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RECOGNITION OF N-FREE POSETS BY USING THE POSET MATRIX

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Abstract. We give a matrix method for the recognition of N-free posets. Since the exact enumeration of N-free posets depends mainly on the constructions of the pairwise nonisomorphic connected posets, several methods for the recognition of N-free posets were considered in the literature. But providing an efficient recognition method for the N-free posets is still an open problem. Here, we introduce the notion of quasi-ordinal sum of poset matrices. We show that the quasi-ordinal sum of poset matrices represents the quasi-ordinal sum of posets. Consequently, this result gives a matrix recognition of the N-free posets.

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1. Introduction

A poset (partially ordered set) is said to be N-*free* if its Hasse diagram does not contain exactly the N-shaped (4-element zigzag) poset as a sub-diagram. The recognition and enumeration of N-free posets have significant importances in the combinatorics of posets, because they play a vital role in the theory of decompositions and linear extensions of posets. The class of series-parallel posets is closed under the direct sum and ordinal sum of posets, that is, any series-parallel poset can be constructed from the singleton poset by using only the direct sum and ordinal sum. Habib and Jegou [5] showed that the class of N-free posets is closed under the direct sum and quasi-ordinal sum of posets. Thus, like series-parallel posets, any N-free poset can be constructed from the singleton poset by using only the direct sum and quasi-ordinal sum. Since the quasi-ordinal sum of posets generalizes the ordinal sum of posets, the class of N-free posets gives a generalization of the class of series-parallel posets. As a result, the class of N-free posets preserves most of the computational tractability properties hold for series-parallel posets. Therefore, several methods for the recognition and enumeration of

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the N-free posets were considered by numerous authors, see [1,4,8,13]. Until today, the numbers of unlabeled (pair-wise nonisomorphic) connected N-free posets are known up to 14 elements, see [8], and disconnected N-free posets are known up to 15 elements, see [13]. See also the integer sequences $A202180$, $A202182$, $A349367$, and $A350783$ in OEIS by Sloane and Plouffe [16].

On the other hand, several incidence matrices were chosen repeatedly in recognizing various classes of posets and graphs, see [2,11,15]. Since the incidence matrices have many computational aspects and hence have classical applications in the adjacent areas, special operations on the incidence matrices were considered in the literature, see [9,10,12]. Mohammad et al. [11,12,14] introduced the notions of ordinal sum, ordinal product, and a composition of square matrices, and described the interpretations of these matrix operations in the case of poset matrices. Recently, poset matrices have gained much attention in the recognitions of several classes of posets and graphs, see [2,12,14]. In this paper, we introduce the notion of quasi-ordinal sum of poset matrices and give an interpretation of this matrix operation in the algebra of posets. Mohammad et al. [11] defined the properties of block of 0s, block of 1s, and complete block of 1s in a poset matrix, and obtained the matrix recognitions of the P-graphs, P-series, and series-parallel posets. Also, Mohammad et al. [12,14] defined the properties of transitive blocks of 1s and transitive blocks of poset matrices in a block poset matrix, and obtained the matrix recognitions of the composite posets, factorable posets, and more generally, decomposable posets. These classical results motivated us to define the property of block of quasi-transitive 1s in a poset matrix, and to obtain the matrix recognition of N-free posets.

In Section 2, we recall some basic terminologies related to the N-free posets and describe the quasi-ordinal sum of the posets. We also recall the definition of poset matrix and briefly describe its association to posets. In Section 3, we define the quasi-ordinal sum of poset matrices and give its interpretation in posets. We mainly show that the quasi-ordinal sum of poset matrices is also a poset matrix, and it represents the poset obtained by the quasi-ordinal sum of the posets represented by the corresponding poset matrices. Here, we also show by using the poset matrix that the class of N-free posets gives a generalization of the class of series-parallel posets. In Section 4, we define the property of block of quasi-transitive 1s in a poset matrix and give the matrix recognition of the N-free posets. We show that a poset matrix represents an N-free poset if and only if it satisfies the block of quasi-transitive 1s property.

2. Preliminaries

A *poset* is a structure $\mathbf{S} = \langle S, \leqslant_S \rangle$ consisting of the nonempty set S with the order relation $\leqslant_S$ on S. For $x, y \in S$, we say that the elements x and y are *incomparable* in $\mathbf{S}$ and we write $x||y$ if neither $x \leqslant_S y$ nor $y \leqslant_S x$. Also, for $x, y \in S$, we say that the element x is *covered by* y (equivalently, y *covers* x) in $\mathbf{S}$ and we write $x \prec y$ if $x \leqslant_S y$ and for some $z \in S$, $x \leqslant_S z \leqslant_S y$ implies either $x = z$ or $z = y$. A poset $\mathbf{S}$

is said to be *finite* if the underlying set S is finite. From now on, every poset is assumed to be finite. Throughout the paper, we use the notations $\mathbf{1}$ for the singleton poset, $\mathbf{C}_n$ $(n \geq 2)$ for the n-element chain posets, $\mathbf{I}_n$ $(n \geq 2)$ for the n-element antichain posets, $\mathbf{Z}_n$ $(n \geq 4)$ for the n-element zigzag posets, and $\mathbf{B}_{m,n}$ $(m \geq 1, n \geq 1)$ for the complete bipartite posets with m minimal elements and n maximal elements. For any posets $\mathbf{R}$ and $\mathbf{S}$, we write $\mathbf{R} + \mathbf{S}$ and $\mathbf{R} \oplus \mathbf{S}$ to denote the direct sum and ordinal sum, respectively. Also, we write briefly $n\mathbf{R}$ for the n-times direct sum $\mathbf{R} + \mathbf{R} + \cdots + \mathbf{R}$ and $\oplus^n \mathbf{R}$ for the n-times ordinal sum $\mathbf{R} \oplus \mathbf{R} \oplus \cdots \oplus \mathbf{R}$. In general, for any posets $\mathbf{S}_i, 1 \leq i \leq n$, we write briefly $\sum_{i=1}^{n} \mathbf{S}_i$ for $\mathbf{S}_1 + \mathbf{S}_2 + \cdots + \mathbf{S}_n$ and $\bigoplus_{i=1}^{n} \mathbf{S}_i$ for $\mathbf{S}_1 \oplus \mathbf{S}_2 \oplus \cdots \oplus \mathbf{S}_n$. A poset is said to be *series-parallel* if it can be decomposed into singletons by using only the direct sum and ordinal sum of posets. All the posets up to 4 elements except $\mathbf{Z}_4$ (the 4-element zigzag poset shown in Figure 1) are series-parallel.

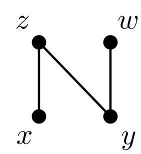

FIGURE 1. Hasse diagram of $\mathbf{Z}_4$, the 4-element zigzag or N-shaped poset.

We also write $\mathbf{R} \cong \mathbf{S}$ whenever the posets $\mathbf{R}$ and $\mathbf{S}$ are order isomorphic. For further basics of posets, we refer the readers to the classical book by Davey and Priestley [3].

2.1. N**-free posets.** A poset is called N*-free* if it does not contain any subset $\{x, y, z, w\}$ such that $x \prec z$, $y \prec z$, $y \prec w$, $x||y$, $x||w$, and $z||w$. In other words, a poset is said to be N*-free* if its Hasse diagram does not contain exactly the N-shaped poset (the 4-element zigzag poset shown in Figure 1) as a sub-diagram. Every series-parallel poset is N-free. The five-element poset, as shown in Figure 2, with the underlying set $\{a, b, c, d, e\}$ and the relations $a \prec d$, $b \prec c$, $b \prec e$, $c \prec d$, $a||b$, $a||c$, $a||e$, $c||e$, and $d||e$, is the least-element N-free poset that is not series-parallel.

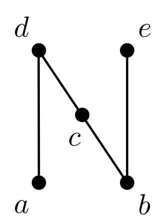

FIGURE 2. Hasse diagram of the least-element N-free poset that is not series-parallel.

The notion of the *quasi-ordinal sum* (*quasi-series composition*) was introduced by Habib and Jegou [5]. Later on, Habib and Möhring [6] studied the complexity properties of some classes of posets induced by the quasi-ordinal sum of posets. We recall the definition of quasi-ordinal sum of posets in the following.

Let $\mathbf{R} = \langle R, \leqslant_R \rangle$ and $\mathbf{S} = \langle S, \leqslant_S \rangle$ be two posets with $R \cap S = \emptyset$. Also, let $A \subseteq \max(\mathbf{R})$ and $B \subseteq \min(\mathbf{S})$ such that $A \neq \emptyset$ and $B \neq \emptyset$. Then, the *quasi-ordinal sum* of the posets $\mathbf{R}$ by A and $\mathbf{S}$ by B, denoted by $(\mathbf{R}, A) \ominus (\mathbf{S}, B)$, is defined to be the poset $\langle R \cup S, \leqslant_\ominus \rangle$ such that for $r, s \in P \cup Q$, we have $r \leqslant_\ominus s$ if and only if one of the following conditions holds.

(1) $r, s \in R$ and $r \leqslant_R s$,

(2) $r, s \in S$ and $r \leqslant_S s$,

(3) $r \in R$, $s \in S$, and there exist $a \in A$, $b \in B$ such that $r \leqslant_R a$ and $b \leqslant_S s$.

For example, all the pairwise nonisomorphic quasi-ordinal sums of the posets $\mathbf{B}_{1,2}$ and $\mathbf{B}_{2,1}$ are the posets $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{S}$ that are shown in Figure 3 by their Hasse diagrams.

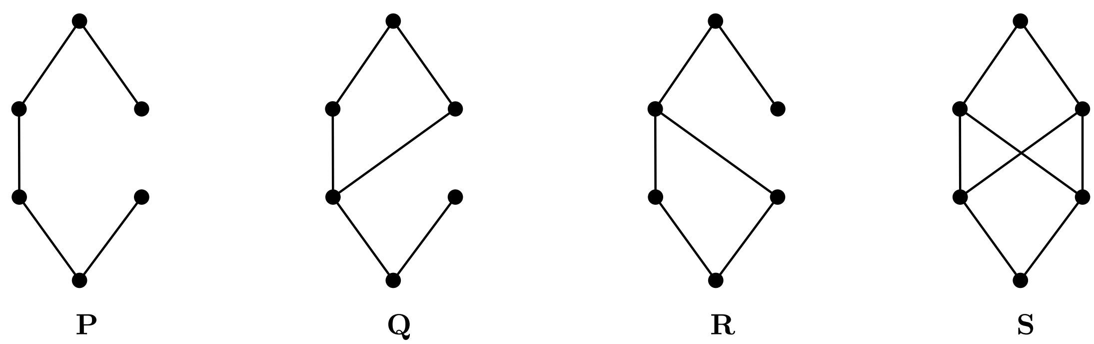

FIGURE 3. Hasse diagrams of all the pairwise nonisomorphic quasi-ordinal sums of the posets $\mathbf{B}_{1,2}$ and $\mathbf{B}_{2,1}$.

Habib and Jegou [5] showed that the class of N-free posets is closed under the direct sum and quasi-ordinal sum of posets including the singleton poset. The authors obtained the following result.

Theorem 2.1. [5] *A poset is N-free if and only if it can be decomposed into the singletons by using only the direct sum and quasi-ordinal sum of posets.*

The above result shows that every direct term and quasi-ordinal term of an N-free poset is N-free. Note that when $A = \max(\mathbf{R})$ and $B = \min(\mathbf{S})$ in the definition of quasi-ordinal sum of posets, then this definition agrees the definition of usual ordinal sum of posets. Since the class of series-parallel posets is closed under the direct sum and ordinal sum of posets including the singleton poset, the class of N-free posets gives a natural generalization of the class of series-parallel posets.

2.2. **Poset matrix.** Throughout this paper, we use the notations $M_{m,n}$ to denote an m-by-n matrix and M_m to denote a square matrix of order m. We use particularly the notations I_n for the n-th order identity matrix, $O_{m,n}$ for the matrix with all entries 1s, and $Z_{m,n}$ for the matrix with all entries 0s. Also, we use the notation C_n to denote the matrix $[c_{ij}]$, $1 \leq i,j \leq n$, defined as $c_{ij} = 1$ for all $i \leq j$ and $c_{ij} = 0$, otherwise. Note here that $I_1 = C_1 = 1$.

Mohammad and Talukder [11] introduced the notion of poset matrix. An upper triangular $(0,1)$-matrix $M_n = [a_{ij}], 1 \leq i,j \leq n$, with all entries 1s in the main diagonal is called a *poset matrix* if and only if M_n is transitive, that is, $a_{ij} = 1$ and $a_{jk} = 1$ imply $a_{ik} = 1$. For every $n \geq 1$, both the matrices I_n and C_n, as defined above, are trivially poset matrices. The matrices R and S in the following example are two nontrivial poset matrices.

Example 2.1.

$$R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Let $M_n = [a_{ij}]$, $1 \leq i,j \leq n$, be any poset matrix. A poset $\mathbf{R} = \langle R, \leqslant \rangle$, where $R = \{x_1, x_2, \ldots, x_n\}$ and x_i corresponds the i-th row (or column) of M_n, is associated to M_n by defining the order relation $\leqslant$ on R such that for all $1 \leq i,j \leq n$, we have $x_i \leqslant x_j$ if and only if $a_{ij} = 1$. Then we say that the poset matrix M_n *represents* the poset $\mathbf{R}$ and vice versa. Trivially, both the poset matrices I_1 and C_1 represent the singleton poset $\mathbf{1}$. The poset matrices R and S, as given in Example 2.1, represent the posets $\mathbf{B}_{1,2}$ and $\mathbf{B}_{2,1}$, respectively. Also, for every $n \geq 2$, the poset matrices I_n and C_n represent the posets $\mathbf{I}_n$ and $\mathbf{C}_n$, respectively. For some $1 \leq i,j \leq n$, interchanges of i-th and j-th rows along with the interchanges of i-th and j-th columns in a poset matrix M_n is called the *(i,j)-relabeling* of M_n. The following result gives an interpretation of the relabeling of poset matrices in posets.

Theorem 2.2. [11] *Any relabeling of a poset matrix is a poset matrix, and it represents the same poset up to isomorphism.*

We now recall the interpretations of direct sum and ordinal sum of matrices in the case of poset matrices. The *direct sum* and *ordinal sum* of the matrices $M_{m,r}$ and $N_{n,s}$ are denoted by $M_{m,r} \oplus N_{n,s}$ and $M_{m,r} \boxplus N_{n,s}$, respectively, and defined as the $(m+n)$-by-$(r+s)$ block matrices as follows:

$$M_{m,r} \oplus N_{n,s} = \begin{bmatrix} M_{m,r} & | & Z_{m,s} \\ -- & . & -- \\ Z_{n,r} & | & N_{n,s} \end{bmatrix} \text{ and } M_{m,r} \boxplus N_{n,s} = \begin{bmatrix} M_{m,r} & | & O_{m,s} \\ -- & . & -- \\ Z_{n,r} & | & N_{n,s} \end{bmatrix}.$$

Here, the terms $M_{m,r}$ and $N_{n,s}$ are called the *direct terms* of the sum $M_{m,r} \oplus N_{n,s}$ and *ordinal terms* of the sum $M_{m,r} \boxplus N_{n,s}$. The notion of the aforesaid ordinal sum of matrices was introduced by Mohammad and Talukder [11]. The authors then gave the interpretations of the direct sum and ordinal sum of poset matrices in posets. They obtained the following results.

Theorem 2.3. [11] *Let M_m represent the poset $\mathbf{R}$ and N_n represent the poset $\mathbf{S}$. Then the matrix $M_m \oplus N_n$ is a poset matrix and it represents the poset $\mathbf{R} + \mathbf{S}$.*

Theorem 2.4. [11] *Let M_m represent the poset $\mathbf{R}$ and N_n represent the poset $\mathbf{S}$. Then the matrix $M_m \boxplus N_n$ is a poset matrix and it represents the poset $\mathbf{R} \oplus \mathbf{S}$.*

3. Quasi-ordinal Sum of Poset Matrices

In this section, we introduce the notion of *quasi-ordinal sum* of poset matrices and give its interpretation in posets. This result gives a generalization of the aforementioned result (Theorem 2.4) obtained for the ordinal sum of poset matrices. In the following, we begin with defining the *diagonal submatrix* of a square matrix.

Definition 3.1. A matrix $M_{p:q} = [b_{ij}]$, $1 \leq i, j \leq q - p + 1$, is said to be a *diagonal submatrix* of the matrix $M_n = [a_{ij}]$, $1 \leq i, j \leq n$, where $1 \leq p \leq q \leq n$, if and only if $b_{ij} = a_{(p+i-1)(p+j-1)}$ for all $1 \leq i, j \leq q - p + 1$.

Note that from now on, by a submatrix we mean a diagonal submatrix. Note also that by a poset matrix we mean a poset matrix in upper triangular form. We now define the *top antichain form* and *bottom antichain form* of poset matrices.

Definition 3.2. A poset matrix M_n is said to be in *top* (analogously, *bottom*) *antichain form* of length r, where $r \leq n$, if and only if the submatrix $M_{1:r}$ is equal to I_r (analogously, $M_{n-r+1:n} = I_r$) and for every relabeling of M_n, the equality $M_{1:s} = I_s$ (analogously, $M_{m-s+1:m} = I_s$) for some $s \leq n$ implies $s \leq r$.

Example 3.1. In the following, the poset matrix A is in top antichain form and A' is in bottom antichain form of the same length 3.

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A' = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Trivially, for any labeling, the poset matrices I_n and C_n are in both top and bottom antichain forms of lengths n and 1, respectively. We see that the following poset matrix B is in top antichain form of length 1, and it is not in bottom antichain form because $B_{5:5} = I_1$ but $\bar{B}_{4:5} = I_2$, where $\bar{B}$ is obtained by taking the $(3,4)$-relabeling of B.

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{(3,4)-relabeling}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \bar{B}$$

On the other hand, in the case of the poset matrix $\bar{B}$, since $\bar{B}_{4:5} = I_2$, and for any $\tilde{B}$ that can be obtained by some relabeling of B, the equality $\tilde{B}_{5-s+1:5} = I_s$ for some $s \leq 5$ implies $s \leq 2$. This shows that the poset matrix $\bar{B}$ is in bottom antichain form of length 2.

Note that if a poset matrix M_n is in the top (analogously, bottom) antichain form of length r, then the submatrix $M_{1:r}$ (analogously, $M_{n-r+1:n}$) of M_n represents the subposet that consists of exactly the minimal (analogously, maximal) elements of the poset represented by M_n. To define the *quasi-ordinal sum* of poset matrices, we need one more definition namely the *quasi-transitive closure* for two poset matrices as given in the following.

Definition 3.3. Let the poset matrix M_m be in bottom antichain form of length p and N_n be in top antichain form of length q. Then an m-by-n $(0,1)$-matrix $T_{m,n}$ is said to be a *quasi-transitive closure* of the matrices M_m and N_n if and only if the following conditions hold.

(1) There exist $p < p_1 < p_2 < \cdots < p_r \leq m$ and $1 \leq q_1 < q_2 < \cdots < q_s \leq q$, such that $T(p_i, q_j) = 1$ for all $1 \leq i \leq r$, $1 \leq j \leq s$.

(2) For every $1 \leq k \leq p$ and $1 \leq i \leq r$, by (1), $M(k, p_i) = 1$ implies $T(k, q_j) = 1$ for all $1 \leq j \leq s$.

(3) For every $1 \leq j \leq s$ and $q < t \leq n$, by (1), $N(q_j, t) = 1$ implies $T(p_i, t) = 1$ for all $1 \leq i \leq r$, and by (2), $N(q_j, t) = 1$ implies $T(k, t) = 1$ for all $1 \leq k \leq p$.

Example 3.2. Consider the poset matrices R and S from Example 2.1. Here, the matrices $R = 1 \boxplus I_2$ and $S = I_2 \boxplus 1$ are respectively in bottom antichain form and top antichain form of the same length 2.

$$R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

We see that any one of the following matrices T, $\bar{T}$, $\tilde{T}$, and $T^{'}$, can be a quasi-transitive closure of the matrices R and S.

$$T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad \bar{T} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad \tilde{T} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad T^{'} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

In the case of the matrix T, we have $p_1 = 3$ and $q_1 = 1$. Then for $T(p_1, q_1) = 1$,

$$R(1, p_1) = 1 \text{ implies } T(1, q_1) = 1,$$

$$S(q_1, 3) = 1 \text{ implies } T(p_1, 3) = 1, \text{ and then}$$

$$T(1, q_1) = S(q_1, 3) = 1 \text{ implies } T(1, 3) = 1.$$

In the case of $\bar{T}$, we have $p_1 = 3$, $q_1 = 1$, and $q_2 = 2$. Then for $\bar{T}(p_1, q_1) = \bar{T}(p_1, q_2) = 1$,

$$R(1, p_1) = 1 \text{ implies } \bar{T}(1, q_1) = \bar{T}(1, q_2) = 1,$$

$$S(q_1, 3) = 1 \text{ or } S(q_2, 3) = 1 \text{ implies } \bar{T}(p_1, 3) = 1, \text{ and finally,}$$

$$\bar{T}(1, q_1) = S(q_1, 3) = 1 \text{ or } \bar{T}(1, q_2) = S(q_2, 3) = 1 \text{ implies } T(1, 3) = 1.$$

Similarly, we have $\tilde{T}$ and $T^{'}$. Also, we see that there are many more matrices like the aforementioned matrices T, $\bar{T}$, $\tilde{T}$, and $T^{'}$ that can be the quasi-transitive closures of the poset matrices R and S.

We observe that the quasi-transitive closure $T^{'}$ of R and S, as in the above example, equals O_3, the 3-by-3 matrix with all entries 1s. Below, we show in general that for any poset matrices M_m and N_n, the matrix $O_{m,n}$ equals a quasi-transitive closure of these matrices.

Lemma 3.1. *Let M_m and N_n be any poset matrices. Then the matrix $O_{m,n}$ equals a quasi-transitive closure of M_m and N_n.*

Proof. Let M_m be in bottom antichain form of length p and N_n be in top antichain form of length q. Also, let $T_{m,n}$ be a quasi-transitive closure of M_m and N_n such that $T(i,j) = 1$ for all $p < i \leq m$ and $1 \leq j \leq q$. Then we have the following.

(1) For every $p < i \leq m$, there exists $1 \leq k \leq p$ such that $M(k,i) = 1$. This implies $T(k,j) = 1$ for all $1 \leq k \leq p$ and $1 \leq j \leq q$.

(2) Analogously, for every $1 \leq j \leq q$, there exists $q < t \leq n$ such that $N(j,t) = 1$. This implies $T(i,t) = 1$ for all $p < i \leq m$ and $q < t \leq n$.

(3) Then for every $1 \leq k \leq p$, $1 \leq j \leq q$, and $q < t \leq n$, by (1) and (2), $T(k,j) = 1$ and $N(j,t) = 1$ imply $T(k,t) = 1$.

Thus $T(i,j) = 1$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. This shows that $T = O_{m,n}$ is a quasi-transitive closure of the matrices M_m and N_n of length $\{p,q\}$. □

Now we define the *quasi-ordinal sum* of two poset matrices.

Definition 3.4. Let the poset matrix M_m be in bottom antichain form of length p and N_n be in top antichain form of length q. Then the *quasi-ordinal sum* of M_m and N_n, denoted by $M_m \boxminus N_n$, is an $(m+n)$-by-$(m+n)$ block matrix defined as follows:

$$M_m \boxminus N_n = \left[\begin{array}{ccc} M_m & | & T_{m,n} \\ -- & . & -- \\ Z_{n,m} & | & N_n \end{array} \right].$$

Here, the matrix $T_{m,n}$ is a quasi-transitive closure of the matrices M_m and N_n.

Note that the matrices M_m and N_n are called the *quasi-ordinal terms* of the sum $M_m \boxminus N_n$. We can generalize the above definition for several poset matrices. Let N_{n_i}, $1 \leq i \leq m$, be all poset matrices. Then, the (i,j)-th block Q_{ij} of the quasi-ordinal sum $Q_r = N_{n_1} \boxminus N_{n_2} \boxminus \cdots \boxminus N_{n_m} = [Q_{ij}]$, $1 \leq i,j \leq m$, where $r = \sum_{i=1}^{m} n_i$, can be expressed as follows:

$$Q_{ij} = \begin{cases} N_{n_i}, & \text{if } i = j, \\ T_{n_i,n_j}, & \text{if } i < j, \\ Z_{n_j,n_i}, & \text{if } i > j. \end{cases} \tag{1}$$

Here, for every $1 \leq i,j \leq m$, the matrix T_{n_i,n_j} is a quasi-transitive closure of the matrices N_{n_i} and N_{n_j}.

Example 3.3. We have the following four different matrices A, B, C, and D each of which gives a quasi-ordinal sum of the poset matrices R and S given in Example 3.2.

$$A = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \qquad B = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$C = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \qquad D = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

We see that any other matrix giving a quasi-ordinal sum of the poset matrices R and S can be obtained by some relabeling of one of the matrices A, B, C, and D.

Recall that the poset matrices R and S (Example 3.2) represent the posets $\mathbf{B}_{1,2}$ and $\mathbf{B}_{2,1}$, respectively. We observe that for $R \boxminus S = [Q_{ij}]$, $1 \leq i, j \leq 2$, the matrix block Q_{12} (the block giving the quasi-transitive closure) equals the matrices T, $\bar{T}$, $\tilde{T}$, and $T^{'}$ (Example 3.1) in the cases of the matrices A, B, C, and D (Example 3.3), respectively. We also observe that A, B, C, and D are all poset matrices, and each of these matrices equals a quasi-ordinal sum $R \boxminus S$. Here, the poset matrices A, B, C, and D represent respectively the posets $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{S}$, as shown in Figure 3, that can be checked immediately from the Hasse diagrams of these posets. In the following, we establish the above observations in general that gives an interpretation of the quasi-ordinal sum of poset matrices in posets.

Theorem 3.1. *Let M_m represent the poset $\mathbf{A}$ and N_n represent the poset $\mathbf{B}$. Then the matrix $M_m \boxminus N_n$ is a poset matrix and it represents the poset $\mathbf{A} \ominus \mathbf{B}$.*

Proof. Let $M_m = [a_{ij}]$, $1 \leq i, j \leq m$, $N_n = [b_{ij}]$, $1 \leq i, j \leq n$, and $M_m \boxminus N_n = Q_{m+n} = [q_{ij}]$, $1 \leq i, j \leq m + n$, with block representation $[Q_{ij}], 1 \leq i, j \leq 2$. Since Q_{m+n} is an upper triangular matrix with entries 1s in the main diagonal, because $Q_{21} = Z_{n,m}$, and M_m and N_n are poset matrices, Q_{m+n} is clearly reflexive and antisymmetric. For transitivity of Q_{m+n}, let $q_{ij} = q_{jk} = 1$ for some i, j, and k where $1 \leq i \leq j \leq k \leq m + n$. Also let $Q_{12} = T_{m,n}$. By the definition of quasi-ordinal sum of poset matrices, $T_{m,n}$ is a quasi-transitive closure of the matrices M_m and N_n. Then we have the cases as follows:

(1) $k \leq m$. Then $i \leq j \leq k \leq m$ implies q_{ij}, q_{jk}, $q_{ik} \in M_m$. Since M_m is transitive, $q_{ik} = 1$.

(2) $k > m$. If $j \leq m$, then $i \leq j \leq m < k$ implies $q_{ij} \in M_m$ and $q_{ik}, q_{jk} \in T_{m,n}$. Since M_m is in bottom antichain form and $q_{jk} = 1$ in $T_{m,n}$, by the definition of quasi-transitive closure, $q_{ij} = 1$ in M_m implies $q_{ik} = 1$ in $T_{m,n}$.

If $j > m$, then we have two cases as follows:

(i) $i \leq m$. Then $i \leq m < j \leq k$ implies $q_{jk} \in N_n$ and $q_{ij}, q_{ik} \in T_{m,n}$. Since N_n is in top antichain form and $q_{ij} = 1$ in $T_{m,n}$, by the definition of quasi-transitive closure, $q_{jk} = 1$ in M_m implies $q_{ik} = 1$ in $T_{m,n}$.

(ii) $i > m$. Then $m < i \leq j \leq k$ implies q_{ij}, q_{jk}, $q_{ik} \in N_n$ and, since N_n is transitive, $q_{ik} = 1$.

Thus Q_{m+n} is transitive, and hence a poset matrix. We now show that $M_m \boxminus N_n$ represents the poset $\mathbf{A} \ominus \mathbf{B}$. Let $\mathbf{A} = \langle A; \leqslant_A \rangle$, where $A = \{x_1, x_2, \ldots, x_m\}$ and $\mathbf{B} = \langle B; \leqslant_B \rangle$, where $B = \{x_{m+1}, x_{m+2}, \ldots, x_{m+n}\}$. We must show that Q_{m+n} represents the poset $\mathbf{A} \ominus \mathbf{B} = \langle A \cup B; \leqslant_{\ominus} \rangle$, where $A \cup B = \{x_1, x_2, \ldots, x_m, x_{m+1}, x_{m+2}, \ldots, x_{m+n}\}$. Let $q_{ij} = 1$ in Q_{m+n} for some i and j where $1 \leq i, j \leq m + n$. Then one of the following holds.

(1) $q_{ij} \in M_m$. Since M_m represents $\mathbf{A}$, we have $x_i, x_j \in A$ and $x_i \leqslant_A x_j$.

(2) $q_{ij} \in N_n$. Since N_n represents $\mathbf{B}$, we have $x_i, x_j \in B$ and $x_i \leqslant_B x_j$.

(3) $q_{ij} \in T_{m,n}$. Since $T_{m,n}$ is a quasi-transitive closure of M_m and N_n, there exist k and t, where $i \leq k, t \leq j$, such that $q_{kt} \in T_{m,n}$ and $q_{kt} = q_{it} = q_{kj} = 1$. Then the following hold.

(a) $q_{ik} \in M_m$ and $q_{ik} = 1$. Since M_m represents $\mathbf{A}$, we have $x_i, x_k \in A$ and $x_i \leqslant_A x_k$.

(b) $q_{tj} \in N_n$ and $q_{tj} = 1$. Since N_n represents $\mathbf{B}$, we have $x_t, x_j \in B$ and $x_t \leqslant_B x_j$.

By the definition of quasi-ordinal sum of posets, $x_i \leqslant_{\ominus} x_j$ in $A \cup B$. Hence Q_{m+n} represents the poset $\mathbf{A} \ominus \mathbf{B}$. □

We can generalize the above result as follows:

Theorem 3.2. *Let M_{m_i}, $1 \leq i \leq n$, be the poset matrices that represent the posets $\mathbf{P}_i$, $1 \leq i \leq n$, respectively. Then $M_{m_1} \boxminus M_{m_2} \boxminus \cdots \boxminus M_{m_n}$ is a poset matrix and it represents the poset $\ominus_{i=1}^{n} \mathbf{P}_i$.*

Proof. The proof follows inductively by Theorem 3.1. □

As an immediate corollary of Theorem 3.1, we prove in the following that the class of N-free posets generalizes the class of series-parallel posets.

Corollary 3.1. *Every series-parallel poset is N-free.*

Proof. Let $\mathbf{S}$ be a series-parallel poset. Then there exist the posets $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{S} \cong \mathbf{A} \oplus \mathbf{B}$. Let M_m represent $\mathbf{A}$ and N_n represent $\mathbf{B}$. Then, by Theorem 2.4, $M_m \boxplus N_n$ is a poset matrix and it represents the poset $\mathbf{A} \oplus \mathbf{B}$. Then we have $M_m \boxplus N_n = [Q_{ij}]$, $1 \leq i, j \leq 2$, such that $Q_{12} = O_{m,n}$. Since $O_{m,n}$ is a quasi-transitive closure of M_m and N_n, by Lemma 3.1, $M_m \boxplus N_n$ equals a quasi-ordinal sum

$M_m \boxminus N_n$ (say) of the matrices M_m and N_n. By Theorem 3.1, $M_m \boxminus N_n$ represents the poset $\mathbf{A} \ominus \mathbf{B}$. Then $M_m \boxplus N_n = M_m \boxminus N_n$ implies $\mathbf{A} \ominus \mathbf{B} \cong \mathbf{A} \oplus \mathbf{B} \cong \mathbf{S}$. This shows that $\mathbf{S}$ is an N-free poset. □

4. Matrix Recognition of the N-free Posets

Mohammad et al. [11,12,14] defined the properties of block of 0s and block of 1s in the poset matrix, and transitive blocks of 1s in a block poset matrix. Then they obtained the matrix recognitions of the series-parallel posets, composite posets, and in general decomposable posets. Let $M_m = [a_{ij}]$, $1 \le i, j \le m$, be a poset matrix. Then M_m has the property of *block of 0s* of length r, where $1 \le r < m$, if and only if $a_{ij} = 0$ for all $1 \le i \le r$ and $r+1 \le j \le m$. Then the authors proved the following result.

Theorem 4.1. [11] *A poset matrix M_m satisfies the property of block of 0s if and only if $M_m = M_{m_1} \oplus M_{m_2} \oplus \cdots \oplus M_{m_n}$ for some m_i, $1 \le i \le n$.*

We now define the property of *block of quasi-transitive 1s* in a poset matrix.

Definition 4.1. A poset matrix Q is said to have the property of *block of quasi-transitive 1s* of length $\{p, q\}$ if and only if there exists a block representation $Q = [Q_{ij}]$, $1 \le i, j \le 2$, such that the following conditions hold.

(1) Q_{11} is a poset matrix in bottom antichain form of length p,
(2) Q_{22} is a poset matrix in top antichain form of length q,
(3) Q_{12} is a quasi-transitive closure of Q_{11} and Q_{22},
(4) $Q_{21} = Z_{n,m}$.

Example 4.1. We see that the following poset matrix $F = [F_{ij}]$, $1 \le i, j \le 2$, does not satisfy the property of block of quasi-transitive 1s, because the matrix block F_{12} is not a quasi-transitive closure in any of the four cases shown below.

$$F = \left[\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

But, the poset matrix $\bar{F}$, obtained by $(1,2)$-relabeling of F, satisfies the property of block of quasi-transitive 1s of length $\{1,3\}$ in the first case and length $\{3,1\}$ in the other case.

$$F \xrightarrow{\text{(1,2)-relabeling}} \left[\begin{array}{c|cccc} 1 & 0 & 1 & 1 & 1 \\ - & - & - & - & - \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ - & - & - & - & - \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] = \bar{F}$$

In the following, we obtain a result regarding the poset matrices satisfying the property of block of quasi-transitive 1s that is analogous to the result obtained in Theorem 4.1.

Theorem 4.2. *A poset matrix satisfies the property of block of quasi-transitive 1s if and only if it can be obtained as the quasi-ordinal sum of two poset matrices.*

Proof. Let the matrix Q be obtained as the quasi-ordinal sum of the poset matrices M_m and N_n, that is, $Q = M_m \ominus N_n$. By Theorem 3.1, Q is a poset matrix. Then Q is upper triangular and has the block representation $Q = [Q_{ij}]$, $1 \leq i, j \leq 2$, such that $Q_{11} = M_m$, $Q_{22} = N_n$, $Q_{12} = T_{m,n}$, and $Q_{21} = Z_{n,m}$ for some matrix $T_{m,n}$. By the definition of quasi-ordinal sum, there exist p and q such that M_m is in bottom antichain form of length p and N_n is in top antichain form of length q, and $T_{m,n}$ is a quasi-transitive closure of the matrices M_m and N_n of length $\{p,q\}$. Thus the matrices M_m, N_n, and $T_{m,n}$ satisfy the first three conditions in Definition 4.1, respectively. Also, the fourth condition in Definition 4.1 is obvious as $Q_{21} = Z_{n,m}$. Thus Q satisfies the property of block of quasi-transitive 1s.

Conversely, we suppose that the matrix Q satisfies the property of block of quasi-transitive 1s of length $\{p,q\}$ for some p and q. Then we similarly show that Q can be obtained as the quasi-ordinal sum of two poset matrices in bottom and top antichain forms of lengths p and q, respectively. $\square$

We observe that both the poset matrices F and $\bar{F}$, as described in Example 4.1, represent the least-element N-free poset (Figure 2) that is not series-parallel. In the following, we establish this result in general that gives a matrix recognition of the N-free posets.

Theorem 4.3. *Let the matrix M_n represent the poset $\mathbf{F} \ncong \mathbf{1}$. Then $\mathbf{F}$ is N-free if and only if M_n can be relabeled in such a form that it satisfies either the block of 0s property or the block of quasi-transitive 1s property, and every term (direct or quasi-ordinal) of M_n until 1 satisfies either the block of 0s property or the block of quasi-transitive 1s property.*

Proof. Let $\mathbf{F} \ncong \mathbf{1}$ be an N-free poset. Then there exist the posets $\mathbf{F}_1$ and $\mathbf{F}_2$ such that either $\mathbf{F} \cong \mathbf{F}_1 + \mathbf{F}_2$ or $\mathbf{F} \cong \mathbf{F_1} \ominus \mathbf{F}_2$. Let $M_{n_{11}}$ and $M_{n_{12}}$ represent the posets $\mathbf{F}_1$ and $\mathbf{F}_2$, respectively. Then either $M_n = M_{n_{11}} \oplus M_{n_{12}}$ by Theorem 2.3 or $M_n = M_{n_{11}} \boxminus M_{n_{12}}$ by Theorem 3.1. There exists either the matrix $Z_{n_{11},n_{12}}$ (as a block of 0s) or the matrix $T_{n_{11},n_{12}}$ (as a block of quasi-transitive 1s) as in the constructions of the direct sum (Theorem 4.1) and the quasi-ordinal sum (Theorem 4.2). These imply that M_n satisfies either the property of block of 0s or the property of block of quasi-transitive 1s. For every $1 \leq i \leq 2$, if $F_i \ncong \mathbf{1}$, we similarly show that the poset matrix $M_{n_{1i}}$ satisfies either the block of 0s property or the block of quasi-transitive 1s property. This is because every direct term and quasi-ordinal term of an N-free poset is also N-free. We can continue the above process to show that every direct term and quasi-ordinal term of M_n until 1 satisfies either the block of 0s property or the block of quasi-transitive 1s property.

Conversely, let M_n can be relabeled in such a form that it satisfies either the block 0s property or the block of quasi-transitive 1s property. Then there exist M_{n_1} and M_{n_2}, where $n_1 + n_2 = n$, such that either $M_n = M_{n_1} \oplus M_{n_2}$ by Theorem 4.1 or $M_n = M_{n_1} \boxminus M_{n_2}$ by Theorem 4.2. Then either $\mathbf{F} \cong \mathbf{F}_{01} + \mathbf{F}_{02}$ or $\mathbf{F} \cong \mathbf{F}_{01} \ominus \mathbf{F}_{02}$, where M_{n_1} and M_{n_2} represent the posets $\mathbf{F}_{01}$ and $\mathbf{F}_{02}$, respectively (Theorem 2.3 and Theorem 3.1). Since every term (direct or quasi-ordinal) $M_{n_i}, 1 \leq i \leq 2$, until 1 satisfies either the block of 0s property or the block of quasi-transitive 1s property, we similarly show that there exist the posets $\mathbf{F}_{i1}$, $\mathbf{F}_{i2}$, $1 \leq i \leq 2$, such that $\mathbf{F}_{0i} \cong \mathbf{F}_{i1} + \mathbf{F}_{i2}$, $1 \leq i \leq 2$, or $\mathbf{F}_{0i} \cong \mathbf{F}_{i1} \ominus \mathbf{F}_{i2}$, $1 \leq i \leq 2$. We can continue the above process to show that the poset $\mathbf{F}$ can be expressed as the sum of singleton posets by using only the direct sum and the quasi-ordinal sum of posets. Therefore, by Theorem 2.1, $\mathbf{F}$ is an N-free poset. $\square$

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] B.I. Bayoumi, M.H. El-Zahar, S.M. Khamis, Asymptotic enumeration of N-free partial orders, Order 6 (1989), 219–225. https://doi.org/10.1007/bf00563522.

[2] G.S. Cheon, B. Curtis, G. Kwon, A. Mesinga Mwafise, Riordan posets and associated incidence matrices, Linear Algebra Appl. 632 (2022), 308–331. https://doi.org/10.1016/j.laa.2021.10.002.

[3] B.A. Davey, H.A. Priestly, Introduction to lattices and order, Cambridge University Press, Cambridge, 2002.

[4] M.H. El-Zahar, S.M. Khamis, Enumeration of series-parallel posets according to heights, J. Egypt. Math. Soc. 8 (2000), 1–7.

[5] M. Habib, R. Jegou, N-free posets as generalizations of series-parallel posets, Discr. Appl. Math. 12 (1985), 279–291. https://doi.org/10.1016/0166-218x(85)90030-7.

[6] M. Habib, R.H. Möhring, On some complexity properties of N-free posets and posets with bounded decomposition diameter, Discr. Math. 63 (1987), 157–182. https://doi.org/10.1016/0012-365x(87)90006-9.

[7] J. Heitzig, J. Reinhold, Counting finite lattices, Algebra Univers. 48 (2002), 43–53. https://doi.org/10.1007/pl00013837.

[8] S.M. Khamis, Height counting of unlabeled interval and N-Free posets, Discrete Math. 275 (2004), 165–175. https://doi.org/10.1016/s0012-365x(03)00106-7.

[9] S.M. Khamis, Recognition of prime posets and one of its applications, J. Egypt. Math. Soc. 14 (2006), 5–13.

[10] H.S. Mehta, U.P. Acharya, Adjacency matrix of product of graphs, Kalpa Publ. Comp. 2 (2017), 158–165.

[11] S.U. Mohammad, M.R. Talukder, Poset matrix and recognition of series-parallel posets, Int. J. Math. Comp. Sci. 15 (2020), 107–125.

[12] S.U. Mohammad, M.R. Talukder, Interpretations of kronecker product and ordinal product of poset matrices, Int. J. Math. Comp. Sci. 16 (2021), 1665–1681.

[13] S.U. Mohammad, M.S. Noor, M.R. Talukder, An exact enumeration of the unlabeled disconnected posets, J. Integer Seq. 25 (2022), 22.5.4.

[14] S.U. Mohammad, M.R. Talukder, S.N. Begum, Recognition of decomposable posets by using the poset matrix, Italian J. Pure Appl. Math. 50 (2023), 455–466.

[15] M. Skandera, B. Reed, Total nonnegativity and (3+1)-free posets, J. Comb. Theory, Ser. A 103 (2003), 237–256. https://doi.org/10.1016/s0097-3165(03)00072-4.

[16] N.J.A. Sloane, S. Plouffe, The on-line encyclopedia of integer sequences, 2022. https://oeis.org.