{p,q}\phi(x) - \phi(px)D_{p,q}\xi(x)}{\xi(px)\xi(qx)} \\ &= \frac{\xi(qx)D_{p,q}\phi(x) - \phi(qx)D_{p,q}\xi(x)}{\xi(px)\xi(qx)}, \quad \xi(px)\xi(qx) \neq 0. \end{aligned} \tag{2.5}$$

Definition 2.3. [20] Let $\phi : [0,b] \to \mathbf{R}$ be a continuous function and $0 < q < p \leq 1$. The definite (p,q)-integral of the ϕ on $[0,b]$ is defined as

$$\int_0^b \phi(x)d_{p,q}x = (p-q)b\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}}\phi\left(\frac{q^j}{p^{j+1}}b\right). \tag{2.6}$$

If $a \in (0,b)$, then the definite (p,q)-integral of ϕ on $[a,b]$ is defined as

$$\int_a^b \phi(x)d_{p,q}x = \int_0^b \phi(x)d_{p,q}x - \int_0^a \phi(x)d_{p,q}x. \tag{2.7}$$

Remark 2.4. Taking $p = 1$, equation (2.6) reduces to the well known Jackson q-integral [14]

$$\int_0^b \phi(x)d_qx = (1-q)b\sum_{j=0}^{\infty} q^j\phi(bq^j). \tag{2.8}$$

Definition 2.5. Let $\phi \in C[a,b] \to \mathbf{R}$, if ϕ is an antiderivative of ϕ and $x \in [a,b]$. Then

$$D_{p,q}\int_a^x \phi(s)d_{p,q}s = \phi(x) \tag{2.9}$$

and

$$\int_a^x D_{p,q}\phi(s)d_{p,q}s = \phi(x) - \phi(a), \tag{2.10}$$

[20].

Definition 2.6. [16]The function ϕ defined on $[a,b]$ is called (p,q)-increasing or (p,q)-decreasing on $[a,b]$, if $\phi(qx) \leq \phi(px) \quad or \quad (\phi(qx) \geq \phi(px))$, for $qx, px \in [a,b]$.

It is easily observed that if the function ϕ is increasing (decreasing), then it is also (p,q)-increasing ((p,q)-decreasing).

Definition 2.7. [20](Fundamental Theorem of (p,q)-Calculus) If $\phi : C[a,b]$ and Φ is an antiderivative of ϕ and defined on $x \in [a,b]$, then we have

$$\Phi(x) = \int_a^x \phi(t)d_{p,q}t. \tag{2.11}$$

Definition 2.8. [25][(p,q)-Hö] Let $\phi, \xi \in I \to \mathbf{R}$, $0 < q > p \leq 1$, $\theta_1, \theta_2 > 1$ such that $\frac{1}{\theta_1} + \frac{1}{\theta_2} = 1$. Then

$$\int_I |\phi(z)\xi(z)| d_{p,q}z \leq \left(\int_I |\phi(z)|^{\theta_1} d_{p,q}z\right)^{\frac{1}{\theta_1}} \left(\int_I |\xi(z)|^{\theta_2} d_{p,q}z\right)^{\frac{1}{\theta_2}}. \tag{2.12}$$

holds.

Taking $\theta_1 = \theta_2 = 2$ then the (p,q)-Cauchy-Bunyakovsky-Schwarz's Integral Inequality is obtained.

Definition 2.9. [17] A function $\phi : I \to R$ is said to be convex if for every $x, y \in I$ and $0 \leq \lambda \leq 1$, the inequality

$$\phi(\lambda x + (1-\lambda)y) \leq \lambda\phi(x) + (1-\lambda)\phi(y) \tag{2.13}$$

holds.

The function ϕ is strictly convex, if for every $x \neq y \in I$ and $0 < \lambda < 1$ the inequality

$$\phi(\lambda x + (1-\lambda)y) < \lambda\phi(x) + (1-\lambda)\phi(y) \tag{2.14}$$

holds.

3. Results and Discussions

Lemma 3.1. *Let $\phi : [a,b] \to \mathbf{R}$ be a differentiable function, such that $D_{p,q}\phi \in L_2[a,b]$ and $0 < q < p \leq 1$. Then*

$$\int_a^b |\phi(px) D_{p,q}\phi(x)| d_{p,q}x \leq \frac{(b-a)}{2(p+q)} \int_a^b |D_{p,q}\phi(x)|^2 d_{p,q}x \tag{3.1}$$

holds. See prove in [1].

Theorem 3.2. *Let $\phi \in [c,d]$ be a (p,q)-differentiable function, such that $D_{p,q}\phi \in L_2[c,d]$ and $0 < q < p \leq 1$. Then*

$$\int_c^d |\phi(pz) D_{p,q}\phi(z)| d_{p,q}z \leq \frac{(2+pq)(d-c)}{4(p+q)^2} \int_c^d |D_{p,q}\phi(z)|^2 d_{p,q}z. \tag{3.2}$$

Proof. Let $[c, td+(1-t)c]$ and $[td+(1-t)c, d]$ be subintervals of $z \in [c,d]$.

Substituting $[c, td+(1-t)c]$ and $[td+(1-t)c, d]$ into inequality (3.1) for $z \in [c,d]$ and $t \in [0,1]$ yields

$$\int_c^{td+(1-t)c} |\phi(pz) D_{p,q}\phi(z)| d_{p,q}z \leq \frac{t(d-c)}{2(p+q)} \int_c^{td+(1-t)c} |D_{p,q}\phi(z)|^2 d_{p,q}z \tag{3.3}$$

and

$$\int_{td+(1-t)c}^d |\phi(pz) D_{p,q}(\phi(z)| d_{p,q}z \leq \frac{(1-t)(d-c)}{2(p+q)} \int_{td+(1-t)c}^d |D_{p,q}\phi(z)|^2 d_{p,q}z. \tag{3.4}$$

Let $z = vd + (1-v)c$, This implies $d_{p,q}z = (d-c)d_{p,q}v$

$$\int_c^{td+(1-t)c} |\phi(pz) D_{p,q}\phi(z)| d_{p,q}z \leq \frac{t(d-c)^2}{2(p+q)} \int_0^t |D_{p,q}\phi(vd+(1-v)c)|^2 d_{p,q}v \tag{3.5}$$

$$\int_{td+(1-t)c}^{d} |\phi(pz)D_{p,q}\phi(z)|d_{p,q}z \leq \frac{(1-t)(d-c)^2}{2(p+q)} \int_{t}^{1} |D_{p,q}\phi(vd+(1-v)c)|^2 d_{p,q}v. \tag{3.6}$$

Adding the inequalities (3.5) and (3.6) and q-integrating the right-hand side over $[0,1]$ with respect to t, we obtain

$$\begin{aligned}\int_{c}^{d} |\phi(pz)D_{p,q}\phi(z)|d_{p,q}z \leq \frac{(d-c)^2}{2(p+q)}\Bigg[&\int_{0}^{1}\int_{0}^{t} t|D_{p,q}\phi(vd+(1-v)c)|^2 d_{p,q}v d_{p,q}t + \\ &\int_{0}^{1}\int_{t}^{1} (1-t)|D_{p,q}\phi(vd+(1-v)c)|^2 d_{p,q}v d_{p,q}t\Bigg]\end{aligned} \tag{3.7}$$

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$$\begin{aligned}\int_{c}^{d} |\phi(pz)D_{p,q}\phi(z)|d_{p,q}z \leq \frac{(d-c)^2}{2(p+q)}\Bigg[&\int_{0}^{1}\int_{qu}^{1} t|D_{p,q}\phi(vd+(1-v)c)|^2 d_{p,q}t d_{p,q}v + \\ &\int_{0}^{1}\int_{0}^{qv} (1-t)|D_{p,q}\phi(vd+(1-v)c)|^2 d_{p,q}t d_{p,q}v\Bigg].\end{aligned} \tag{3.8}$$

Now

$$\begin{aligned}\int_{qv}^{1} t d_{p,q}t &= \int_{0}^{1} t d_{p,q}t - \int_{0}^{qv} t d_{p,q}t \\ &= (p-q)\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}}.\frac{q^j}{p^{j+1}} - (p-q)qv\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}}.\frac{q^j}{p^{j+1}}(qv) \\ &= (p-q)\sum_{j=0}^{\infty}\frac{q^{2j}}{p^{2(j+1)}} - (p-q)\sum_{j=0}^{\infty}\frac{q^{2j}}{p^{2(j+1)}}(qv)^2 \\ &= (p-q)\frac{1}{p^2}\frac{p^2}{p^2-q^2} - (p-q)\frac{1}{p^2}\frac{p^2(qv)^2}{p^2-q^2} \\ &= \frac{1}{p+q} - \frac{(qv)^2}{p+q} \\ &= \frac{1-(qv)^2}{p+q}.\end{aligned} \tag{3.9}$$

Also,

$$\begin{aligned}\int_{0}^{qv} (1-t)d_{p,q}t &= \int_{0}^{qv} 1 d_{p,q}t - \int_{0}^{qv} t d_{p,q}t \\ &= (p-q)qv\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}}.\left(\frac{q^j}{p^{j+1}}qv\right)^0 - (p-q)qv\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}}.\frac{q^j}{p^{j+1}}(qv) \\ &= (p-q)qv\sum_{j=0}^{\infty}\frac{q^j}{p^{j+}} - (p-q)\sum_{j=0}^{\infty}\frac{q^{2j}}{p^{2(j+1)}}(qv)^2\end{aligned}$$

$$= (p-q)\frac{1}{p}\frac{p(qv)}{p-q} - (p-q)\frac{1}{p^2}\frac{p^2(qv)^2}{p^2-q^2}$$
$$= qu - \frac{(qv)^2}{p+q}$$
$$= \frac{pqv + q^2v - (qv)^2}{p+q}. \tag{3.10}$$

Substituting (3.9) and (3.10) into (3.8) yields

$$\int_c^d |\phi(z)D_{p,q}\phi(z)|d_{p,q}z \leq \frac{(d-c)^2}{2(p+q)^2}\Bigg[\int_0^1 (1-(qv)^2)|D_{p,q}\phi(vd+(1-v)c)|^2 d_{p,q}v + \int_0^1 (pqv + q^2v - (qv)^2)|D_{p,q}\phi(vd+(1-v)c)|^2 d_{p,q}v\Bigg]. \tag{3.11}$$

Simplifying, we obtain

$$\int_c^d |\phi(pz)D_{p,q}\phi(z)|d_{p,q}z \leq \frac{(d-c)^2}{2(p+q)^2}\times \int_0^1 (1+pqv+q^2v-2(qv)^2)|D_{p,q}\phi(vd+(1-v)c)|^2 d_{p,q}v. \tag{3.12}$$

Let $z = vd + (1-v)c$. $d_{p,q}z = (d-c)d_{p,q}v$. Implying that $d_{p,q}v = \frac{1}{d-c}d_{p,q}z$ and $v = (z-c)/(d-c)$. It implies

$$\int_c^d |\phi(pz)D_{p,q}\phi(z)|d_{p,q}z \leq \frac{(d-c)}{2(p+q)^2}\times \int_c^d \left(1 + pq\left(\frac{z-c}{d-c}\right) + q^2\left(\frac{z-c}{d-c}\right) - 2q^2\left(\frac{z-c}{d-c}\right)^2\right)|D_{p,q}\phi(z)|^2 d_{p,q}z. \tag{3.13}$$

Let

$$L(z) = 1 + pq\left(\frac{z-c}{d-c}\right) + q^2\left(\frac{z-c}{d-c}\right) - 2q^2\left(\frac{z-c}{d-c}\right)^2$$

Since $z \in [c,d]$. Let $z = \frac{c+d}{2}$

Implying that

$$L\left(\frac{c+d}{2}\right) = 1 + pq\left(\frac{\frac{c+d}{2}-c}{d-c}\right) + q^2\left(\frac{\frac{c+d}{2}-c}{d-c}\right) - 2q^2\left(\frac{\frac{c+d}{2}-c}{d-c}\right)^2$$
$$= 1 + pq\frac{(d-c)}{2(d-c)} + q^2\frac{(d-c)}{2(d-c)} - 2q^2\frac{(d-c)^2}{4(d-c)^2}$$
$$= 1 + \frac{1}{2}pq + \frac{1}{2}q^2 - \frac{1}{2}q^2$$
$$= 1 + \frac{pq}{2}. \tag{3.14}$$

Substituting (3.14) into (3.13) yields

$$\int_c^d |\phi(pz)D_{p,q}\phi(z)|d_{p,q}z \leq \frac{(d-c)}{2(p+q)^2}\int_c^d \frac{2+pq}{2}|D_{p,q}\phi(z)|^2 d_{p,q}z. \tag{3.15}$$

This gives

$$\int_c^d |\phi(pz)D_{p,q}\phi(z)|d_{p,q}z \leq \frac{(2+pq)(d-c)}{4(p+q)^2}\int_c^d |D_{p,q}\phi(z)|^2 d_{p,q}z,$$

which complete the proof. □

Remark 3.3. In theorem 3.2, taking $p = 1$ and limit as $q \to 1$, we recapture (1.2) which is proved in [22].

Lemma 3.4. *Let $\phi : [a,b] \to \mathbf{R}$ be an differentiable function, such that for $(D_{p,q}\phi) \in L_2[a,b]$, $\phi(a) = \phi(b) = 0$, $0 < q < p \leq 1$. Then*

$$\int_a^b |\phi(px)D_{p,q}\phi(x)|d_{p,q}x \leq \frac{(b-a)}{p+q}\int_a^b |D_{p,q}\phi(x)|^2 d_{p,q}x \tag{3.16}$$

holds. See prove in [1].

Theorem 3.5. *Let $\phi \in [c,d]$ be a (p,q)-differentiable function, such that $D_{p,q}\phi \in L_2[c,d]$, $\phi(c) = 0 \quad (or \quad \phi(d) = 0)$ and $0 < q < p \leq 1$. Then*

$$\int_c^d |\phi(pz)D_{p,q}\phi(z)|d_{p,q}z \leq \frac{(2+pq)(d-c)}{2(p+q)^2}\int_c^d |D_{p,q}\phi(z)|^2 d_{p,q}z. \tag{3.17}$$

Proof. Let $[c, td+(1-t)c]$ and $[td+(1-t)c, d]$ be subintervals of $z \in [c,d]$.
Substituting $[c, td+(1-t)c]$ and $[td+(1-t)c, d]$ into inequality (3.4) for $z \in [c,d]$ and $t \in [0,1]$ yields

$$\int_c^{td+(1-t)c} |\phi(pz)D_{p,q}\phi(z)|d_{p,q}z \leq \frac{t(d-c)}{(p+q)}\int_c^{td+(1-t)c} |D_{p,q}\phi(z)|^2 d_{p,q}z \tag{3.18}$$

and

$$\int_{td+(1-t)c}^d |\phi(pz)D_{p,q}\phi(z)|d_{p,q}z \leq \frac{(1-t)(d-c)}{(p+q)}\int_{td+(1-t)c}^d |D_{p,q}\phi(z)|^2 d_{p,q}z. \tag{3.19}$$

Let $z = vd + (1-v)c$, then $d_{p,q}z = (d-c)d_{p,q}v$.

$$\int_c^{td+(1-t)c} |\phi(pz)D_{p,q}\phi(z)|d_{p,q}z \leq \frac{t(d-c)^2}{(p+q)}\int_0^t |D_{p,q}\phi(vd+(1-v)c)|^2 d_{p,q}v \tag{3.20}$$

$$\int_{td+(1-t)c}^d |\phi(pz)D_{p,q}\phi(z)|d_{p,q}z \leq \frac{(1-t)(d-c)^2}{(p+q)}\int_t^1 |D_{p,q}\phi(vd+(1-v)c)|^2 d_{p,q}v. \tag{3.21}$$

Adding the inequalities (3.20) and (3.21) and q-integrating the right-hand side over $[0,1]$ with respect to t, we obtain

$$\begin{aligned}\int_c^d |\phi(pz)D_{p,q}\phi(z)|d_{p,q}z \leq \frac{(d-c)^2}{(p+q)}\Bigg[&\int_0^1\int_0^t t|D_{p,q}\phi(vd+(1-v)c)|^2 d_{p,q}v d_{p,q}t + \\ &\int_0^1\int_t^1 (1-t)|D_{p,q}\phi(vd+(1-v)c)|^2 d_{p,q}v d_{p,q}t\Bigg]\end{aligned} \tag{3.22}$$

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$$\int_c^d |\phi(pz)D_{p,q}\phi(z)|d_{p,q}z \le \frac{(d-c)^2}{(p+q)}\Bigg[\int_0^1\int_{qu}^1 t|D_{p,q}\phi(vd+(1-v)c)|^2 d_{p,q}t d_{p,q}v + \int_0^1\int_0^{qv}(1-t)|D_{p,q}\phi(vd+(1-v)c)|^2 d_{p,q}t d_{p,q}v\Bigg]. \tag{3.23}$$

Now

$$\begin{aligned}\int_{qv}^1 t d_{p,q}t &= \int_0^1 t d_{p,q}t - \int_0^{qv} t d_{p,q}t\\ &= (p-q)\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}}.\frac{q^j}{p^{j+1}} - (p-q)qv\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}}.\frac{q^j}{p^{j+1}}(qv)\\ &= (p-q)\sum_{j=0}^{\infty}\frac{q^{2j}}{p^{2(j+1)}} - (p-q)\sum_{j=0}^{\infty}\frac{q^{2j}}{p^{2(j+1)}}(qv)^2\\ &= (p-q)\frac{1}{p^2}\frac{p^2}{p^2-q^2} - (p-q)\frac{1}{p^2}\frac{p^2(qv)^2}{p^2-q^2}\\ &= \frac{1}{p+q} - \frac{(qv)^2}{p+q}\\ &= \frac{1-(qv)^2}{p+q}.\end{aligned} \tag{3.24}$$

Also,

$$\begin{aligned}\int_0^{qv}(1-t)d_{p,q}t &= \int_0^{qv} 1 d_{p,q}t - \int_0^{qv} t d_{p,q}t\\ &= (p-q)qv\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}}.\left(\frac{q^j}{p^{j+1}}qv\right)^0 - (p-q)qv\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}}.\frac{q^j}{p^{j+1}}(qv)\\ &= (p-q)qv\sum_{j=0}^{\infty}\frac{q^j}{p^{j+}} - (p-q)\sum_{j=0}^{\infty}\frac{q^{2j}}{p^{2(j+1)}}(qv)^2\\ &= (p-q)\frac{1}{p}\frac{p(qv)}{p-q} - (p-q)\frac{1}{p^2}\frac{p^2(qv)^2}{p^2-q^2}\\ &= qu - \frac{(qv)^2}{p+q}\\ &= \frac{pqv+q^2v-(qv)^2}{p+q}.\end{aligned} \tag{3.25}$$

Substituting (3.24) and (3.25) into (3.23) yields

$$\int_a^b |\phi(z)D_{p,q}\phi(z)|d_{p,q}z \le \frac{(b-a)^2}{(p+q)^2}\Bigg[\int_0^1(1-(qv)^2)|D_{p,q}\phi(vb+(1-v)a)|^2 d_{p,q}v + \int_0^1(pqv+q^2v-(qv)^2)|D_{p,q}\phi(vb+(1-v)a)|^2 d_{p,q}v\Bigg]. \tag{3.26}$$

Simplifying, we obtain

$$\int_a^b |\phi(pz)D_{p,q}\phi(z)|d_{p,q}z \leq \frac{(b-a)^2}{(p+q)^2}\times \int_0^1 (1+pqv+q^2v-2(qv)^2)|D_{p,q}\phi(vb+(1-v)a)|^2 d_{p,q}v. \tag{3.27}$$

Let $z = vb + (1-v)a$. $d_{p,q}z = (b-a)d_{p,q}v$. Implying that $d_{p,q}v = \frac{1}{b-a}d_{p,q}z$ and $v = (z-a)/(b-a)$. It implies

$$\int_a^b |\phi(pz)D_{p,q}\phi(z)|d_{p,q}z \leq \frac{(b-a)}{(p+q)^2}\times \int_a^b \left(1+pq\left(\frac{z-a}{b-a}\right)+q^2\left(\frac{z-a}{b-a}\right)-2q^2\left(\frac{z-a}{b-a}\right)^2\right)|D_{p,q}\phi(z)|^2 d_{p,q}z. \tag{3.28}$$

Let

$$L(z) = 1+pq\left(\frac{z-a}{b-a}\right)+q^2\left(\frac{z-a}{b-a}\right)-2q^2\left(\frac{z-a}{b-a}\right)^2$$

Since $z \in [a,b]$. Let $z = \frac{a+b}{2}$

Implying that

$$\begin{aligned} L\left(\frac{a+b}{2}\right) &= 1+pq\left(\frac{\frac{a+b}{2}-a}{b-a}\right)+q^2\left(\frac{\frac{a+b}{2}-a}{b-a}\right)-2q^2\left(\frac{\frac{a+b}{2}-a}{b-a}\right)^2 \\ &= 1+pq\frac{(b-a)}{2(b-a)}+q^2\frac{(b-a)}{2(b-a)}-2q^2\frac{(b-a)^2}{4(b-a)^2} \\ &= 1+\frac{1}{2}pq+\frac{1}{2}q^2-\frac{1}{2}q^2 \\ &= 1+\frac{pq}{2}. \end{aligned} \tag{3.29}$$

Substituting (3.29) into (3.28) yields

$$\int_c^d |\phi(pz)D_{p,q}\phi(z)|d_{p,q}z \leq \frac{(d-c)}{(p+q)^2}\int_c^d \frac{2+pq}{2}|D_{p,q}\phi(z)|^2 d_{p,q}z. \tag{3.30}$$

This gives

$$\int_c^d |\phi(pz)D_{p,q}\phi(z)|d_{p,q}z \leq \frac{(2+pq)(d-c)}{2(p+q)^2}\int_c^d |D_{p,q}\phi(z)|^2 d_{p,q}z.$$

This completes the proof. □

Conclusion

In this work, (p,q)-analogues of a generalized Opial's Inequality is established. The basic definitions of (p,q)-calculus and the principles of convex functions were employed to obtain the results. The Opial type integral inequalities have a great interest in their own right and also have important applications in ordinary differential equations, number theory, Quantum Theory, Quantum Computing and among others. (p,q)-Cauchy-Schwartz and (p,q)-Hölder's integral inequalities were also applied to prove the theorems. It is hoped that these results will be very useful to the mathematics community.

Authors' Contributions

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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