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DISCONNECTED MULTI-EFFECT DOMINATION IN GRAPHS

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Abstract. In this paper, disconnected multi-effect domination, a novel domination model in graphs is introduced. Let $G = (V, E)$ be a finite, nontrivial, simple, and undirected graph without isolated vertex. A dominating subset $D \subseteq V$ is a disconnected multi-effect dominating set in G if for every vertex $v \in D$, $|\,\mathrm{N}(v) \cap (V - D)| \geq 2$ and $G[D]$ is disconnected subgraph. The minimum cardinality over all disconnected multi-effect dominating sets in G is the disconnected multi-effect domination number of G denoted by $\gamma_{dm}(G)$. Some bounds and properties of disconnected multi-effect domination are studied with respect to order, size, minimum degree, and the maximum degree of a graph. A disconnected multi-effect domination number is determined for a recognized graphs.

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1. Introduction

Let $G = (V, E)$ be a graph has no isolated vertices with size $m = |E|$ and order $n = |V|$. The number of edges that incident on a vertex v determines the degree of v, which is denoted by $\deg(v)$. A vertex with degree 0 is an isolated vertex, and a vertex with degree 1 is an end or pendant vertex. The minimum and maximum degrees, denoted by $\delta(G)$ and $\Delta(G)$, respectively. $G[D]$ is the subgraph of G induced by the vertices in set D and the edges incident between them. Regarding terms to graph theory, we refer to [23]. One of the regions of graph theory that is expanding the swiftest is the study of domination problems, a thorough analysis on the fundamentals of domination is provided like [13, 15]. A set $D \subseteq V$ is a dominating set if each vertex in $V - D$ is adjacent to a vertex in D. The minimal dominating set D of G is has no proper subset as a dominating set. The domination number $\gamma(G)$ is the cardinality of the minimum dominating set D of G. Different varieties of dominating

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models have emerged depending on the aim of domination due to how important in different kinds of applications as [1,2,4–6,9,10,12,14,20–22,24]. Domination theory has been developing, with researchers delving into different kinds of domination from the prominent researchers who have contributed to this continuing discourse. T. W. Haynes et. al. [15] studied the fundamentals of domination in graphs. A survey of stratified domination in graphs was conducted by T. W. Haynes et. al. [16]. However, total domination stability in graphs were given by M. A. Henning et. al. [17]. Pitchfork domination in graphs are established in 2020 by Manal N. Al-Harere and M. A. Abdlhusein [11]. Doubly connected pitchfork domination are established in 2020 by Mohammed A. Abdlhusein and M. N. Al-Harere [8]. Mohammed A. Abdlhusein introduced stability of inverse pitchfork domination in 2021 [3]. In 2022, some modified types of pitchfork domination and its inverse are given by M. A. Abdlhusein and M. A1-Harere [7]. In 2023, constructing new topological graph with several properties are established by Z. N. Jwair et. al. [18,19]. In this work the disconnected multi-effect domination is discussed. A novel domination model in graphs is introduced. Certain limitations on the disconnected multi-effect domination number correlated with a graph order, size, minimum degree, and maximum degree, and other characteristics are discussed for any graph has this type of domination. Additionally, disconnected multi-effect domination number is determined for some newly updated and known graphs. This raises the question of whether or not any finite, simple, undirected graph G has no isolated vertices has a disconnected multi-effect domination? has been addressed and resolved.

2. Disconnected multi-effect domination in graphs

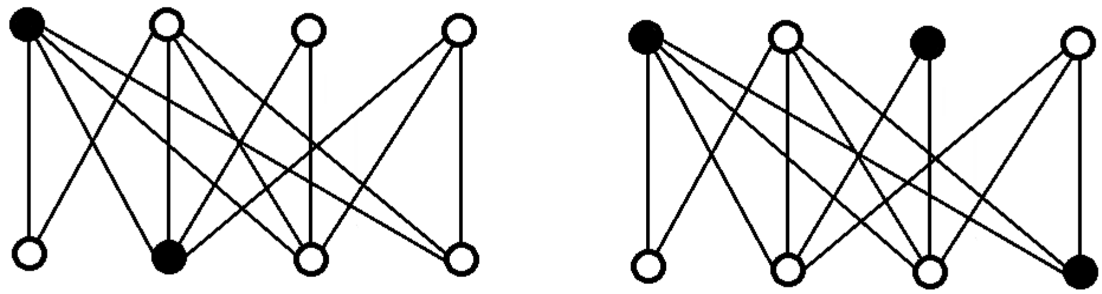
In this section, disconnected multi-effect domination is given with its bounds and properties. The order of graph, minimum degree, maximum degree, and other characteristics are discussed for any graph has this type of domination.

Definition 2.1. *Let $G = (V, E)$ be a finite, nontrivial, simple, and undirected graph without isolated vertex. A dominating subset $D \subseteq V$ is a disconnected multi-effect dominating set in G if for every vertex $v \in D$, $|\,\mathrm{N}(v) \cap (V - D)| \geq 2$ and $G[D]$ is disconnected subgraph, and denoted by $DMEDS$. For example, see Fig 1.*

Definition 2.2. *A disconnected multi-effect dominating set D of G is minimal if it has no proper disconnected multi-effect dominating subset and denoted by MDMEDS.*

Definition 2.3. *A minimal disconnected multi-effect dominating set D of G is said to be minimum if it has the smallest order among all minimal disconnected multi-effect dominating sets in G.*

Definition 2.4. *The minimum cardinality over all disconnected multi-effect dominating sets in G is the disconnected multi-effect domination number of G denoted by $\gamma_{dm}(G)$. Such set is denoted by γ_{dm}-set.*



(a) *Minimum dominating set* (b) *Minimum disconnected multi-effect dominating set*

FIGURE 1. The dominating set and disconnected multi-effect dominating set.

In Fig. 1 (a) there are two vertices in D dominates all other vertices of $V - D$ of a graph G where $G[D]$ is connected graph. In the Fig. 1 (b) there are three vertices in D dominates all other vertices of $V - D$ of a graph G and $G[D]$ is disconnected graph.

Observation 2.1. *For any graph $G = (n, m)$ with disconnected multi-effect dominating set D and disconnected multi-effect domination number $\gamma_{dm}(G)$, we have:*

1. *The order of G is $n \geq 4$.*
2. $|D| \geq 2$.
3. $|V - D| \geq 2$.
4. $\delta(G) \geq 1$ *and* $\Delta(G) \geq 2$.
5. $\deg(v) \geq 2 \quad \forall v \in D$.
6. *Each support vertex belongs to D.*
7. $\gamma(G) \leq \gamma_{dm}(G)$

Theorem 2.1. *Let G be a graph and D be a $DMEDS$ of a graph G. If any of the following conditions holds, then D is a MDMEDS:*

1. $|N(v) \cap D| = 1, \forall v \in V - D$.
2. *$G[D]$ is a null graph.*
3. *Each vertex in D is a support vertex.*

Proof. Suppose that D be any $DMEDS$ in a graph G. Assume that D is not $MDMEDS$. Then, there is at least one vertex $v \in D$ such that $D - \{v\}$ is a $MDMEDS$. There are many cases to discussed as follows:

Case 1. Assume that the first condition holds, then for any vertex $u \in V - D$ which is dominated by vertex v, it is not dominated by any vertex in $D - \{v\}$. Then, $D - \{v\}$ is not $DMEDS$.

Case 2. Assume that the second condition holds, then v is not adjacent with any vertex of D since $G[D]$ is a null graph. Therefore, v is not dominated by any vertex from $D - \{v\}$. Then, $D - \{v\}$ is not

$DMEDS$.

Case 3. Assume that the third condition holds, then the proof is similar to proof in Case 1 where u is an end vertex. □

Theorem 2.2. *The size of any graph $G = (n, m)$ having disconnected multi-effect dominating set D and disconnected multi-effect domination number $\gamma_{dm}(G)$ has boundaries as:*

$$2\gamma_{dm}(G) \leq m \leq \binom{n}{2} - \gamma_{dm} + 1$$

Proof. Assume that D be a γ_{dm} - set of a graph G, then:

Case 1. Assume that $G[D]$ and $G[V - D]$ are two null graphs to be G has as few edges as possible. According to the definition of the $DMED$, there is at least two edges from each vertex of D to $V - D$. Then, the number of edges between D and $V - D$ equal to $2|D| = 2\gamma_{dm}(G)$. Therefore, $m \geq 2\gamma_{dm}(G)$ this is the lower bound in general.

Case 2. Assume that $G[D]$ are union of complete subgraph and isolated vertex, $G[D] = K_{t-1} \cup K_1$ such that $|E(G[D])| = m_1$, and $G[V - D]$ is a complete subgraph to be G have maximum number of edges. Where the number of edges of $G[D]$ and $G[V - D]$ equal to m_1 and m_2 respectively. So

$$m_1 = \frac{|D-1||D-2|}{2} = \frac{(\gamma_{dm}-1)(\gamma_{dm}-2)}{2}$$
$$m_2 = \frac{|V-D||V-D-1|}{2} = \frac{(n-\gamma_{dm})(n-\gamma_{dm}-1)}{2}$$

According to the definition of $DMED$, there is at most $|V - D|$ edges from each vertex of D to $V - D$, such that every vertex in D dominates all vertices of $V - D$. Then, the number of edges between D and $V - D$ equals to $|D||V - D| = \gamma_{dm}(n - \gamma_{dm}) = m_3$. Then, the number of edges of G equals to $m \leq m_1 + m_2 + m_3$

$$\leq \frac{(\gamma_{dm}-1)(\gamma_{dm}-2)}{2} + \frac{(n-\gamma_{dm})(n-\gamma_{dm}-1)}{2} + n\gamma_{dm} - \gamma_{dm}^2$$
$$\leq \binom{n}{2} - \gamma_{dm} + 1$$

In general, this is the upper bound.

The lower bound is sharp for $G = P_5$ where $\gamma_{dm}(P_5) = 2$ and $m = 4$ see Fig. 3 (e) and the upper bound is sharp for $G = F_3$ where $\gamma_{dm}(F_3) = 2$ and $m = 5$ see Fig. 11 (a). □

3. Disconnected multi-effect domination of some graphs

The disconnected multi-effect domination model is studied for some graphs such as: path, cycle, star graph, complete graph, complete bipartite graph, wheel graph, barbell graph, the corresponding barbell graph, fan graph and double fan graph.

Proposition 3.1. *The path graph $P_n, (n \leq 4)$ has no disconnected multi-effect domination.*

Proof. Since $\deg(v_i) \leq 2\ \forall v \in P_i, i = 2, 3, 4$. When $n = 2$, there is one vertex dominates one vertex in $V - D$. When $n = 3$, there is one vertex v_2 dominates two vertices in $V - D$, but $G[D]$ is connected graph. When $n = 4$, if $D = \{v_1, v_4\}$, then each one of them dominates one vertex. If $D = \{v_1, v_3\}$, then v_1 dominates one vertex and v_3 dominates two vertices. If $D = \{v_2, v_3\}$, then each one of them dominates one vertex. If $D = \{v_2, v_4\}$, then v_2 dominates two vertices and v_4 dominates one vertex. Everything above is a contraction of our definition, so $\bar{P}_i$ has no disconnected multi-effect domination. For example, see Fig 2.

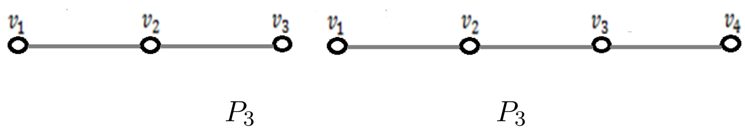


FIGURE 2. P_3 and P_4 has no disconnected multi-effect domination.

□

Theorem 3.2. *For any path graph $P_n, (n \geq 5)$ we have $\gamma_{dm}(P_n) = \lceil \frac{n}{3} \rceil$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of a path graph of order n and let $D \subseteq V(P_n)$ such that:

$$D = \begin{cases} \{v_{3i-1}, i = 1, 2, \ldots, \frac{n}{3}\} & \text{if } n \equiv 0(\text{mod}3) \\ \{v_{3i-1}, i = 1, 2, 3, \ldots, \lfloor \frac{n}{3} \rfloor - 1\} \cup \{v_{n-3}, v_{n-1}\} & \text{if } n \equiv 1(\text{mod}3) \\ \{v_{3i-1}, i = 1, 2, \ldots, \frac{n-2}{3}\} \cup \{v_{n-1}\} & \text{if } n \equiv 2(\text{mod}3) \end{cases}$$

To prove D is the $DMEDS$ in path graph. So, we will discuss three cases:

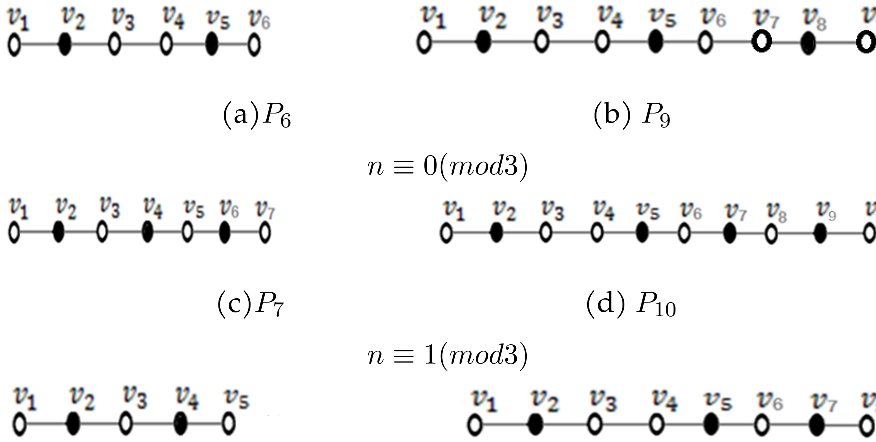
Case 1: If $n \equiv 0(\text{mod}3)$. Let $D = \{v_{3i-1}, i = 1, 2, \ldots, \frac{n}{3}\}$, since every vertex in D adjacent with exactly two vertices, then it dominates exactly two vertices and $G[D]$ is disconnected graph. Thus, D is the $DMEDS$ and $\gamma_{\text{dm}}(P_n) = \frac{n}{3}$.

Case 2: If $n \equiv 1(\text{mod}3)$. Let $D = \{v_{3i-1}, i = 1, 2, 3, \ldots, \lfloor \frac{n}{3} \rfloor - 1\} \cup \{v_{n-3}, v_{n-1}\}$, since every vertex in D adjacent exactly two vertices, then it dominates exactly two vertices and $G[D]$ is disconnected graph. Thus, D is the $DMEDS$ and $\gamma_{\text{dm}}(P_n) = \lceil \frac{n}{3} \rceil$.

Case 3: If $n \equiv 2(\text{mod}3)$. Let $D = \{v_{3i-1}, i = 1, 2, \ldots, \frac{n-2}{3}\} \cup \{v_{n-1}\}$, since every vertex in D adjacent exactly two vertices, then it dominates exactly two vertices and $G[D]$ is disconnected graph. Thus, D is the $DMEDS$ and $\gamma_{\text{dm}}(P_n) = \lceil \frac{n}{3} \rceil$.

To prove that D is a minimum $DMEDS$ in each the previous cases. Let D' is a $DMEDS$ in G, such that $|D'| < |D|$, then there exist one or more vertices of $V - D$ don't dominated by any vertex of D'

or $G\left[D'\right]$ is connected graph. This contraction with the concept of the $DMEDS$. Hence, D' is not $DMEDS$ and D is the minimum $DMEDS$. For example, see Fig 3.



(a)P_6 (b) P_9

$n \equiv 0(mod3)$

(c)P_7 (d) P_{10}

$n \equiv 1(mod3)$

(e)P_5 (f) P_8

$n \equiv 2(mod3)$

Figure 3. A minimum disconnected multi-effect dominating set of P_n

□

Theorem 3.3. *Given a cycle graph C_n, then:*

$$\gamma_{dm}\left(C_n\right) = \left\lceil \frac{n}{3} \right\rceil \qquad \textit{for } n \geq 4$$

Proof. To prove D is the $DMEDS$ in cycle graph. Let $v_1, v_2, \ldots, v_n$ be the vertices of a cycle graph of order n and let $D \subseteq V\left(C_n\right)$ such that

$$D = \begin{cases} \left\{v_{3i-2}, i = 1, \ldots, \frac{n}{3}\right\} & \text{if } n \equiv 0(\text{mod}3) \\ \left\{v_{3i-2}, i = 1, 2, \ldots, \frac{n-1}{3}\right\} \cup \left\{v_{n-1}\right\} & \text{if } n \equiv 1(\text{mod}3) \\ \left\{v_{3i-2}, i = 1, 2, \ldots, \frac{n+1}{3}\right\} & \text{if } n \equiv 2(\text{mod}3) \end{cases}$$

So, the following three cases are discussed:

Case1: If $n \equiv 0(\text{mod}3)$. Let $D = \left\{v_{3i-2}, i = 1, 2, \ldots, \frac{n}{3}\right\}$, then every $v \in D$ dominates exactly two vertices and $G[D]$ is disconnected graph. Then, D is a $DMEDS$.

Case 2: If $n \equiv 1(\text{mod}3)$. Let $D = \left\{v_{3i-2}, i = 1, 2, \ldots, \frac{n-1}{3}\right\} \cup \left\{v_{n-1}\right\}$, then each vertex of D dominates two vertices and $G[D]$ is disconnected graph. Therefore, D is a $DMEDS$.

Case 3: If $n \equiv 2(\text{mod}3)$. Let $D = \left\{v_{3i-2}, i = 1, 2, \ldots, \frac{n+1}{3}\right\}$, then every vertex of D dominates two vertices and $G[D]$ is disconnected graph. Hence, D is a $DMEDS$. Thus, in all above cases D is a $\gamma_{dm}-$set and $\gamma_{dm}\left(C_n\right) = \left\lceil \frac{n}{3} \right\rceil$.

The set D is a minimum $DMEDS$ in each the previous three cases and the proof of it similar to the proof of Theorem 3.2. For example, see Fig 4.

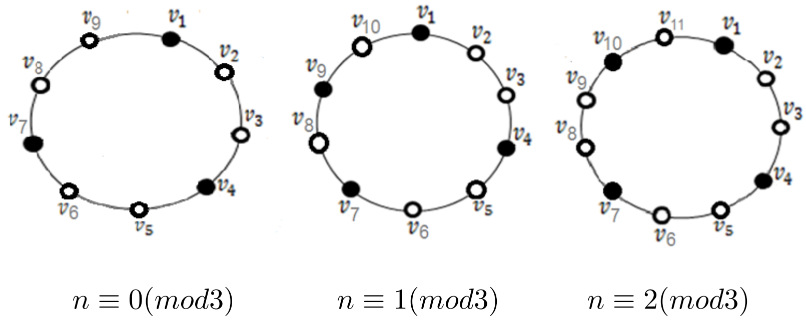


FIGURE 4. A minimum disconnected multi-effect dominating set of C_n

□

Proposition 3.4. *A star graph $S_n (n \geq 3)$ has no disconnected multi-effect domination.*

Proof. According to the definition the star graph is a bipartite graph of the form $K_{1,n}$. Let $u \in S_n$ be a support vertex adjacent with pendants vertices $v_1, v_2, \ldots, v_n$, then if $u \in D$, it dominates three or more end vertices. But $G[D]$ is connected graph. If $u \notin D$, then every vertex of the $n \geq 3$ end vertices dominate only v which is a contradiction. Hence, S_n has no disconnected multi-effect dominating set. □

Proposition 3.5. *Every complete graph K_n has no disconnected multi-effect dominating set.*

Proof. Since every vertex in K_n is joined with all other vertices, so that every vertex in disconnected multi-effect dominating set D can be dominates two or more vertices. Then, there is one vertex in D that dominates all vertices of $V - D$ in the complete graph K_n, but $G[D]$ is connected graph. Hence, K_n has no disconnected multi-effect dominating set. For example, see Fig 5.

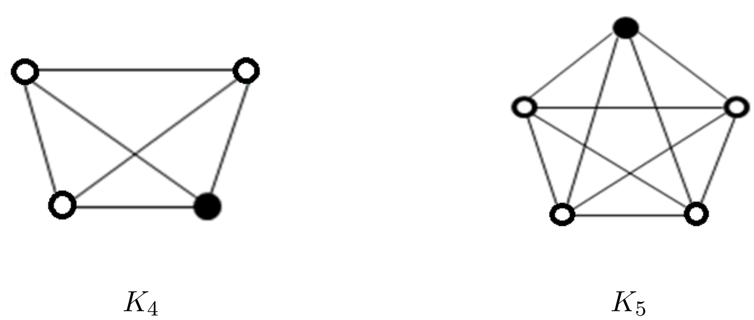


FIGURE 5. A complete graph K_n has no disconnected multi-effect domination.

□

Theorem 3.6. *The complete bipartite graph $K_{n,m}$ has a disconnected multi-effect dominating set if and only if n and $m \geq 2$, where $\gamma_{dm}(K_{n,m}) = \min\{n, m\}$*

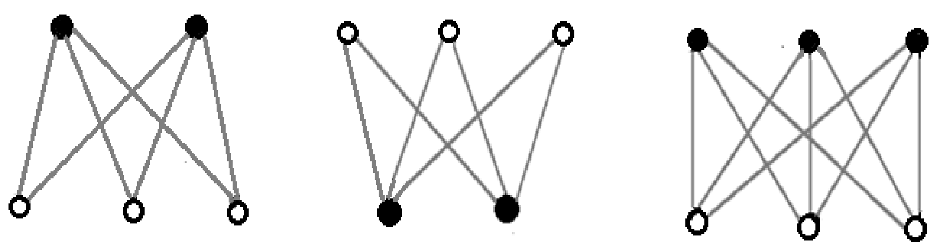
Proof. suppose that U_1 and U_2 be the two sets of vertices of $K_{n,m}$, such that $|U_1| = n$ and $|U_2| = m$. So, the following three cases are discussed:

Case 1. If $n < m$, then D must be contains, n vertices of U_1 where every vertex in U_1 will dominates all the m vertices, that belong to $V - D$, and $G[D]$ is disconnected graph. Hence $\gamma_{dm}(K_{n,m}) = n$

Case 2. If $m < n$, then D must be contains, m vertices of U_2 where each vertex in D will dominates all vertices in U_1, that belong to $V - D$, and $G[D]$ is disconnected graph. Hence, $\gamma_{dm}(K_{n,m}) = m$

Case 3. If $n = m$, then D must be contains, n vertices of U_1 or m vertices of U_2 where each vertex in D will dominates every vertices in $V - D$, and $G[D]$ is disconnected graph. Hence $\gamma_{dm}(K_{n,m}) = n$ or m.

The set D is a minimum $DMEDS$ in all the previous three cases and the proof of it similar to the proof of Theorem 3.2. For example, see Fig 6.
If $n = 1$ and $m \geq 1$, there is one vertex in U_1 will dominates all the m vertices, but $G[D]$ is connected graph. Hence, $K_{n,m}$ has no disconnected multi-effect domination.



$K_{2,3}$ $K_{3,2}$ $K_{3,3}$

FIGURE 6. A minimum disconnected multi-effect dominating set of $K_{n,m}$

□

Theorem 3.7. *If G is a wheel graph $W_n (n \geq 3)$, then*

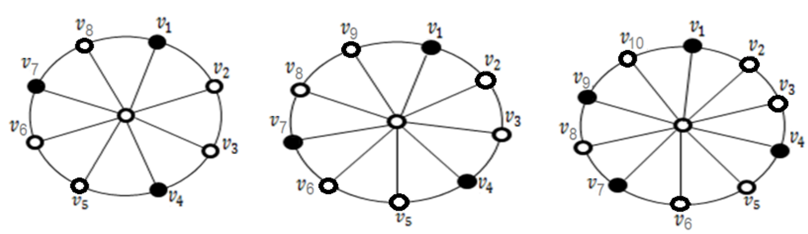
$$\gamma_{dm}(W_n) = \left\lceil \frac{n}{3} \right\rceil \quad \textit{for } n \geq 4$$

Proof. Since the wheel graph $W_n = C_n + K_1$. Let $v_1, v_2, \ldots, v_{n+1}$ be the vertices of W_n where $\deg(v_i) = 3$ for all $i = 1, 2, \ldots, n$ and $\deg(v_{n+1}) = n$. The three cases that follow can be obtained based on n in order to select a set D.

$$D = \begin{cases} \{v_{3i-2}, i = 1, 2, \dots, \frac{n}{3}\} & \text{if } n \equiv 0(\text{mod}3) \\ \{v_{3i-2}, i = 1, 2, \dots, \lfloor \frac{n}{3} \rceil - 1\} \cup \{v_{n-1}\} & \text{if } n \equiv 1(\text{mod}3) \\ \{v_{3i-2}, i = 1, 2, \dots, [\frac{n}{3}]\} & \text{if } n \equiv 2(\text{mod}3) \end{cases}$$

Case 1. If $n \equiv 0(\text{mod}3)$, assume that D has one vertex for from each three consecutive vertices of C_n. Hence, $D = \{v_{3i-2}, i = 1, 2, \dots, \frac{n}{3}\}$. Every vertex in D dominates three vertices, v_{n+1} and another two vertices adjacent with it, and $G[D]$ is disconnected graph. Therefore, D is $\gamma_{dm}-$ set and $\gamma_{dm} = |D| = \frac{n}{3}$.
Case 2. If $n \equiv 1(\text{mod}3)$, assume that $D = \{v_{3i-2}, i = 1, 2, \dots, [\frac{n}{3}] - 1\} \cup \{v_{n-1}\}$. Each vertex in D dominates three vertices, v_{n+1} and another two vertices adjacent with it, but there are two vertices v_1, v_{n-1} of D dominate v_n, v_{n+1} and another vertex, and $G[D]$ is disconnected graph. Hence, D is γ_{dm} - set and $\gamma_{dm} = |D| = \lceil \frac{n}{3} \rceil$.
Case 3. If $n \equiv 2(\text{mod}3)$. Let $D = \{v_{3i-2}, i = 1, 2, \dots, [\frac{n}{3}]\}$. Every vertex in D dominates three vertices, v_{n+1} and another two vertices adjacent with it, but there are two vertices v_1, v_{n-1} of D dominate v_n, v_{n+1} and another vertex, and $G[D]$ is disconnected graph. Hence, D is $\gamma_{dm}-$ set and $\gamma_{dm} = |D| = \lceil \frac{n}{3} \rceil$.

The set D is a minimum $DMEDS$ in each the previous three cases and the proof of it similar to the proof of Theorem 3.2. For example, see Fig 7.



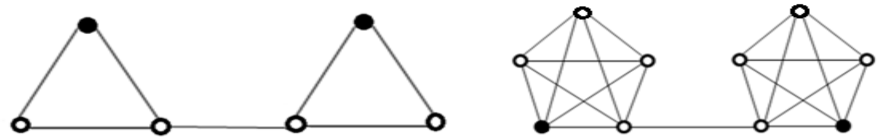
$n \equiv 2(mod3)$ $\quad$ $n \equiv 0(mod3)$ $\quad$ $n \equiv 1(mod3)$

FIGURE 7. A minimum disconnected multi-effect dominating set of W_n

□

Proposition 3.8. *For the barbell graph $B_{n,n} (n \geq 3)$, has disconnected multi-effect domination and $\gamma_{dm}(B_{n,n}) = 2$.*

Proof. Since $B_{n,n}$ have two copies of K_n joined by a bridge, and since K_n has no disconnected multi effect domination by Proposition 3.5. Then, D must be contains two non-adjacent vertices of $B_{n,n}$, and $G[D]$ is disconnected graph. So that every copy of complete graph contains one vertex from D that dominates the $n-1$ vertices of complete graph K_n. When the bridge must be lies on two vertices belong to $V - D$. For example, see Fig 8.



$B_{3,3}$ $B_{5,5}$

FIGURE 8. A minimum disconnected multi-effect dominating set of $B_{n,n}$

□

Definition 3.1. [11] *The corresponding barbell graph $B_n^n (n \geq 3)$ is a graph created by joined a bridge between each pair of corresponding vertices in two copies of the complete graph K_n (see Fig. 9), such that $V\left(B_{n,n}^c\right) = 2n$ and $E\left(B_{n,n}^c\right) = 2\binom{n}{2} + n$.*

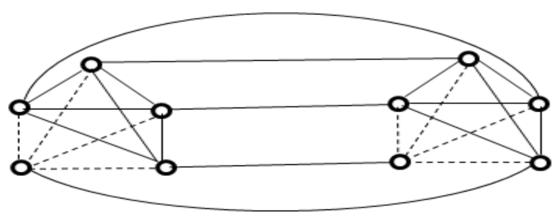
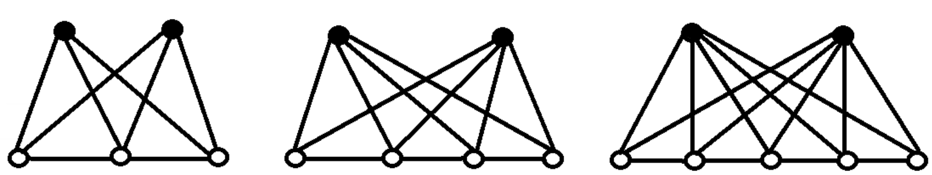


FIGURE 9. The corresponding barbell graph.

Proposition 3.9. *The corresponding barbell graph $B_{n,n}^c (n \geq 3)$, has disconnected multi-effect domination and $\gamma_{dm}\left(B_{n,n}^c\right) = 2$*

Proof. Since $B_{n,n}^c$ have two copies of K_n created by joined a bridge between every two corresponding vertices, and since K_n has no disconnected multi effect domination by Proposition 3.5. Then, D must be contains two non-adjacent vertices of $B_{n,n}^c$ and $G[D]$ is disconnected graph. Such that every copy of complete graph must contains one vertex in D dominates other vertices of K_n. For example, see Fig 10.



$B_{4,4}^c$ $B_{5,5}^c$

FIGURE 10. A minimum disconnected multi-effect dominating set of $B_{n,n}^C$.

□

Proposition 3.10. *If G is a fan graph F_n. Then F_n has disconnected multi-effect domination, where* $\gamma_{dm}(F_n) = \begin{cases} 2 & \text{if } n = 3 \\ \lceil \frac{n}{3} \rceil & \text{if } n \geq 4 \end{cases}$

Proof. Since the fan graph $F_n = P_n + K_1$, let $v_1, v_2, \ldots, v_{n+1}$ be the vertices of F_n where $\deg(v_1) = \deg(v_n) = 2$ and $\deg(v_i) = 3$ for $i = 2, 3, \ldots, n-1$ and $\deg(v_{n+1}) = n$. The four cases that follow can be obtained based on n in order to select a set D.

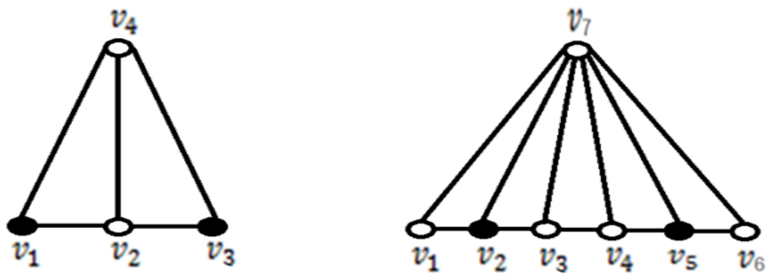
$$D = \begin{cases} \{v_1, v_3\} & \text{if } n = 3 \\ \left\{v_{3i-1}, i = 1, 2, \ldots, \frac{n}{3}\right\} & \text{if } n \equiv 0(\text{mod}3) \\ \left\{v_{3i-1}, i = 1, 2, \ldots, \frac{n-1}{3}\right\} \cup \{v_n\} & \text{if } n \equiv 1(\text{mod}3) \\ \left\{v_{3i-1}, i = 1, 2, \ldots, \frac{n+1}{3}\right\} & \text{if } n \equiv 2(\text{mod}3) \end{cases}$$

Case 1: If $n = 3$, let $D = \{v_1, v_3\}$, since there is one vertex in K_1 is joined all vertices of P_3. Then, each vertex of D dominate v_2 and v_4. Then, every vertex $v \in D$ dominates exactly two vertices and $G[D]$ is disconnected graph. Therefore, D is γ_{dm} - set and $\gamma_{dm} = |D| = 2$.

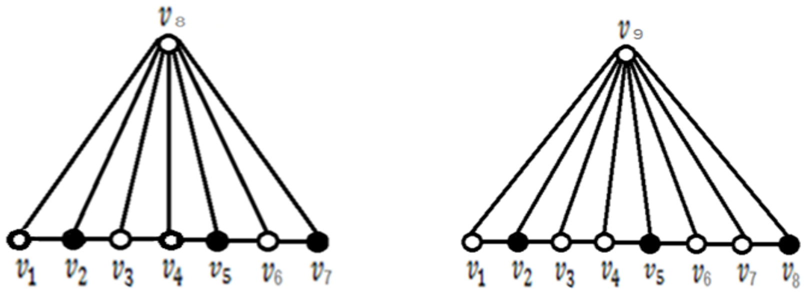
Case 2: If $n \equiv 0(\text{mod}3)$, assume that D has one vertex for from each three consecutive vertices of P_n. Hence, $D = \left\{v_{3i-1}, i = 1, 2, \ldots, \frac{n}{3}\right\}$. Every vertex in D dominates three vertices, v_{n+1} and another two vertices adjacent with it, and $G[D]$ is disconnected graph. Therefore, D is $\gamma_{dm}-$ set and $\gamma_{dm} = |D| = \left[\frac{n}{3}\right]$.

Case 3: If $n \equiv 1(\text{mod}3)$. Let $D = \left\{v_{3i-1}, i = 1, 2, \ldots, \frac{n-1}{3}\right\} \cup \{v_n\}$, every vertex in D dominates three vertices except the last vertex where it dominates two vertices. So, each vertex in D dominates two or three vertices and $G[D]$ is disconnected graph. Thus, D is $\gamma_{dm}-$ set and $\gamma_{dm} = |D| = \lceil \frac{n}{3} \rceil$.

Case 4: If $n \equiv 2(\text{mod}3)$. Let $D = \left\{v_{3i-1}, i = 1, 2, \ldots, \frac{n+1}{3}\right\}$, every vertex in D dominates three vertices except the last vertex where it dominates two vertices. So, each vertex in D dominates two or three vertices and $G[D]$ is disconnected graph. Thus, D is $\gamma_{dm}-$ set and $\gamma_{dm} = |D| = \lceil \frac{n}{3} \rceil$. The set D is a minimum $DMEDS$ in all the previous four cases and the proof of it similar to the proof of Theorem 3.2. For example, see Fig 11.



(a) F_3 (b) $n \equiv 0(mod3)$



(c) $n \equiv 1(mod3)$ (d) $n \equiv 2(mod3)$

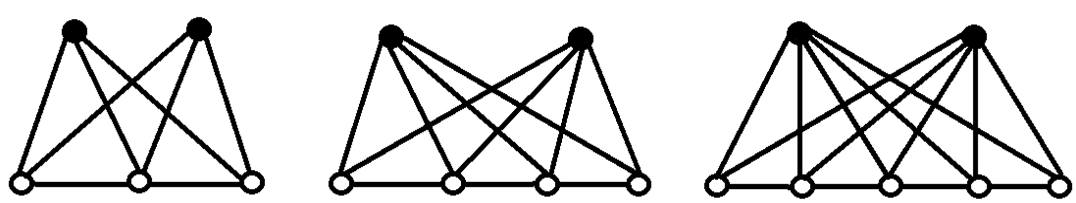
FIGURE 11. A minimum disconnected multi-effect dominating set of F_n.

□

Proposition 3.11. *Let G be the double fan graph $\left(P_n + \bar{k}_2\right)$, then $\gamma_{dm}\left(P_n + \bar{K}_2\right) = 2$ if $n \geq 2$.*

Proof. Let $D = \{v_1, v_2\} = V\left(\bar{K}_2\right)$, since every vertex in $\bar{K}_2$ is joined with all vertices in P_n. Then each vertex of D dominates all vertices in P_n. So $\gamma_{dm}\left(P_n + \bar{K}_2\right) = 2$

To proof that D is a minimum $DMEDS$. Let D' is a $DMEDS$ in graph G, such that $|D'| < |D|$, then there exist one or more vertices in $V - D$ don't dominated by any vertex of D' and $G[D']$ is connected graph. Hence, D' is not $DMEDS$ and D is the minimum $DMEDS$. For example, see Fig 12.



$P_3 + \bar{K}_2$ $P_4 + \bar{K}_2$ $P_5 + \bar{K}_2$

FIGURE 12. A minimum disconnected multi-effect dominating set of $P_n + \bar{K}_2$.

□

4. Conclusion:

Here, a novel kind of domination is known as "disconnected multi-effect domination" is presented. A relationship is determined between the disconnected multi-effect domination number and the order, size, minimum degree and maximum degree of the graph. This work creates a variety of standard graphs and some modified ones that allow the domination number to be calculated.

Authors' Contributions

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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