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ABSTRACT RESULTS CONCERNING COHEN STRONGLY (p, σ)-CONTINUOUS OPERATORS

NAWEL ABDESSELAM, AMAR BOUGOUTAIA*, YOUCEF SEDDIK HAMIDOU, AMAR BELACEL

Department of Mathematics, Laboratory of Pure and Applied Mathematics, University of Laghouat, Algeria

*Corresponding author: amarbou28@gmail.com

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Abstract. In this paper, we introduce an abstract finding that delineates the correspondence between specific sets of linear operators and the class of Cohen strongly (p, σ)-continuous operators. We expand our argument to encompass multilinear operators, consequently establishing alternative descriptions for the class of Cohen strongly(p, σ)-continuous multilinear operators.

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1. Notation and background

Motivated by the observation that the class of p-summing linear operators lacks closure under conjugation, Pietsch [11, p.338] demonstrated that while the identity from ℓ_1 to ℓ_2 is absolutely 2-summing, its conjugate from ℓ_2 to ℓ_∞ is not 2-summing. Cohen [5] introduced the class $\mathcal{D}_p$ of strongly p-summing linear operators, establishing in his work that the conjugate of a strongly p-summing operator is a p^*-summing operator, satisfying the condition $1/p + 1/p^* = 1$. In [4], Campos presented abstract results, highlighting instances of overlap between Cohen's space and other operator spaces. The interpolated operator ideal $\Pi_{p,\sigma}$, as introduced by Matter [1], is defined for (p, σ)-absolutely continuous operators, where $1 \leq p < \infty$ and $0 \leq \sigma < 1$. It serves as an intermediary between the ideal of absolutely p-summing linear operators and the ideal of all continuous operators. Subsequently, Achour et al. in [1] introduced the ideal $\mathcal{D}_p^\sigma$ of strongly (p, σ)-continuous linear operators to investigate the adjoints of (p, σ)-absolutely continuous linear operators. They further constructed a new multi-ideal using the composition method from this ideal, demonstrating the corresponding Pietsch domination theorem and presenting a tensorial representation for this multi-ideal. In this paper, we establish an abstract

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result derived from the Full General Pietsch Domination Theorem [10, Theorem 4.6], which holds immediate relevance concerning the class of strongly (p,σ)-continuous operators. Furthermore, we extend this result to the multilinear case, enabling the establishment of alternative definitions for Cohen strongly (p,σ)-continuous multilinear operators, drawing inspiration from techniques employed in [4]. We adopt standard Banach space notation, where n and m are positive integers, $E, E_1, ..., E_m, F$ denote Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. E^* represents the topological dual of E, and B_E denotes the closed unit ball of E. The Banach space of all continuous m-linear operators from $E_1 \times \cdots \times E_m$ into F, equipped with the supremum norm, is denoted by $\mathcal{L}(E_1, ..., E_m; F)$. When $F = \mathbb{K}$ and if $E_1 = \cdots = E_m = E$, we respectively write $\mathcal{L}(E_1, ..., E_m)$ and $\mathcal{L}(^m E; F)$. Let's define the classical sequence spaces we'll be working with:

- $\ell_p(E)$: Represents absolutely p-summable E-valued sequences with the usual norm $\|\cdot\|_p$.
- $\ell_{p,w}(E)$: Represents weakly p-summable E-valued sequences with the norm

$$\|(x_i)_{i=1}^{\infty}\|_{\ell_{p,w}(E)} = \sup_{\varphi \in B_{E^*}} \left(\sum_{i=1}^{\infty} |\langle x_i, \varphi \rangle|^p \right)^{\frac{1}{p}};$$

The space $\ell^{p\sigma}(E)$ of (p,σ)-weakly summable sequences was introduced in [8] in order to provide a characterization of the class of (p,σ)-absolutely continuous operators. Now, let's recall some properties of this space:

For a fixed sequence $(x_i)_{i=1}^n$ in E, where $1 \leq p < \infty$ and $0 \leq \sigma < 1$, we define

$$\delta_{p\sigma}\left((x_i)_{i=1}^n\right) = \sup_{x^* \in B_{E^*}} \left(\sum_{i=1}^{n} \left(|\langle x_i, x^* \rangle|^{1-\sigma} \|x_i\|^{\sigma} \right)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}},$$

It is clear that

$$\|(x_i)_{i=1}^n\|_{\frac{p}{1-\sigma},w} \leq \delta_{p\sigma}\left((x_i)_{i=1}^n\right) \leq \|(x_i)_{i=1}^n\|_{\left(\frac{p}{1-\sigma}\right)}, \tag{1.1}$$

for all $(x_i)_{i=1}^n \in \ell_{\left(\frac{p}{1-\sigma}\right)}(E)$.

We start by introducing various classes of operator ideals. Let (Π_p, π_p) be the ideal of p-absolutely summing operators for $1 \leq p < \infty$. The notion of (p,σ)-absolutely continuous operators was initiated by Matter [9], such that a linear operator $T \in \mathcal{L}(E,F)$ is (p,σ)-absolutely continuous operator if there exist a Banach space G and an operator $S \in \Pi_p(E,G)$ such that

$$\|Tx\| \leq \|x\|^{\sigma} \|Sx\|^{1-\sigma}, x \in E. \tag{1.2}$$

In such case, we put $\pi_{p,\sigma} = \inf \pi_p(S)^{1-\sigma}$, taking the infimum over all Banach spaces G and $S \in \Pi_p(E,G)$ such that (1.2) holds. We denote by $(\Pi_{p,\sigma}, \pi_{p,\sigma})$ the Banach ideal of (p,σ)-absolutely continuous linear operators [9]. Clearly $\Pi_{p,0}$ coincides with the ideal Π_p. If $0 \leq \sigma \leq 1$ and $1 \leq p \leq \infty$, we put $\Pi_{\infty,\sigma} = \Pi_{p,1} = \mathcal{L}$.

The following definition of strongly (p,σ)-continuous m-linear operator is due to Achour et al. in [1].

• Let $1 \leq p, r < \infty$ and $0 \leq \sigma < 1$ such that $\frac{1}{r} + \frac{1-\sigma}{p^*} = 1$. An m-linear mapping $T \in \mathcal{L}(E_1, ..., E_m; F)$ is strongly (p, σ)-continuous if there is a constant $C > 0$ such that for any $x_1^j, ..., x_n^j \in E_j, 1 \leq j \leq m$ and any $y_1^*, ..., y_n^* \in F^*$, we have

$$\left\| \left(\left\langle T\left(x_1^j, ..., x_n^j\right); y_i^* \right\rangle \right)_{i=1}^n \right\|_1 \leq C \left(\sum_{i=1}^n \prod_{j=1}^m \left\| x_i^j \right\|^r \right)^{\frac{1}{r}} \delta_{p^*\sigma}\left((y_i^*)_{i=1}^n \right). \tag{1.3}$$

The collection of all strongly (p, σ)-continuous m-linear maps $E_1 \times \cdots \times E_m$ to F will be denoted $\mathcal{D}_p^{m,\sigma}(E_1, ..., E_m; F)$, that is readily seen to be a subspace of $\mathcal{L}(E_1, ..., E_m; F)$. The least C for which (1.3) holds will be written $\|T\|_{\mathcal{D}_p^{m,\sigma}}$. This define a norm on the space $\mathcal{D}_p^{m,\sigma}(E_1, ..., E_m; F)$. Clearly $\mathcal{D}_p^{m,0}$ coincides with the ideal $\mathcal{D}_p^m$, where $(\mathcal{D}_p^m, d_p^m)$ be the ideal of Cohen strongly p-summing m-linear operators (see [2]).

Definition 1.1. [6] *Let $1 < p, q < \infty$ and $0 \leq \sigma < 1$ be such that $\frac{1}{p} + \frac{1-\sigma}{q} = 1$. They define the space $\ell^{q\sigma}\langle E\rangle$ to be the space of all sequences $(x_i)_{i=1}^\infty$ in E such that $\left|\sum_{i=1}^\infty |\phi_i(x_i)|\right| < \infty$ for all $\phi_i \in \ell^{q\sigma}(E^*)$. In this case we say that $(x_i)_{i=1}^\infty$ is strongly (p, q, σ)-summable. In addition, we have by [6, Theorem 1] the inclusions*

$$\ell_p\langle E\rangle \subset \ell^{q\sigma}\langle E^*\rangle \subset \ell_p(E).$$

Where $\ell_p\langle E\rangle$ the space of Cohen strongly p-summing sequences in E (See [5]).

As in the classical cases, the natural way of presenting the summability properties of the strongly (p, σ)-continuous operators is by defining the corresponding operator between adequate sequence spaces.

Theorem 1.1. [6, Theorem 3] *Let $1 < r, p < \infty$ and $0 \leq \sigma < 1$ be such that $\frac{1}{r} + \frac{1-\sigma}{p^*} = 1$. Then, an operator $T \in \mathcal{L}(E, F)$ is strongly (p, σ)-continuous if and only if $\widehat{T}(\ell_r(X)) \subset \ell_r^{p^*\sigma}\langle F\rangle$ .where the operator $\widehat{T}$ is defined by $\widehat{T}((x_i)_{i=1}^n) = T(x_i)_{i=1}^n$ for all $(x_i)_{i=1}^n \in \ell^{p\sigma}(E)$.*

Abstract findings regarding strongly (p, σ)-continuous operators

1.1. **The linear case.** In this section, we show that the class of strongly (p, σ)-continuous operators can be delineated through various inequalities. Consequently, we derive several coincidental and inclusive outcomes between specific classes of linear operators.

Let $1 \leq q_0, q_1, p, r < \infty$ and $0 \leq \sigma < 1$ be such that $1/r + (1-\sigma)/p^* = 1$. We denote by $\mathcal{C}_{(q_0,q_1;p)}^\sigma(E, F)$ the class of all operators $T \in \mathcal{L}(E, F)$ such that there exists a constant $C > 0$ satisfying

$$\left(\sum_{i=1}^n |\phi_i(T(x_i))|^{q_0} \right)^{\frac{1}{q_0}} \leq C \left\| (x_i)_{i=1}^n \right\|_{q_1} \delta_{p^*\sigma}\left((\phi_i)_{i=1}^n \right) \tag{1.4}$$

for all positive integers n, and all $(x_i)_{i=1}^n \subset E$ and $(\phi_i)_{i=1}^n \subset F^*$.

Let

$$\Gamma_p^\sigma = \left\{ (q_0, q_1) \in [1, \infty) \times (1, \infty) : \frac{1}{q_0} = \frac{1}{q_1} + \frac{1-\sigma}{p^*} \right\}.$$

According to Definition 1.1, it is evident that

$$\mathcal{D}_p^\sigma(E, F) \subset \bigcup_{(q_0,q_1)\in\Gamma_p^\sigma} \mathcal{C}^\sigma_{(q_0,q_1;p)}(E, F).$$

Proposition 1.1. *If* $1 \le q_0, q_1, p < \infty$ *and* $0 \le \sigma < 1$, *and if* $p < q_1$, *then* $\mathcal{C}^\sigma_{(1,q_1;p)}(E, F) = \{0\}$.

Proof. The statement is clear if either E or $F = \{0\}$. We proceed assuming $E \neq \{0\}$ and $F \neq \{0\}$. Let

$$(\lambda_i)_{i=1}^\infty = (\alpha_i \beta_i)_{i=1}^\infty \notin \ell_1 \tag{1.5}$$

with $(\alpha_i)_{i=1}^\infty \in \ell_{\left(\frac{p^*}{1-\sigma}\right)}$ and $(\beta_i)_{i=1}^\infty \in \ell_{q_1}$. We have

$$\begin{aligned}\sum_{i=1}^n |\varphi(T(\lambda_i x))| &= \sum_{i=1}^n |\varphi(T(\alpha_i\beta_i x))| \\ &= \sum_{i=1}^n |\alpha_i \varphi(T(\beta_i x))|\end{aligned}$$

By (1.4), we have

$$\sum_{i=1}^n |\varphi(T(\lambda_i x))| \le C \left\|(\beta_i x)_{i=1}^n\right\|_{q_1} \delta_{p^*\sigma}\left((\alpha_i\varphi)_{i=1}^n\right).$$

Thus by (1.1)

$$\begin{aligned}|\varphi(T(x))| \sum_{i=1}^n |\lambda_i| &\le C\|x\| \left\|(\beta_i)_{i=1}^n\right\|_{q_1} \delta_{p^*\sigma}\left((\alpha_i\varphi)_{i=1}^n\right) \\ &\le C\|x\| \left\|(\beta_i)_{i=1}^n\right\|_{q_1} \left\|(\alpha_i\varphi)_{i=1}^n\right\|_{\left(\frac{p^*}{1-\sigma}\right)} \\ &\le C\|x\| \left\|(\beta_i)_{i=1}^n\right\|_{q_1} \|\varphi\| \left\|(\alpha_i)_{i=1}^n\right\|_{\left(\frac{p^*}{1-\sigma}\right)}.\end{aligned}$$

Taking the supremum thereafter, it follows that

$$\|T\| \sum_{i=1}^n |\lambda_i| \le C \left\|(\beta_i)_{i=1}^n\right\|_{q_1} \left\|(\alpha_i)_{i=1}^n\right\|_{\frac{p^*}{1-\sigma}}.$$

Therefore, we conclude that if $T \neq 0$ then $(\lambda_i)_{i=1}^\infty \in \ell_1$, which contradicts assertion (1.5). □

To obtain the main result of this section, we require a brief review of the Pietsch Domination Theorem as presented by Pellegrino et al. [10].

Let $E_1, ..., E_m, F$ and $F_1, ..., F_k$ be non-empty sets, and let $\mathcal{H}$ be a family of mappings from $E_1 \times \cdots \times E_m$ to F. Additionally, let $K_1, .., K_t$ be compact Hausdorff topological spaces, $G_1, ..., G_t$ be Banach spaces, and suppose that the mappings

$$\begin{cases} R_j : K_j \times F_1 \times \cdots \times F_k \times G_j \to [0, +\infty),\ j = 1, ..., t \\ S : \mathcal{H} \times F_1 \times \cdots \times F_k \times G_1 \times ... \times G_t \to [0, +\infty) \end{cases}$$

satisfy

(1) For each $x^l \in F_l$ and $b \in G_j$, where $(j,l) \in \{1,...,t\} \times \{1,...,k\}$, the mapping

$$\begin{array}{rccc} (R_j)_{x^1,...,x^k,b}: & K_j & \longrightarrow & [0,+\infty) \\ & \varphi & \longmapsto & R_j\left(\varphi, x^1,...,x^k,b\right) \end{array}$$

is continuous.

(2) The following inequalities holds

$$\begin{cases} R_j\left(\varphi, x^1,...,x^k, \eta_j b^j\right) \leq \eta_j R_j\left(\varphi, x^1,...,x^k, b^j\right) \\ S\left(f, x^1,...,x^k, \alpha^1 b^1,...,\alpha^t b^t\right) \geq \alpha^1 \cdots \alpha^1 S\left(f, x^1,...,x^k, b^1,...,b^t\right) \end{cases}$$

Definition 1.2. *If $0 \leq p_1,...,p_t, q < \infty$, with $\frac{1}{q} = \frac{1}{p_1} + \cdots + \frac{1}{p_t}$, a mapping $f : E_1 \times \cdots \times E_m \to F$ in $\mathcal{H}$ is said to be $R_1,...,R_t$-S-abstract $(p_1,...,p_t)$-summing if there is a constant $C > 0$ so that*

$$\left(\sum_{i=1}^{n} S(f, x_i^1,...,x_i^k, b_i^1,...,b_i^t)^q\right)^{\frac{1}{q}} \leq C \prod_{j=1}^{t} \sup_{\varphi \in K_j} \left(\sum_{i=1}^{n} R_j\left(\varphi, x_i^1,...,x_i^k, b_i^j\right)^{p_j}\right)^{\frac{1}{p_j}}, \tag{1.6}$$

for all $(x_i^s)_{i=1}^n \subset F_s$, $\left(b_i^j\right)_{i=1}^n \subset G_j, n \in \mathbb{N}$ and $(s,j) \in \{1,...,k\} \times \{1,...,t\}$.

Theorem 1.2. *[10, Theorem 4.6] A map $f \in \mathcal{H}$ is $R_1,...,R_t$-S-abstract $(p_1,...,p_t)$-summing if, and only if there is a constant $C > 0$ and Borel probability measures μ_j on K_j with the weak star topology such that*

$$S(f, x^1,...,x^k, b^1,...,b^t) \leq C \prod_{j=1}^{t} \left(\int_{K_j} R_j\left(\varphi, x^1,...,x^k, b^j\right)^{p_j} d\mu_j\right)^{\frac{1}{p_j}}, \tag{1.7}$$

for all $x^l \in E_l, l \in \{1,...,k\}$ and $b^j \in G_j$ with $j = 1,...,t$.

Given all the conditions and notations mentioned above, we will prove the following theorem. It will be sufficient to consider $k = 1$ and $t = 2$ in Definition 1.2 and Theorem 1.2.

Theorem 1.3. *Let $f : X \to Y$ be a mapping belonging to $\mathcal{H}$, and let $0 < q_0, q_1, p_0, p_1, p < \infty$ and $0 \leq \sigma < 1$ be such that $\frac{1}{q_0} = \frac{1}{q_1} + \frac{1-\sigma}{p^*}$ and $\frac{1}{p_0} = \frac{1}{p_1} + \frac{1-\sigma}{p^*}$. If $(R_1)_{(x,b)}(.)$ is constant for each x and for each b, then the following statements are equivalent*

(1) *f is $R_1, R_2 - S$-abstract $(q_1, \frac{p^*}{1-\sigma})$-summing;*
(2) *f is $R_1, R_2 - S$-abstract $(p_1, \frac{p^*}{1-\sigma})$-summing.*

Proof. According to [10], f is R_1, R_2-S-abstract $(q_1, \frac{p^*}{1-\sigma})$-summing if and only if, there exists a constant $C > 0$ and Borel probability measures μ_j on $K_j, j = 1, 2$, such that

$$S(f, x, b^1, b^2) \leq C \prod_{j=1}^{2} \left(\int_{K_j} R_j\left(\varphi, x, b^j\right)^{p_j} d\mu_j\right)^{\frac{1}{p_j}}.$$

Given the hypothesis that for every fixed $\varphi \in K_1$:

$$\left(\int_{K_1} R_1\left(\varphi, x, b^1\right)^{q_1} d\mu_1\right)^{\frac{1}{q_1}} = R_1\left(\varphi, x, b^1\right),$$

then, in our case, f is R_1, R_2-S-abstract $(q_1, \frac{p^*}{1-\sigma})$-summing if and only if, there exists a constant $C > 0$ and Borel probability measure μ on K_2 such that

$$S(f, x, b^1, b^2) \leq C \prod_{j=1}^{2} \left(\int_{K_j} R_j\left(\varphi, x, b^j\right)^{p_j} d\mu_j\right)^{\frac{1}{p_j}} = C R_1\left(\varphi, x, b^1\right) . \left(\int_{K_2} R_2\left(\varphi, x, b^2\right)^{\frac{p^*}{1-\sigma}} d\mu\right)^{\frac{1-\sigma}{p^*}} . \tag{1.8}$$

On the other hand, using the same reasoning, f is R_1, R_2-S-abstract $(p_1, \frac{p^*}{1-\sigma})$-summing if and only if there exists a constant $C > 0$ and a Borel probability measure μ in K_2 such that

$$S(f, x, b^1, b^2) \leq C R_1\left(\varphi, x, b^1\right) . \left(\int_{K_2} R_2\left(\varphi, x, b^2\right)^{\frac{p^*}{1-\sigma}} d\mu\right)^{\frac{1-\sigma}{p^*}} .$$

This expression corresponds exactly to the one given by (1.8) . □

The theorem above implies the equality $\mathcal{D}_p^\sigma(E, F) = \mathcal{C}_{(q_0, q_1; p)}^\sigma(E, F)$ for all $(q_0, q_1) \in \Gamma_p^\sigma$, which leads to various ways of characterizing the class of strongly (p, σ)-continuous operators. Now, we can establish the following corollary:

Corollary 1.1. *For $0 \leq \sigma < 1$ and all $(q_0, q_1), (p_0, p_1) \in \Gamma_p^\sigma$, where $\frac{1}{q_0} = \frac{1}{q_1} + \frac{1-\sigma}{p^*}$ and $\frac{1}{p_0} = \frac{1}{p_1} + \frac{1-\sigma}{p^*}$, we have*

$$\mathcal{C}_{(q_0, q_1; p)}^\sigma(E, F) = \mathcal{C}_{(p_0, p_1; p)}^\sigma(E, F).$$

and

$$\mathcal{C}_{(q_0, q_1; p)}^\sigma(E, F) = \mathcal{C}_{(1, p; p)}^\sigma(E, F) = \mathcal{D}_p^\sigma(E, F).$$

Proof. Let $T \in \mathcal{C}_{(q_0, q_1; p)}^\sigma(E, F)$. Since $(q_0, q_1) \in \Gamma_p^\sigma$, then, $\frac{1}{q_0} = \frac{1}{q_1} + \frac{1-\sigma}{p^*}$. For the parameters

$$\begin{cases} t = 2, k = 1 \\ E_1 = \{0\}, \ X_1 = E, Y = F \\ \mathcal{H} = \mathcal{L}\left(E, F\right) \\ p_0 = q_0, \ p_1 = q_1, \ p_2 = \frac{p^*}{1-\sigma} \\ G_1 = E, \ G_2 = F^* \\ K_1 = \{0\}, \ K_2 = B_{F^{**}} \\ S\left(T, 0, x, \varphi\right) = \left|\varphi\left(T\left(x\right)\right)\right| \\ R_1\left(\vartheta, 0, x\right) = \|x\| \\ R_2\left(\phi, 0, \varphi\right) = \left|\phi\left(\varphi\right)\right|^{1-\sigma} \|\varphi\|^\sigma, \end{cases}$$

we have

$$\begin{aligned}\left(\sum_{i=1}^{n}\left(S(f,x_i,b_i^1,b_i^2)\right)^{p_0}\right)^{1/p_0} &= \left(\sum_{i=1}^{n}\left(S(f,x_i,b_i^1,b_i^2)\right)^{q_0}\right)^{1/q_0} \\ &= \left(\sum_{i=1}^{n}\left|\varphi_i\left(T\left(x_i\right)\right)\right|^{q_0}\right)^{1/q_0},\end{aligned}$$

and

$$\begin{aligned}&\prod_{j=1}^{2}\sup_{\varphi\in K_j}\left(\sum_{i=1}^{n}R_j\left(\varphi,x_i,b_i^j\right)^{p_j}\right)^{\frac{1}{p_j}} \\ =\ & \sup_{\vartheta\in K_1}\left(\sum_{i=1}^{n}R_1\left(\vartheta,0,x_i\right)^{q_1}\right)^{\frac{1}{q_1}}\cdot\sup_{\phi\in K_j}\left(\sum_{i=1}^{n}R_2\left(\phi,0,\varphi_i\right)^{p^*/(1-\sigma)}\right)^{(1-\sigma)/p^*} \\ =\ & \left(\sum_{i=1}^{n}\left\|x_i\right\|^{q_1}\right)^{\frac{1}{q_1}}\cdot\sup_{\phi\in B_{F^{**}}}\left(\sum_{i=1}^{n}\left(\left|\phi\left(\varphi_i\right)\right|^{1-\sigma}\left\|\varphi_i\right\|^{\sigma}\right)^{p^*/(1-\sigma)}\right)^{(1-\sigma)/p^*} \\ =\ & \left\|\left(x_i\right)_{i=1}^{n}\right\|_{q_1}\delta_{p^*\sigma}\left(\left(\varphi_i\right)_{i=1}^{n}\right).\end{aligned}$$

Thus, by [10, Definition 4.4], it follows that T is R_1, R_2-S-abstract $(q_1, \frac{p^*}{1-\sigma})$-summing and, by Theorem 2.4, T is R_1, R_2-S-abstract $(p_1, \frac{p^*}{1-\sigma})$-summing. Therefore, exists a constant $C > 0$ such that

$$\left(\sum_{i=1}^{n}\left|\varphi_i\left(T\left(x_i\right)\right)\right|^{p_0}\right)^{\frac{1}{p_0}} \leq C\left\|\left(x_i\right)_{i=1}^{n}\right\|_{p_1}\delta_{p^*\sigma}\left(\left(\phi_i\right)_{i=1}^{n}\right)$$

for all positive integers n and for all $(x_i)_{i=1}^{n} \subset E$ and $(\phi_i)_{i=1}^{n} \subset F^{**}$, so $T \in \mathcal{C}^{\sigma}_{(p_0,p_1;p)}(E,F)$, we find that

$$\mathcal{C}^{\sigma}_{(q_0,q_1;p)}(E,F) \subset \mathcal{C}^{\sigma}_{(p_0,p_1;p)}(E,F).$$

The other inclusion is obtained by the same argument. □

1.2. **The multilinear case.** Here's an alternative definition for the concept of Cohen strongly (p,σ)-continuous multilinear operators as established by Achour et al. [1, Definition 4.1]. Let $1 < p, r < \infty$ and $0 \leq \sigma < 1$ be such that $1/r + (1-\sigma)/p^* = 1$.

Theorem 1.4. *For $T \in \mathcal{L}(E_1, ..., E_m; F)$, the following statements are equivalent:*

(1) *There exists a constant $C > 0$ such that*

$$\left\|\left(\left\langle T\left(x_1^j,\ldots,x_n^j\right);y_i^*\right\rangle\right)_{i=1}^{n}\right\|_1 \leq C\left(\sum_{i=1}^{n}\prod_{j=1}^{m}\left\|x_i^j\right\|^r\right)^{1/r}\delta_{p^*\sigma}\left(\left(y_i^*\right)_{i=1}^{n}\right),$$

for all $x_i^j \in E_j$ and $y_i^* \in F^*$ such that $i = 1, ..., n$ and $j = 1, ..., m$.

(2) *There exists a constant $C > 0$ such that*

$$\left\|\left(\left\langle T\left(x_1^j,\ldots,x_n^j\right);y_i^*\right\rangle\right)_{i=1}^{\infty}\right\|_1 \leq C\left(\sum_{i=1}^{\infty}\left\|x_i^1\right\|^{mr}\right)^{1/mr}\cdots\left(\sum_{i=1}^{\infty}\left\|x_i^m\right\|^{mr}\right)^{1/mr}\delta_{p^*\sigma}\left(\left(y_i^*\right)_{i=1}^{\infty}\right), \quad (1.9)$$

whenever $\left(x_i^j\right)_{i=1}^{\infty} \in \ell_p^m(E_j), j = 1, ..., m$ *and* $(y_i^*)_{i=1}^{\infty} \in \ell_{\left(\frac{p^*}{1-\sigma}\right)}(F^*)$.

(3) *There is a constant* $C > 0$ *such that*

$$\left\|\left(\left\langle T\left(x_1^j, \ldots, x_n^j\right); y_i^*\right\rangle\right)_{i=1}^{n}\right\|_1 \leq C\left(\sum_{i=1}^{n}\|x_i^1\|^{mr}\right)^{1/mr} \cdots \left(\sum_{i=1}^{n}\|x_i^m\|^{mr}\right)^{1/mr} \delta_{p^*\sigma}\left((y_i^*)_{i=1}^{n}\right),$$

for all $x_i^j \in E_j$ *and* $y_i^* \in F^*$ *such that* $i = 1, ..., n$ *and* $j = 1, ..., m$.

This theorem provides an alternative characterization for Cohen strongly (p, σ)-continuous multilinear operators.

Proof. $(1) \Rightarrow (2)$ Given T is Cohen strongly (p, σ)-continuous, the mapping

$$\begin{array}{rccc} \widetilde{T}: & \ell^{p^*\sigma}(Y^*) \times \ell_p^m(E_1) \times \cdots \times \ell_p^m(E_m) & \rightarrow & \ell_1 \\ & \left((y_i^*)_{i=1}^{\infty}, (x_i^1)_{i=1}^{\infty}, ..., (x_i^m)_{i=1}^{\infty}\right) & \rightarrow & \left(y_i^*\left(T\left(x_i^1, ..., x_i^m\right)\right)\right)_{i=1}^{\infty} \end{array}$$

is well-defined and $(m+1)$-linear. A straightforward calculation shows that $\widetilde{T}$ has closed graph and hence is continuous. Therefore,

$$\begin{aligned} \left\|\left(\left\langle T\left(x_1^j, \ldots, x_n^j\right); y_i^*\right\rangle\right)_{i=1}^{\infty}\right\|_1 &= \left\|\widetilde{T}\left((y_i^*)_{i=1}^{\infty}, (x_i^1)_{i=1}^{\infty}, ..., (x_i^m)_{i=1}^{\infty}\right)\right\|_1 \\ &\leq \left\|\widetilde{T}\right\| \|(x_i^1)_{i=1}^{\infty}\|_{mp} \cdots \|(x_i^m)_{i=1}^{\infty}\|_{mp} \, \delta_{p^*\sigma}\left((y_i^*)_{i=1}^{\infty}\right). \end{aligned}$$

$(2) \Rightarrow (1)$ and $(2) \Rightarrow (3)$ are immediate.

$(3) \Rightarrow (2)$ Let $\left(x_i^j\right)_{i=1}^{\infty} \in \ell_p^m(E_j), j = 1, ..., m$ and $(y_i^*)_{i=1}^{\infty} \in \ell_{\left(\frac{p^*}{1-\sigma}\right)}(F^*)$. Then

$$\begin{aligned} \left\|\left(\left\langle T\left(x_1^j, \ldots, x_n^j\right); y_i^*\right\rangle\right)_{i=1}^{\infty}\right\|_1 &= \sup_n \left\|\left(\left\langle T\left(x_1^j, \ldots, x_n^j\right); y_i^*\right\rangle\right)_{i=1}^{n}\right\|_1 \\ &\leq \sup_n \left(C\prod_{j=1}^{m}\sum_{i=1}^{n}\left\|x_i^j\right\|^{mr}\right)^{1/mr} \delta_{p^*\sigma}\left((y_i^*)_{i=1}^{\infty}\right) \\ &= C\left(\prod_{j=1}^{m}\sum_{i=1}^{\infty}\left\|x_i^j\right\|^{mr}\right)^{1/mr} \delta_{p^*\sigma}\left((y_i^*)_{i=1}^{\infty}\right) \\ &< \infty. \end{aligned}$$

The smallest C such that (1.9) is satisfied, denoted by $\|T\|_{\mathcal{D}_p^{m,\sigma}}$, defines a norm on $\mathcal{D}_p^{m,\sigma}\left(E_1, ..., E_m; F\right)$. □

As a result of Theorem 1.4 and the subsequent theorem, akin to the linear scenario, we can delineate the category of Cohen strongly (p, σ)-continuous multilinear operators through several inequalities. To achieve this, we must broaden the abstract outcome of Theorem 1.3.

Theorem 1.5. *Let* $f: E_1 \times \cdots \times E_m \rightarrow F$ *be a mapping belonging to* $\mathcal{H}$, *and let*

$$0 < p, p_0, q_0, p_1, ..., p_{t-1}, q_1, ..., q_{t-1} < \infty$$

and $0 \leq \sigma < 1$, *be such that*

$$\frac{1}{p_0} = \frac{1}{p_1} + \cdots + \frac{1}{p_{t-1}} + \frac{1-\sigma}{p^*}$$

and

$$\frac{1}{q_0} = \frac{1}{q_1} + \cdots + \frac{1}{q_{t-1}} + \frac{1-\sigma}{p^*}.$$

If $(R_j)_{x_1,...,x_k,b}(.)$ *is constant for all* $x_1, ..., x_k, b$ *and for all* $1 \leq j \leq t-1$, *then the following statements are equivalent*

(1) f *is* $R_1, ..., R_t$*-*S*-abstract* $\left(p_1, ..., p_{t-1}, \frac{p^*}{1-\sigma}\right)$*-summing,*
(2) f *is* $R_1, ..., R_t$*-*S*-abstract* $\left(q_1, ..., q_{t-1}, \frac{p^*}{1-\sigma}\right)$*-summing.*

Corollary 1.2. *For* $T \in \mathcal{L}(E_1, ..., E_m; F)$ *and* $0 \leq \sigma < 1$ *with* $\frac{1}{q_0} = \frac{1}{q_1} + \cdots + \frac{1}{q_m} + \frac{1-\sigma}{p^*}$, *the following statements are equivalent:*

(1) *There exists a constant* $C > 0$ *such that*

$$\sum_{i=1}^{n} \left|\left\langle T\left(x_i^1, ..., x_i^m\right), y_i^*\right\rangle\right| \leq C \left(\sum_{i=1}^{n} \left\|x_i^1\right\|^{mr}\right)^{1/mr} \cdots \left(\sum_{i=1}^{n} \left\|x_i^m\right\|^{mr}\right)^{1/mr} \delta_{p^*\sigma}\left((y_i^*)_{i=1}^n\right),$$

for all $x_i^j \in E_j$ *and* $y_i^* \in F^*$ *such that* $i = 1, ..., n$ *and* $j = 1, ..., m$.

(2) *There exists a constant* $C > 0$ *such that*

$$\left(\sum_{i=1}^{n} \left|\left\langle T\left(x_i^1, ..., x_i^m\right), y_i^*\right\rangle\right|^{q_0}\right)^{1/q_0} \leq C \left(\sum_{i=1}^{n} \left\|x_i^1\right\|^{q_1}\right)^{1/q_1} \cdots \left(\sum_{i=1}^{n} \left\|x_i^m\right\|^{q_m}\right)^{1/q_m} \delta_{p^*\sigma}\left((y_i^*)_{i=1}^n\right), \quad (1.10)$$

for all $x_i^j \in E_j$ *and* $y_i^* \in F^*$ *such that* $i = 1, ..., n$ *and* $j = 1, ..., m$.

From Corollary 1.2, it's feasible to derive further characterizations for the class of Cohen strongly (p, σ)-continuous multilinear operators using sequences. Equivalently, one can establish an inequality akin to that presented by (1.10), albeit with infinite sums:

For $1 \leq p \leq \infty$ and $0 \leq \sigma < 1$, with $\frac{1}{q_0} = \frac{1}{q_1} + \cdots + \frac{1}{q_m} + \frac{1-\sigma}{p^*}$, an operator $T \in \mathcal{L}(E_1, ..., E_m; F)$ is Cohen strongly (p, σ)-continuous if $\left(\left\langle T\left(x_i^1, ..., x_i^m\right), y_i^*\right\rangle\right)_{i=1}^{\infty} \in \ell_{q_0}$ whenever $\left(x_i^j\right)_{i=1}^{\infty} \in \ell_{q_j}(E_j), j = 1, ..., m$, and $(y_i^*)_{i=1}^{\infty} \in \ell_{\left(\frac{p^*}{1-\sigma}\right)}(F^*)$.

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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