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NILPOTENT GRAPH OF A SEMIRING

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Abstract. In this paper, a new kind of graph on a commutative semiring is introduced. Let S be a commutative semiring with unity. The nilpotent graph of S, indicated by $\Gamma_{\mathrm{N}}(S)$, is a graph with vertex set $Z_N(S)^* = \{0 \neq a \in S \mid ab \in N(S) \text{ for some } 0 \neq b \in S\}$; and two vertices a and b are adjacent if and only if $ab \in N(S)$, where $N(S)$ is the set of all nilpotent elements of S. In this article, we investigate the simple properties of these graphs to relate the combinatorial properties of $\Gamma_{\mathrm{N}}(S)$ to the algebraic properties of the semiring S. We determine the diameter besides the girth of $\Gamma_{\mathrm{N}}(S)$. We also study the diameter of matrix algebras. We prove that if F is a semifield and $n \geq 3$, then $\operatorname{diam}(\Gamma_{\mathrm{N}}(M_n(F))) = 2$.

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1. Introduction

Semirings are helpful tools for resolving issues in a variety of information sciences besides applied mathematics fields, including automata, coding, graph, and optimization theories, besides computer program analysis. This is because the structure of semirings offers an algebraic method for analyzing and modeling the important variables in these fields.

In the past several years, the study of algebraic structures with graph possessions has gained a lot of attention and produced many intriguing findings in addition to intriguing questions. Assigning a graph to a ring is the subject of multiple works, for instance see, [1–10]. In addition, there are several papers on assigning a graph to semirings, for instance [11–17]. More papers study domination of graphs such as [21–26]. Every graph in this paper is simple with no loops and multiple edges. Zero-divisor graph of commutative semiring S was first described by [11]. It is represented by the graph $\Gamma(S)$ plus has two distinct vertices (x besides y) that are adjacent if $xy = 0$, with the vertex set $Z(S)^*$ is the set of nonzero zero-divisors in S. In contemporary times, a great deal of research has been done on

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zero-divisor graph of rings; see, for instance, [1–4]. Zero-divisor graph of a semiring has been studied in [11,12]. A graph structure on a ring R was defined by Chen in [5]. Its vertices are all of the elements in R, with the exception of two unique vertices x and y are adjacent if $xy \in N(R)$, wherever $N(R)$ is set of all nilpotent elements in R.

Throughout this paper, S is semiring with unity. A semiring is a set S equipped with binary operations + and $\cdot$ where $(S, +)$ is a commutative monoid with identity element 0 , besides $(S, \cdot)$ is a monoid with identity element 1 . Too, operations + and $\cdot$ are joined by distributivity and 0 annihilates S. A semiring is commutative if $ab = ba$ for all $a, b \in S$. Throughout section two of this paper presume S is commutative semiring with unity. The simplest examplee to commutative semirings is $\{0, 1\}$ the Boolean semiring, in which $1+ 1 = 1 \cdot 1 = 1$. Besides, the set of nonnegative integers (or reals) with the standard operations in addition and multiplication, is commutative semiring. A non-empty subset I in S is named an ideal in S if the next two conditions hold: (i) $x + y \in I$ for $x, y \in I$ (ii) $sx \in I$ for $s \in S$ besides $x \in I$. An ideal I in S is named k-ideal (subtractive ideal) if $z, z + w \in I$, then $w \in I$. $\{0\}$ is k-ideal in S by page 66 of [18]. S is named a subtractive semiring if every ideal in S is subtractive ideal. A semiring S is named semidomain whenever $a, b \in S$ with $ab = 0$ involves that either $a = 0$ or $b = 0$ [12]. A semifield is a semiring where a group under multiplication is formed by non-zero members [12] besides ([18], p. 52). We refer to Golan [18] for definitions of semiring theory and related terminology.

The nilpotent graph of rings was introduced in [6] besides [7]. Here, alike to [6] we define $\Gamma_{\mathrm{N}}(S)$ the nilpotent graph to semiring S. Presume S is a commutative semiring with zero element 0 . An element $a \in S$ is named nilpotent if there exists positive integer n such that $a^n = 0$, besides we symbolize $N(S)$ to be the set of all nilpotent elements in S. We represent $X\backslash\{0\}$ for each $X \subseteq S$ by X^*. The vertex set in $\Gamma_{\mathrm{N}}(S)$ is $Z_N(S)^*$, wherever $Z_N(S) = \{x \in S \mid xy \in N(S)$ for certain $y \in S^*\}$, besides two different vertices x besides y in $Z_N(S)^*$ are adjacent if and only if $xy \in N(S)$, similarly, $yx \in N(S)$. Relaxed to see that the usual zerodivisor graph $\Gamma(S)$ introduced by [11] besides [12] is subgraph in $\Gamma_{\mathrm{N}}(S)$. A semiring S is (von Neuman) regular in case for any $x \in S$ there is $y \in S$ with $xyx = x$. For semiring S, presume $Z(S)$ indicate the set of all zero-divisors in S, besides presume $|X|$ indicate the cardinality of the subset X in S. A semiring S is named reduced if S has no non-zero nilpotent elements. A semiring S is non-reduced if $N(S) \neq 0$. As a general rule, $N(S)$ is not an ideal in S. Alike to [6] we say S is named a null semiring if $S^2 = \{0\}$. Note when S is reduced, at that time $\Gamma_{\mathrm{N}}(S)$ is truly the general zero-divisor groph $\Gamma(S)$, as a result we focus our chief attention on non-reduced semirings.

A graph with vertex set $V(G)$ is denoted by G. A path from x toward y is series of adjacent vertices $x - x_1 - x_2 - \cdots - x_n - y.G$ is connected if a path connects each of G's two unique vertices; if not, it is disconnected. For $x, y \in V(G)$ with $x \neq y, d(x, y)$ indicates the length of shortest path from x into y, if such a path does not exist, one uses the convention $d(x, y) = \infty$. The diameter to G is defined as $\operatorname{diam}(G) = \sup\{d(x,y) \mid x$ besides y are vertices of $G\}$. For any $x \in V(G), \deg(x)$ symbolizes the number of edges incident with x, named the degree to x. A cycle is a path that starts besides ends at the same vertex, has no edges that are repeated, besides has different vertices at every point except the starting and finishing vertices. The girth of G, symbolized by $\operatorname{gr}(G)$, is the length of shortest cycle in $G(\operatorname{gr}(G) = \infty$ if G contains no cycles). When a graph's vertex set can be divided into two subsets, X and Y, such that each edge has one end in X plus one end in Y, the graph is named bipartite. Any bipartite graph with two partitions (X and Y) in which any vertex in X is linked toward any vertex in Y is said to be complete. A bipartite graph with part sizes 1 other than n for a given positive integer n is called a star graph. Apart from the notion of graph theory, we resort to Bondy in addition to Murty [19] for any ambiguous terminology.

For a semiring S, we symbolize using $M_n(S), S^n$ besides I, the semiring to all $n \times n$ matrices, the set to $n \times 1$ matrices over S besides the identity matrix, respectively. Too, for i besides $j, 1 \leq j, i \leq n$, one usage E_{ij} to indicate the element of $M_n(S)$ whose (i,j)-entry is 1 and other entries are 0.

In Section 2, in addition to studying nilpotent graphs over commutative semirings, we generalize conclusions from [6]. We show that if S be a commutativ semiring, at that time the graph $\Gamma_{\mathrm{N}}(S)$ is connected besides $\operatorname{diam}(\Gamma_{\mathrm{N}}(S)) \leq 3$. Too, if $\Gamma_{\mathrm{N}}(S)$ has cycle, at that time $\operatorname{gr}(\Gamma_{\mathrm{N}}(S)) \leq 4$. In addition, $\operatorname{gr}(\Gamma_{\mathrm{N}}(S)) = 3$ whenever S iis nonreduced.

In Section 3, we study the concept of nilpotent graphs to matrix semirings over semifields. We fix the diameter of the nilpotent graph to matrix semirings. We show that if F is a semifield and $n \geq 3$, then $\operatorname{diam}(\Gamma_{\mathrm{N}}(M_n(F))) = 2$. We show that if F is a semifield, at that time $\operatorname{diam}(\Gamma_{\mathrm{N}}(M_n(F))) \leq 3$.

2. The nilpotent graphs of commutative semirings $\Gamma_{\mathrm{N}}(S)$

This section is primarily meant to illustrate the connectedness, girth, besides diameter to $\Gamma_{\mathrm{N}}(S)$ to commutative semirings in addition to ordinary semirings. We start by the following definition.

Definition 2.1. *Let S be a commutative semiring with unity. The nilpotent graph of S, indicated by $\Gamma_{\mathrm{N}}(S)$, is an undirected simple graph with vertex set $Z_N(S)^* = \{0 \neq x \in S \mid xy \in N(S)$ for certain $0 \neq y \in S\}$; and two different vertices x and y are adjacent if and only if $xy \in N(S)$, where $N(S)$ is the set of all nilpotent elements of S.*

It is commonly understood that if and only if $\Gamma_{\mathrm{N}}(R)$ is empty, then a ring R is domain, see ([6], Remark 1). A similar result holds for semidomain in the next remark.

Remark 2.1. *A semiring S is semidomain if and only if $\Gamma_{\mathrm{N}}(S)$ is empty. Actually, if S is semidomain, at that time $Z_N(S)^* = \emptyset$, besides as a result $\Gamma_{\mathrm{N}}(S)$ is empty. In opposition, if $\Gamma_{\mathrm{N}}(S)$ is empty, at that time by definition,* S *have no nonzero zero-divisors according to $\Gamma_{\mathrm{N}}(S)$. As a result S is semidomain.*

Presume $S = \mathbb{Z}_p$, besides $T = \mathbb{N}_0$ are the semiring of integers $\mathrm{mod} p$ wherever p is prime number, besides the semiring of non-negative integers by Remark 2.1 , one has $\Gamma_{\mathrm{N}}(S)$ besides $\Gamma_{\mathrm{N}}(T)$ contain no vertices.

Remark 2.2. *Presume S is regular commutative semiring with identity, then any element in $S/Z(S)$ is unit. Actually, note for $x \in S$, there is $y \in S$ with $xyx = x$. Now if $x \notin Z(S)$, at that time $xy = 1$, henceforth x is unit.*

Remark 2.3. *Presume S is commutative semiring with unity or regular semiring. When S is reduced, at that time $|S| \leq |Z(S)|^2 = |Z_N(S)|^2$, in any case $|Z_N(S)^*| = |S| - 1$. Really, when S is reduced, at that time $\Gamma_{\mathrm{N}}(S)$ is really zerro-divisor graph $\Gamma(S)$. Thus, one has $|S| \leq |Z(S)|^2 = |Z_N(S)|^2$ via Rem 1 of [1]. If S is nonreduced commutative semiring, now r besides x are adjacent, for $r \in S^*$ besides $x \in N(S)^*$. As a result $Z_N(S) = S$. At present we presume S is nonreduced regular semiring. Presume r is a random element in S^*. If $r \in Z(S)$, now $Z_N(S) = S$ as $Z(S) \subseteq Z_N(S)$, besides so we are done. else, via Rem 2.2, r is unit besides one indicates an inverse element using r^{-1}. Presume x is nonzero nil-element. It is relaxed to see this $r^{-1}x \neq 0, r$, besides r is adjacentt toward $r^{-1}x$. As a result $r \in Z_N(S)$, besides thus $|Z_N(S)| = |S|$.*

The next two theorems can be thought of as a generalization of Theorems 2.1 besides 2.2 of [6] or Theorems 2.1 besides 2.2 of [7].

Theorem 2.1. *Let S be a commutative semiring with unity. Now the next hold:*

(1) *$\Gamma_{\mathrm{N}}(S)$ is connected.*
(2) $\mathrm{diam}\,(\Gamma_{\mathrm{N}}(S)) \leq 3$.
(3) *When $\Gamma_{\mathrm{N}}(S)$ contains a cycle, at that time $\mathrm{gr}\,(\Gamma_{\mathrm{N}}(S)) \leq 4$. Besides, $\mathrm{gr}\,(\Gamma_{\mathrm{N}}(S)) = 3$, whenever S is non-reduced.*

Proof. First one shown (1) besides (2). Presume $x, y \in Z_N(S)^*$ besides $x \neq y$. Now there exist $x_1, y_1 \in Z_N(S)^*$ such that $xx_1, yy_1 \in N(S)$ utilizing the $\Gamma_{\mathrm{N}}(S)$ definition.

Case 1: If $xy \in N(S)$, at that time x besides y are adjacent besides henceforth $d(x, y) = 1$.

Case 2: Presume $xy \notin N(S)$. If $x_1y_1 \neq 0$, then $x - x_1y_1 - y$ is the shortest trail from x to y besides as a result $d(x, y) = 2$. If $x_1y_1 = 0$, at that time $x - x_1 - y_1 - y$ is a trail (not necessarily the shortest trail) from x to y besides as a result $d(x, y) \leq 3$. The proof of (1) and (2) is thus concluded.

(3) When S a reduced, at that time $\Gamma_{\mathrm{N}}(S)$ a truly thee zero-divisor graph $\Gamma(S)$. So when $\Gamma_{\mathrm{N}}(S)$ contains cyclee, now by (1.4) of [2], $\mathrm{gr}\,(\Gamma_{\mathrm{N}}(S)) \leq 4$. At this time we presume S is nonreduced besides $\Gamma_{\mathrm{N}}(S)$ contains aa cycle of size n, wherever $n \geq 4$. Without loss of generalization, presume $x_0 - x_1 - \cdots - x_{n-1} - x_0$ is such a cycle. Examine the next two instances:

Case 1: Presume all x_i 's are non-nilpotent elements. As S is nonreduced, there is nonzero nilpOtent element a with $a - x_{i-1} - x_i - a$ is triangle, wherever $1 \leq i \leq n$. Thus $\mathrm{gr}\,(\Gamma_{\mathrm{N}}(S)) = 3$

Case 2: Presume nilpotency of at least one of x_i 's, give or take x_0. Note x_0 is adjacent to each x_i, wherever $1 \le i \le n-1$, as a result $x_0 - x_i - x_{i+1} - x_0$ is triangle, wherever $1 \le i \le n-2$, henceforth $\mathrm{gr}\,(\Gamma_{\mathrm{N}}(S)) = 3$. □

An ideal K in commutative semiring S is prime if and only if $zw \in K$ involves that $z \in K$ or $w \in K$ ([18], Corollary 7.6). Given now an overt description for a nonreduced commutative semiring where $\Gamma_{\mathrm{N}}(S)$ contains no cycles.

Theorem 2.2. *Presume S is a non-reduced commutative semiring with unity besides $\boldsymbol{\Gamma}_{\mathrm{N}}(\boldsymbol{S})$ is not a singleton. Then the next claims are now equivalent*

(1) $\mathrm{gr}\,(\Gamma_{\mathrm{N}}(S)) = \infty$;
(2) $\Gamma_{\mathrm{N}}(S)$ *is a star graph;*
(3) *S is either null semiring of order 3 , or $|N(S)| = 2$ besides $N(S)$ is prime ideal in S.*

Proof. $(1) \Leftrightarrow (2)$. Obviously, when $\Gamma_{\mathrm{N}}(S)$ is star groph now its diameter is ∞. On the other hand, if $\mathrm{gr}\,(\Gamma_{\mathrm{N}}(S)) = \infty$, at that time $\Gamma_{\mathrm{N}}(S)$ is star graph since it has a vertex that is next toward very other vertices for a nonreduced commutative semiring.

$(1) \Leftrightarrow (3)$. The sufficiency is readily apparent. Assume that $\mathrm{gr}\,(\Gamma_{\mathrm{N}}(S)) = \infty$ for the opposite direction. If $|N(S)| \ge 3$, at that time $\Gamma_{\mathrm{N}}(S)$ contains triangle, a illogicality. Henceforth $|N(S)^*| \le 2$. Presume the two scenarios listed below:

Case 1: Presume $|N(S)^*| = 2$, at that time it is obvious this $S = N(S)$. Presume $S = \{0, a, b\}$. If $a^2 \neq 0$, at that time $a^2, a, a + a^2$ are pairwise different elements in S^*, a conflict (note that when $a + a^2 = 0$, then a is not nilpotet). So $a^2 = b^2 = 0$. If $ab \neq 0$, then assume without loss that $ab = a$. Since $0 = ab^2 = abb = ab = a$, a conflict occurs. As a result $ab = 0$ besides S is null semiring.

Case 2: Presume $|N(S)^*| = 1$. Set $x, y \in S \backslash N(S)$. If $x = y$, at that time $xy = x^2 \notin N(S)$, for otherwise $x \in N(S)$, an illogicality. If $x \neq y$, at that time $xy \notin N(S)$, for otherwise, there is a triangle $x - y - a - x$, where $a \in N(S)^*$, an illogicality again. $N(S)$ is as a result prime ideal. □

We shall go through the typical rings' zero–divisor graphs in the next. The next theorem is like to Theorem 2.3 in [7].

Theorem 2.3. *Presume S is a regular semiring with identity 1. At that time $\Gamma_{\mathrm{N}}(S)$ is connectoed besides* $\mathrm{diam}\,(\Gamma_{\mathrm{N}}(S)) \le 3$.

Proof. Observe that if S is reduced, it now becomes special case to $\Gamma(S)$ of Thm 2.3 in [3] according to the definition of $\Gamma_{\mathrm{N}}(S)$. At this time presume S is non-reduced. At that time $Z_N(S) = S$ using Rem 2.4. As S is regular, one see thatt any elementt in S is exactly either unit or of $Z(S)$ via Rem 2.3. Presume n is the lowest positive integer with $x^n = 0$ and x be a nonzero nilpotent element. One establishes the existence of a path from r_1 to r_2 of length no more than 3 for every two different vertices, r_1 plus r_2, in $\Gamma_{\mathrm{N}}(S)$. One examines next cases

Case 1: $r_1, r_2 \in S \backslash Z(S)$. Note that $xr_1^{-1} \neq 0, r_1$, and $r_2^{-1}x^{n-1} \neq 0, r_2$. It is easy to see $r_1 - xr_1^{-1} - r_2^{-1}x^{n-1} - r_2$ is a path from r_1 to r_2 and as a result diam $(\Gamma_{\mathrm{N}}(S)) \leq 3$.

Case 2: $r_1 \in S \backslash Z(S), r_2 \in Z(S)^*$. If r_2 is left zeroo-divisor, at that time there exists a nonzero element b with $r_2 b = 0$. If $bx = 0$, at that time $r_1 - xr_1^{-1} - b - r_2$ is a path from r_1 toward r_2. If $bx \neq 0$, then $r_1 - x^{n-1}r_1^{-1} - bx - r_2$ is a path from r_1 toward r_2. Likewise, one can show it as soon as r_2 is right zero-divisor.

Case 3: $r_1, r_2 \in Z(S)^*$. If $r_1 r_2 \in N(S)$, then $r_1 - r_2$ is a path from r_1 to r_2. Now presume that $r_1 r_2 \notin N(S)$. If there is $a, b \in S^*$ with $r_1 a = br_2 = 0$, then one obtains a path $r_1 - a - b - r_2$ when $ab = 0$ or a path $r_1 - ab - r_2$ when $ab \neq 0$. If there exist $a, b \in S^*$ such that $r_1 a = r_2 b = 0$ and $br_2 \neq 0$ then there is a path $r_1 - abr_2 - br_2 - r_2$ when $abr_2 \neq 0$ or a path $r_1 - a - br_2 - r_2$ when $abr_2 = 0$. So, $\Gamma_{\mathrm{N}}(S)$ is joined besides diam $(\Gamma_{\mathrm{N}}(S)) \leq 3$. □

Example 2.1. *An inspection will shows that a set $SP_4 = \{0, 1, 2, b\}$ equipped with operations + besides · defined as: is a semiring with unity (which is is not ring) see [13]. $V(\Gamma_{\mathrm{N}}(SP_4)) = \{2, b\}$. Here, $2 \cdot b = 0 \in N(S)$.*

+	0	1	2	b
0	0	1	2	b
1	1	2	1	2
2	2	1	2	1
b	b	2	1	0

.	0	1	2	b
0	0	0	0	0
1	0	1	2	b
2	0	2	2	0
b	0	b	0	b

As a result, $2 - b$ is a path from 2 to b. So, 2 and b are adjacent in $\Gamma_{\mathrm{N}}(SP_4)$. As a result diam $(\Gamma_{\mathrm{N}}(SP_4)) = 1$

FIGURE 1. $\Gamma_{\mathrm{N}}(SP_4)$

Example 2.2. *$\{2, 4, 6\}$ is the set of all nontrivial nilpptent elements in $\mathbb{Z}_8$ in commutative semiring $\mathbb{Z}_8$. Henceforth $N(\mathbb{Z}_8) = \{2, 4, 6\}$. Thus, $V(\Gamma_{\mathrm{N}}(\mathbb{Z}_8)) = \{2, 4, 6\} \cup \{1, 3, 5, 7\} = \{1, 2, 3, 4, 5, 6, 7\}$, since $1 \cdot 4 = 3 \cdot 4 = 5 \cdot 4 = 7 \cdot 4 = 4 \in N(\mathbb{Z}_8)$. It is clear that any vertex x of $\{2, 4, 6\}$ is adjacent to every vertex of $\Gamma_{\mathrm{N}}(\mathbb{Z}_8) \backslash \{x\}$. Clearly, $\Gamma_{\mathrm{N}}(\mathbb{Z}_8)$ is connected graph besides diam $(\Gamma_{\mathrm{N}}(\mathbb{Z}_8)) = 2$.*

Theorem 2.4. *Assume I is non-zero proper ideal in S. Now $\Gamma_{\mathrm{N}}(I)$ is complete whenever $\Gamma_{\mathrm{N}}(S)$ is complete.*

Proof. Presume S is semiring besides I be non-zero proper ideal in S. Presume x, y are any two vertices in $\Gamma_{\mathrm{N}}(I)$. Now $x \cdot z, y \cdot w \in N(I)$, for certain $0 \neq z, w$ belong to $I \leq S$. So, $x, y \in V(\Gamma_{\mathrm{N}}(S))$. As $\Gamma_{\mathrm{N}}(S)$ is complete now x, y are adjacent in $\Gamma_{\mathrm{N}}(S)$. As a result, x, y are adjacent in $\Gamma_{\mathrm{N}}(I)$. Consequently, $\Gamma_{\mathrm{N}}(S_1)$ is complete graph. □

3. The nilpotent graph of matrix algebras

In this section, we show that if F is a semifield besides $n \geq 3$, at that time $\operatorname{diam}(\Gamma_{\mathrm{N}}(M_n(F))) = 2$. We show that if F is a semifield, then $\operatorname{diam}(\Gamma_{\mathrm{N}}(M_2(F))) \leq 3$. In this section the semiring not necessary to be commutative. Begin by the next remark.

Remark 3.1. *If $S = M_n(F)$, where F is a semifield besides $n \geq 2$, then all nonzero element in S is vertex in $\Gamma_{\mathrm{N}}(S)$. Really, if A is aa non-singular matrix, at that time A adjacent toward $A^{-1}E_{1n}$ besides as a result $A \in V(\Gamma_{\mathrm{N}}(S))$. Too, if A is a singular matrix, at that time $AY = 0$ for certain $0 \neq Y \in S$. As a result $A \in V(\Gamma_{\mathrm{N}}(S))$.*

The next two theorems can be thought of as a generalization of ([8], Theorems 1 and 2).

Theorem 3.1. *If F is a semifield besides $n \geq 3$, then $\operatorname{diam}(\Gamma_{\mathrm{N}}(M_n(F))) = 2$.*

Proof. Presume that $A, B \in M_n(F)$ and $C = [0 \mid X]$, where $X \in F^n$. Then $AC = [0 \mid AX]$ and $BC = [0 \mid BX]$. Presume that $W_1 = \{X \in F^n \mid A_n X = 0\}$ besides $W_2 = \{X \in F^n \mid B_n X = 0\}$, where A_n besides B_n are the nth rows of A besides B, respectively. Both W_1 besides W_2 are subspaces of F^n. One has $\dim W_i \geq n-1$, for $i = 1, 2$. As $n \geq 3$, there is $0 \neq X_0 \in W_1 \cap W_2$. Presume $C = [0 \mid X_0]$. Obviously, C is adjacent to both A besides B. Now $\operatorname{diam}(\Gamma_{\mathrm{N}}(M_n(F))) \leq 2$. However, E_{nn} besides I are two non-adjacent vertices in $\Gamma_{\mathrm{N}}(M_n(F))$. As a result $\operatorname{diam}(\Gamma_{\mathrm{N}}(M_n(F))) = 2$. □

Theorem 3.2. *If F is a semifield, at that time $\operatorname{diam}(\Gamma_{\mathrm{N}}(M_2(F))) \leq 3$.*

Proof. Presume $A, B \in M_2(F)$ and X be a nilpotent matrix in $M_2(F)$. We have the next cases:
Case 1: A besides B are non-singular matrices. Now $A - XA^{-1} - B^{-1}X - B$ is a path.
Case 2: While B, A are a singular and non-singular matrix, respectively. Now $BY = 0$ for some $0 \neq Y$. If $YX = 0$, at that time $A - XA^{-1} - Y - B$ is a trail (path). If $YX \neq 0$, at that point $A - XA^{-1} - YX - B$ is a trail.
Case 3: A besides B a singularr matricess. If AB is nilpotent, now $A - B$ is trail. Else, there is $Y, X \neq 0$ with $AX = 0$ besides $YB = 0$. If $XY = 0$, now $A - X - Y - B$ is a trail. Too, $A -\ XY - B$ is trail (path), in place of $XY \neq 0$. As a result $\operatorname{diam}(\Gamma_{\mathrm{N}}(M_2(F))) \leq 3$. □

Theorem 3.3. *Let S be an additively regular subtractive semisimple finite semiring, then $\operatorname{diam}(\Gamma_{\mathrm{N}}(S)) \leq 3$.*

Proof. As S is finit semisimple semiring, he is an Artinian besides as a result the Jacobson radical to S, $J(S)$ is nilpotent by Theorem 4.4 in [20]. We have the next cases:

Case 1: $J(S) \neq 0$. Presume $0 \neq x \in J(S)$. Evidently, all vertex in $\Gamma_{\mathrm{N}}(S)$ is adjacent to x besides as a result $\operatorname{diam}(\Gamma_{\mathrm{N}}(M_2(S))) \leq 2$.

Case 2: $J(S) = 0$. As S is additively regular subtractive semisimple semiring using Theorem 4.14 of [20], $S \cong D_1 \times \cdots \times D_n \times M_{n_1}(T_1) \times \cdots \times M_{n_t}(T_t)$ for suitable additively regular division semirings $D_1, \ldots, D_n$ besides division rings $T_1, \ldots, T_r (n \geq 0, t \geq 0)$. Since S is a finite semiring by using Theorem 4 in [8], $\operatorname{diam}(\Gamma_{\mathrm{N}}(S)) \leq 3$. By the proof of Theorem 3.3 the following result is obvious. □

Corollary 3.1. *If S is an additively regular subtractive semisimple finite semiring with $J(S) \neq 0$. Then* $\operatorname{diam}(\Gamma_{\mathrm{N}}(S)) \leq 2$.

4. Conclusion

This work defines and studies the nilpotent graph of a commutative semiring $S, \Gamma_{\mathrm{N}}(S)$, an undirected graph. We examined the girth, diameter, besides connectedness of $\Gamma_{\mathrm{N}}(S)$. We observed that if S is commutative semiring then $\Gamma_{\mathrm{N}}(S)$ is connected and $\operatorname{diam}(\Gamma_{\mathrm{N}}(S)) \leq 3$. If $\Gamma_{\mathrm{N}}(S)$ contains a cycle, then $\operatorname{gr}(\Gamma_{\mathrm{N}}(S)) \leq 4$. Besides, if S is nonreduced, then $\operatorname{gr}(\Gamma_{\mathrm{N}}(S)) \leq 3$.

Authors' Contributions

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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