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EFFICIENTLY ADDRESSING FRACTIONAL-ORDER POPULATION DIFFUSION EQUATIONS: KAMAL RESIDUAL POWER SERIES METHOD

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Abstract. In this study, we utilized the Kamal residual power series method to solve the fractional-order population diffusion equation in the Caputo sense. This method combines the residual power series method with the Kamal transformation integral. The procedure starts by defining the approximate solution of a power series with unknown coefficients; the residual function is then constructed. By imposing the condition, the coefficients can be easily calculated, and finally, the approximate series solution is found. Three different figures were used to evaluate the strategy's accuracy and effectiveness. This method offers a significant advantage: it negates the requirement for computing Adomian polynomials, considering perturbation processes, or performing linearization.

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Key words and phrases. residual power series method; population diffusion equations; fractional derivative; Kamal transform; approximation solutions.

1. Introduction

The field of fractional calculus is a significant area of mathematics that has been extensively studied. Research into fractional calculus has rapidly progressed, providing scholars with new tools to solve complex problems and facilitating the development of more accurate mathematical models. Mathematical models involving fractional-order derivatives, both linear and nonlinear, have been attracting the attention of academics in many application fields. In particular, nonlinear fractional-order differential equations are known to be extremely complex and difficult to conquer. One of these sorts of equations is the population diffusion model.

The Kamal transformation is an essential technique utilized in a variety of applications. It was first introduced by A. Kamal in 2016 to solve linear ordinary differential equations with constant

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coefficients [1]). Since then, this Laplace-like transform has grown in popularity among scholars for solving a variety of mathematical models of scientific problems, including partial integrodifferential equations [8], delay differential equations [19], and second-order linear Volterra integrodifferential equation systems [20]. The Kamal transformation is not only useful for classical calculus problems but also for problems related to the field of fractional calculus.

Oderinu et al. [18] utilized the Kamal transform in solving the linear time-fractional differential equations. Samar and Saxena [22], [23] applied the Kamal transform to obtain the solutions to some fractional differential equations and generalized fractional kinetic equations. Johansyah et al. [11] proposed the approximate solution of the Riccati fractional differential equation and Economic Growth Model by using the Kamal transform method. Khandelwal [13] presented the solution to the non-homogenous fractional ordinary differential equation by using the Kamal transform. Johansyah et al. [10] used the Combined Adomian Decomposition Method with Kamal Integral Transformation and the Kamal integral transform to solve differential equations of fractional order.

The residual power series method (RPSM) is a tool that can be used to solve scientific issues simply and powerfully. The semi-analytic method has been gaining more and more attention from scientists and engineers for the past decade. In 2013, Mohammed H. and Al-Smadi [5] used the RPSM to solve initial value problems of linear and nonlinear first-order differential equations, whereas Omar et al. [2] applied this technique to gain the solution for initial value problems of linear and nonlinear high-order differential equations. Moreover, Amit et al. [16] have shown that the RPSM is a reliable method and is easily applied to all types of fractional nonlinear problems arising in science and technology. Later in 2016, Fairouz et al. [24] obtained the solution of time fractional reaction-diffusion equations by RPSM, while Amit et al. [16] solved the fractional Sharma-Tasso-Olever equations. Moreover, Iryna et al. [15] found the solution of the Fredholm integral equations by using RPSM. Jaradat et al. [9] applied RPSM to find the solution of time-fractional Drinfeld-Sokolov-Wilson equations. In 2021, Marwan Alquran et al. [4] used the RPSM method to solve a system of n autonomic equations with nonlinear fractional dimensions.

This work is motivated by the study of Zhang et al. in [26]. The primary objective of this investigation is to utilize the Kamal integral transform and RPSM to accomplish the initial value problem for fractional-order population diffusion equations,

$$D_t^{\alpha} u(x,y,t) = (u^2(x,y,t))_{xx} + (u^2(x,y,t))_{yy} + \sigma(u(x,y,t)), \tag{1.1}$$

$$u(x,y,0) = g(x,y), \tag{1.2}$$

$t > 0,\ 0 < \alpha \leqslant 1$. Here, D_t^{α} is the Caputo fractional derivative with respect to t, u represents population density, and $\sigma(u)$ indicates population births and deaths.

2. Basic concepts of fractional calculus, Kamal transform and residual power series

This section provides the definition of Caputo fractional derivatives, the Riemann-Liouville fractional integral, and its important properties. Moreover, the Kamal transform definition and the related concept of residue power series are reviewed.

Definition 2.1. [14] *The Euler gamma function is defined by*

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt,\ z\in\mathbb{C},\ \textit{with}\ \Re(z)>0.$$

Definition 2.2. [14] *The Riemann-Liouville fractional integral of function $u:(0,\infty)\to\mathbb{R}$, for $\alpha\in\mathbb{R}^+$ is defined by*

$$I^\alpha u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}u(\tau)d\tau$$

and $I^\alpha u(t)=u(t)$ if $\alpha=0$.

Definition 2.3. [14] *The Caputo fractional derivative of function $u:(0,\infty)\to\mathbb{R}$, of order α is defined as*

$$^cD^\alpha u(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^\infty(t-\tau)^{n-\alpha-1}u^{(n)}(\tau)d\tau,\quad n-1<\alpha<n$$

and $^cD^\alpha u(t)=u^{(n)}(t)$ if $\alpha=n\in\mathbb{N}$.

For the convenience of writing, we will commence employing D^α in the place of $^cD^\alpha u(t)$. The important properties of Caputo fractional derivatives and Riemann-Liouville fractional integrals are briefly mentioned below.

(1) $I^\alpha t^\beta=\frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}t^{\alpha+\beta},\ \beta>-1.$

(2) $D^\alpha\mu=0$ for any constant μ.

(3) $D^\alpha t^\beta=\begin{cases}\dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}, & \beta\geqslant\alpha,\\ 0, & \beta<\alpha.\end{cases}$

(4) $D^\alpha I^\alpha u(t)=u(t)$,

(5) $I^\alpha D^\alpha u(t)=u(t)-\sum_{k=0}^{n-1}\frac{u^{(k)}(0)}{k!}t^k$

Definition 2.4. [14] *Mittag-Leffler function is the generalization of exponential function denoted by $E_\alpha(z)$ defined as*

$$E_\alpha(z)=\sum_{k=0}^\infty\frac{z^k}{\Gamma(k\alpha+1)},\ \alpha\in\mathbb{R}^+,\ z\in\mathbb{C}$$

Consider set of functions defined as follows

$$\mathcal{A}=\left\{u(t):\exists M,k_1,k_2>0,|u(t)|<Me^{\frac{|t|}{k_j}},\ \text{if}\ t\in(-1)^j\times[0,\infty)\right\}$$

Definition 2.5. *The Kamal transform of function $u(t) \in \mathcal{A}$ is defined by*

$$\mathcal{K}[u(t)] = U(v) = \int_0^\infty u(t)e^{-\frac{t}{v}}dt, \ t \geqslant 0, \ k_1 \leqslant v \leqslant k_2.$$

Additionally, the inverse of the Kamal transformation is denoted by $\mathcal{K}^{-1}[U(v)] = u(t), \ t \geqslant 0$. The operators $\mathcal{K}(\cdot)$ and $\mathcal{K}^{-1}(\cdot)$ are called the Kamal operator and the inverse Kamal operator, respectively. In this case, the variable v is the factor of the variable t in the function u argument. This converges over a certain interval. The Kamal transform exists if $u(t)$ for $t \geqslant 0$ is of exponential order, piecewise continuous, and function; otherwise, it does not exist.

Remark 2.1. *The operators $\mathcal{K}$ and $\mathcal{K}^{-1}$ are linear, and the Kamal transform of fundamental functions is shown in* [1].

Theorem 2.1. [12] *If $n \in \mathbb{N}$ where $n-1 < \alpha \leqslant n$ and $\mathcal{K}[u(t)] = U(v)$, the Kamal transformation of the Caputo fractional derivative of order $\alpha > 0$ is defined as*

$$\mathcal{K}[{}^cD^\alpha u(t)] = \frac{U(v)}{v^\alpha} - \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{v^{\alpha-k-1}}, \quad n-1 < \alpha \leqslant n.$$

Definition 2.6. [7] *If $n \in \mathbb{N}$ where $n-1 < \alpha \leqslant n$. A power series expansion of the form*

$$\sum_{n=0}^{\infty} a_n(t-t_0)^{n\alpha} = a_0 + a_1(t-t_0)^\alpha + a_2(t-t_0)^{2\alpha} + \ldots, t \leqslant t_0$$

is called fractional power series about t_0.

Theorem 2.2. [7] *Suppose that $u(t)$ has a fractional power series representation at $t = t_0$ of the form*

$$u(t) = \sum_{n=0}^{\infty} a_n(t-t_0)^{n\alpha}, \quad t_0 \leqslant t < t_0 + R.$$

If $D^{n\alpha}u(t), \ n = 0, 1, 2, \ldots$ are continuous on (t_0, t_0+R), then $a_n = \frac{D^{n\alpha}u(t)}{\Gamma(n\alpha+1)}$.

Definition 2.7. [7] *A power series of the form*

$$\sum_{n=0}^{\infty} f_n(x)(t-t_0)^{n\alpha} = f_0(x) + f_1(x)(t-t_0)^\alpha + f_2(x)(t-t_0)^{2\alpha} + \ldots$$

is called multiple fractional power series about $t = t_0$, where t is a variable and f_m's are functions of x called the coefficients of the series.

Theorem 2.3. [7] *Suppose that $u(x,t)$ has a multiple fractional power series representation at $t = t_0$ of the form*

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty} f_n(x)(t-t_0)^{n\alpha},$$

$0 < n-1 < \alpha \leqslant n$, $x \in I$, $t_0 \leqslant t < t_0 + R$. *If* $D_t^{n\alpha} u(x,t)$, $n = 0,1,2,\ldots$ *are continuous on* $I \times (t_0, t_0 + R)$, *then*

$$f_n(x) = \frac{D_t^{n\alpha} u(x,t)}{\Gamma(n\alpha + 1)},\ n = 0,1,2,\ldots$$

Here $D_t^{n\alpha}(\cdot) = \frac{\partial^{n\alpha}}{\partial t^{n\alpha}}(\cdot) = \frac{\partial^{\alpha}}{\partial t^{\alpha}}(\frac{\partial^{\alpha}}{\partial t^{\alpha}}(\ldots(\frac{\partial^{\alpha}}{\partial t^{\alpha}}(\cdot))))$ $(n - times)$, *and* $R = \min_{c \in I} R_c$, *in which* R_c *is the radius of convergence of the fractional power series* $\sum_{k=0}^{\infty} f_n(c)(t - t_0)^{n\alpha}$.

According to the convergence of the classic residual power series method, there is a real number $0 < \lambda < 1$, such that $||u_{n+1}(x,t)|| \leqslant \lambda ||u_n(x,t)||$, $t \in (t_0, t_0 + R)$.

3. Main results

This section contains the discussion of Kamal RPSM (KRPSM) in solving the fractional diffusion population equations as well as the examples that support the proposed concepts.

3.1. **Implementation of KRPSM in the Fractional Population Diffusion Model.** Consider the fractional population diffusion equation

$$D_t^{\alpha} u(x,y,t) = (u^2(x,y,t))_{xx} + (u^2(x,y,t))_{yy} + \sigma(u(x,y,t)), \tag{3.1}$$

subjects to the initial condition

$$u(x,y,0) = g(x,y).$$

Applying the Kamal transform to (3.1), using linearity and utilizing the Kamal transform of fractional derivative, one finds that

$$\begin{aligned} \frac{1}{v^{\alpha}} \mathcal{K}\{u(x,y,t)\} - \frac{1}{v^{\alpha-1}} u(x,y,0) &= \mathcal{K}\{(u^2(x,y,t))_{xx}\} \\ &+ \mathcal{K}\{(u^2(x,y,t))_{yy}\} + \mathcal{K}\{\sigma(u(x,y,t))\}. \end{aligned}$$

Imposing the initial condition and rearranging the equation leads to

$$\begin{aligned} \mathcal{K}\{u(x,y,t)\} &= v^{\alpha} \mathcal{K}\Big\{(u^2(x,y,t))_{xx}\Big\} + v^{\alpha} \mathcal{K}\Big\{(u^2(x,y,t))_{yy}\Big\} \\ &+ v^{\alpha} \mathcal{K}\Big\{\sigma(u(x,y,t))\Big\} + v g(x,y). \end{aligned}$$

After taking the inverse Kamal transform on both sides of the equation, we get

$$\begin{aligned} u(x,y,t) &= \mathcal{K}^{-1}\Big\{v^{\alpha} \mathcal{K}\Big\{(u^2(x,y,t))_{xx}\Big\}\Big\} + \mathcal{K}^{-1}\Big\{v^{\alpha} \mathcal{K}\Big\{(u^2(x,y,t))_{yy}\Big\}\Big\} \\ &+ \mathcal{K}^{-1}\Big\{v^{\alpha} \mathcal{K}\Big\{\sigma(u(x,y,t))\Big\}\Big\} + g(x,y). \end{aligned}$$

Suppose the solution to the problem is expressed in the infinite series

$$u(x,y,t) = \sum_{n=0}^{\infty} f_n(x,y) \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)}. \tag{3.2}$$

According to the RPSM, we define the residual function as follows :

$$\mathrm{Res}_{\infty}(x,y,t) = u(x,y,t) - g(x,y) - \mathcal{K}^{-1}\Big\{v^{\alpha}\mathcal{K}\Big\{(u^2(x,y,t))_{xx}\Big\}\Big\} - \mathcal{K}^{-1}\Big\{v^{\alpha}\mathcal{K}\Big\{(u^2(x,y,t))_{yy}\Big\} - \mathcal{K}^{-1}\Big\{v^{\alpha}\mathcal{K}\Big\{\sigma(u(x,y,t))\Big\}\Big\}, \tag{3.3}$$

and the kth truncated series of (3.2) is denoted by

$$u_k(x,y,t) = \sum_{n=0}^{k} f_n(x,y)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}.$$

Since the solution satisfied the initial condition, it leads to $f_0(x,y) = g(x,y)$. Hence, the $k-$order approximated solution became

$$u_k(x,y,t) = g(x,y) + \sum_{n=1}^{k} f_n(x,y)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{3.4}$$

Moreover, the $k-$residual function is given by

$$\mathrm{Res}_{k}(x,y,t) = u_k(x,y,t) - g(x,y) - \mathcal{K}^{-1}\Big\{v^{\alpha}\mathcal{K}\Big\{(u_k^2(x,y,t))_{xx}\Big\}\Big\} - \mathcal{K}^{-1}\Big\{v^{\alpha}\mathcal{K}\Big\{(u_k^2(x,y,t))_{yy}\Big\} - \mathcal{K}^{-1}\Big\{v^{\alpha}\mathcal{K}\Big\{\sigma(u_k(x,y,t))\Big\}\Big\}.$$

In order to compute the coefficients $f_n(x,y),\ n = 1,\ 2,\ 3,\ \dots$ in equation (3.4), the recurrence relation is defined as

$$\begin{cases} u_0(x,y,t) = g(x,y) \\ \mathrm{Res}_{k}(x,y,t) = u_k(x,y,t) - g(x,y) - \mathcal{K}^{-1}\Big\{v^{\alpha}\mathcal{K}\Big\{(u_{k-1}^2(x,y,t))_{xx}\Big\}\Big\} \\ \qquad - \mathcal{K}^{-1}\Big\{v^{\alpha}\mathcal{K}\Big\{(u_{k-1}^2(x,y,t))_{yy}\Big\} - \mathcal{K}^{-1}\Big\{v^{\alpha}\mathcal{K}\Big\{\sigma(u_{k-1}(x,y,t))\Big\}\Big\}, \end{cases} \tag{3.5}$$

and the coefficients $f_1(x,y),\ f_2(x,y),\ \dots,\ f_k(x,y)$ are simply obtained by imposing the condition [26]

$$t^{-k\alpha}\cdot \mathrm{Res}_k(x,y,t)\Big|_{t=0} = 0. \tag{3.6}$$

3.2. **Illustrative examples.** This section provides an example of solving the fractional population diffusion equation by using the KRPSM.

Example 3.1. [3] *Consider the fractional population diffusion equation*

$$D_t^{\alpha}u(x,y,t) = (u^2(x,y,t))_{xx} + (u^2(x,y,t))_{yy} + hu(x,y,t), \tag{3.7}$$

where h is constant, subjects to initial condition $u(x,y,0) = \sqrt{xy}$. The exact solution of this problem when $\alpha = 1$ is $u(x,y,t) = \sqrt{xy}e^{ht}$.

Note that $g(x,y)=\sqrt{xy}$ and $\sigma(u)=hu(x,y,t)$. The recurrence relation for this problem is given by

$$\begin{aligned} u_0(x,y,t) &= \sqrt{xy},\\ \mathrm{Res}_k(x,y,t) &= u_k(x,y,t)-\sqrt{xy}-\mathcal{K}^{-1}\Big\{v^{\alpha}\mathcal{K}\{(u_{k-1}^2(x,y,t))_{xx}\}\Big\}\\ &\quad -\mathcal{K}^{-1}\Big\{v^{\alpha}\mathcal{K}\{(u_{k-1}^2(x,y,t))_{yy}\}\Big\}\\ &\quad -\mathcal{K}^{-1}\Big\{v^{\alpha}h\mathcal{K}\{u_{k-1}(x,y,t)\}\Big\},\ k\geqslant 1. \end{aligned} \tag{3.8}$$

For $k=1$, the 1-st truncated series solution is

$$u_1(x,y,t)=f_0(x,y)+f_1(x,y)\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

and the relation becomes

$$\begin{aligned} \mathrm{Res}_1(x,y,t) &= u_1(x,y,t)-\sqrt{xy}-\mathcal{K}^{-1}\Big\{v^{\alpha}\mathcal{K}\{(u_0^2(x,y,t))_{xx}\}\Big\}\\ &\quad -\mathcal{K}^{-1}\Big\{v^{\alpha}\mathcal{K}\{(u_0^2(x,y,t))_{yy}\}\Big\}-\mathcal{K}^{-1}\Big\{v^{\alpha}h\mathcal{K}\{u_0(x,y,t)\}\Big\}\\ &= f_1(x,y)\frac{t^{\alpha}}{\Gamma(\alpha+1)}-h\sqrt{xy}\frac{t^{\alpha}}{\Gamma(\alpha+1)}. \end{aligned}$$

Searching for $f_1(x,y)$ by using the condition (3.6) with $k=1$, $t^{-\alpha}\mathrm{Res}_1(x,y,t)\big|_{t=0}=0$, we obtained $f_1(x,y)=h\sqrt{xy}$. Hence, the first-order approximate solution of the equation is

$$u_1(x,y,t)=\sqrt{xy}+h\sqrt{xy}\frac{t^{\alpha}}{\Gamma(\alpha+1)}.$$

For $k=2$, the 2-nd truncated series solution is

$$u_2(x,y,t)=\sqrt{xy}+\frac{h\sqrt{xy}\,t^{\alpha}}{\Gamma(\alpha+1)}+\frac{f_2(x,y)t^{2\alpha}}{\Gamma(2\alpha+1)}.$$

Substitute into (3.8), and it finds that

$$\begin{aligned} \mathrm{Res}_2(x,y,t) &= u_2(x,y,t)-\sqrt{xy}-\mathcal{K}^{-1}\Big\{v^{\alpha}\mathcal{K}\{(u_1^2(x,y,t))_{xx}\}\Big\}\\ &\quad -\mathcal{K}^{-1}\Big\{v^{\alpha}K\{(u_1^2(x,y,t))_{yy}\}\Big\}-\mathcal{K}^{-1}\Big\{v^{\alpha}h\mathcal{K}\{u_1(x,y,t)\}\Big\}\\ &= \frac{h\sqrt{xy}\,t^{\alpha}}{\Gamma(1+\alpha)}+\frac{f_2(x,y)t^{2\alpha}}{\Gamma(1+2\alpha)}\\ &\quad -\mathcal{K}^{-1}\left\{v^{\alpha}\mathcal{K}\Big\{\left[f_0^2+\frac{2f_0f_1t^{\alpha}}{\Gamma(1+\alpha)}+\frac{f_1^2t^{2\alpha}}{\Gamma^2(1+\alpha)}\right]_{xx}\Big\}\right\}\\ &\quad -\mathcal{K}^{-1}\left\{v^{\alpha}\mathcal{K}\Big\{\left[f_0^2+\frac{2f_0f_1t^{\alpha}}{\Gamma(1+\alpha)}+\frac{f_1^2t^{2\alpha}}{\Gamma^2(1+\alpha)}\right]_{yy}\Big\}\right\}\\ &\quad -\mathcal{K}^{-1}\left\{v^{\alpha}h\left(\mathcal{K}\left\{f_0\right\}+\mathcal{K}\Big\{f_1\frac{t^{\alpha}}{\Gamma(1+\alpha)}\Big\}\right)\right\}\\ &= \frac{f_2(x,y)t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{h^2\sqrt{xy}t^{2\alpha}}{\Gamma(2\alpha+1)}. \end{aligned}$$

In order to find $f_2(x,y)$, the condition (3.6) is applied for $k=2$,

$$t^{-2\alpha}\text{Res}_2(x,y,t)\big|_{t=0}=0.$$

It obtains

$$f_2(x,y)=h^2\sqrt{xy}.$$

Therefore, the second-order approximate solution is written as

$$u_2(x,y,t)=\sqrt{xy}+\sqrt{xy}\frac{ht^{\alpha}}{\Gamma(\alpha+1)}+\sqrt{xy}\frac{h^2t^{2\alpha}}{\Gamma(2\alpha+1)}.$$

By processing in a similar way, one find that

$$f_k(x,y)=h^k\sqrt{xy},\ k=0,1,2,\dots,$$

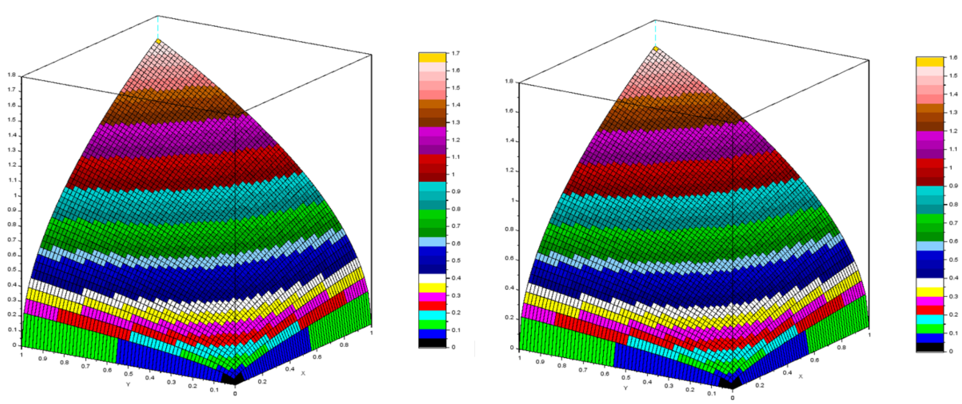
and the k-order approximate solution is

$$\begin{aligned} u_k(x,y,t) &= \sqrt{xy}+\frac{h\sqrt{xy}\,t^{\alpha}}{\Gamma(\alpha+1)}+\frac{h^2\sqrt{xy}\,t^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots+\frac{h^k\sqrt{xy}\,t^{k\alpha}}{\Gamma(k\alpha+1)}\\ &= \sqrt{xy}\sum_{n=0}^{k}\frac{(ht^{\alpha})^n}{\Gamma(n\alpha+1)}. \end{aligned}$$

When $k\to\infty$, the solution to this problem trends to the exact one,

$$u(x,y,t)=\lim_{k\to\infty}u_k(x,y,t)=\sqrt{xy}E_{\alpha}(ht^{\alpha}).$$

Note that if $\alpha=1$ this solution reduces to $u(x,y,t)=\sqrt{xy}e^{ht}$ which is related to the outcome of [3].



$u(x,y,t)_{exact}$ $u_2(x,y,t)_{KRPSM}$

FIGURE 1. The 3D graph shows a comparison of the exact solution and the approximate solution obtained $(u_2(x,y,t)_{KRPSM})$ when $x\in[0,1], y\in[0,1], h=0.5, t=1$ and $\alpha=1$ for example 3.1

x	y	$u(x,y,t)^{\text{Exact}}$	$u_2(x,y,t)^{\text{KRPSM}}$	Absolute Error			
				RPSM	KRPSM	ERPSM	HPM
0.1	0.1	0.1648721271	0.1625000000	2.37×10^{-3}	2.37×10^{-3}	2.37×10^{-3}	2.37×10^{-3}
	0.2	0.2331643981	0.2298097039	3.35×10^{-3}	3.35×10^{-3}	3.35×10^{-3}	3.35×10^{-3}
	0.3	0.2855669010	0.2814582562	4.11×10^{-3}	4.11×10^{-3}	4.11×10^{-3}	4.11×10^{-3}
	0.4	0.3297442542	0.3250000000	4.74×10^{-3}	4.74×10^{-3}	4.74×10^{-3}	4.74×10^{-3}
	0.5	0.3686652837	0.3633610463	5.30×10^{-3}	5.30×10^{-3}	5.30×10^{-3}	5.30×10^{-3}
0.3	0.1	0.2855669010	0.2814582562	4.11×10^{-3}	4.11×10^{-3}	4.11×10^{-3}	4.11×10^{-3}
	0.2	0.4038525842	0.3980420832	5.81×10^{-3}	5.81×10^{-3}	5.81×10^{-3}	5.81×10^{-3}
	0.3	0.4946163813	0.4875000000	7.12×10^{-3}	7.12×10^{-3}	7.12×10^{-3}	7.12×10^{-3}
	0.4	0.5711338018	0.5629165125	8.22×10^{-3}	8.22×10^{-3}	8.22×10^{-3}	8.22×10^{-3}
	0.5	0.6385470025	0.6293597938	9.19×10^{-3}	9.19×10^{-3}	9.19×10^{-3}	9.19×10^{-3}
0.5	0.1	0.3686652837	0.3633610463	5.30×10^{-3}	5.30×10^{-3}	5.30×10^{-3}	5.30×10^{-3}
	0.2	0.5213714443	0.5138701198	7.50×10^{-3}	7.50×10^{-3}	7.50×10^{-3}	7.50×10^{-3}
	0.3	0.6385470025	0.6293597938	9.19×10^{-3}	9.19×10^{-3}	9.19×10^{-3}	9.19×10^{-3}
	0.4	0.7373305676	0.7267220927	1.06×10^{-2}	1.06×10^{-2}	1.06×10^{-2}	1.06×10^{-2}
	0.5	0.8243606355	0.8125000000	1.19×10^{-2}	1.19×10^{-2}	1.19×10^{-2}	1.19×10^{-2}
1	0.1	0.5213714443	0.5138701198	7.50×10^{-3}	7.50×10^{-3}	7.50×10^{-3}	7.50×10^{-3}
	0.2	0.7373305676	0.7267220927	1.06×10^{-2}	1.06×10^{-2}	1.06×10^{-2}	1.06×10^{-2}
	0.3	0.9030418312	0.8900491559	1.30×10^{-2}	1.30×10^{-2}	1.30×10^{-2}	1.30×10^{-2}
	0.4	1.0427428890	1.0277402396	1.50×10^{-2}	1.50×10^{-2}	1.50×10^{-2}	1.50×10^{-2}
	0.5	1.1658219910	1.1490485194	1.68×10^{-2}	1.68×10^{-2}	1.68×10^{-2}	1.68×10^{-2}

FIGURE 2. The table depics a comparison of the exact solution and an 2-order approximate solution obtained from KRPSM, RPSM [25], ERPSM [26] and HPM [17] when $t = \alpha = 1$ and $h = 0.5$ for the example 3.1

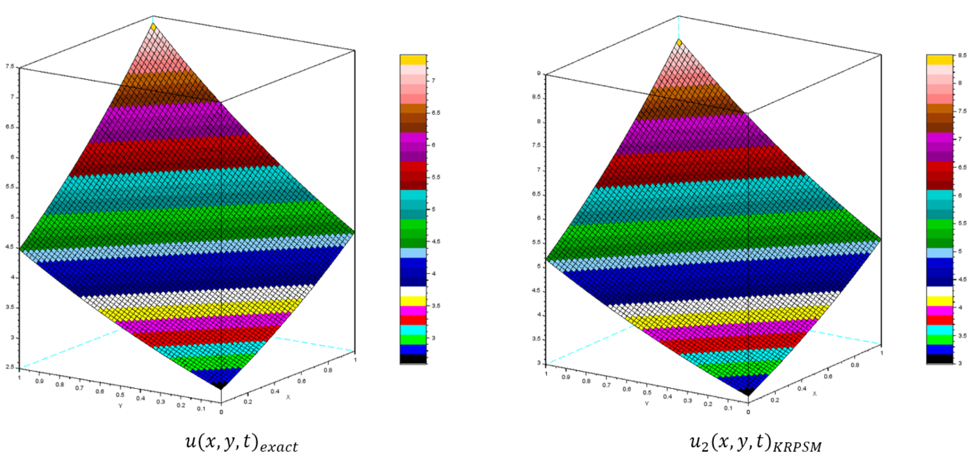


FIGURE 3. The 3D graph compares the exact solution and the approximate solution obtained $(u_2(x,y,t)_{KRPSM})$ when $x \in [0,1], y \in [0,1], h = 1, r = 2, t = 1$ and $\alpha = 1$ for example 3.2

Example 3.2. [3] *Next, consider the fractional population diffusion equations in the form*

$$D_t^\alpha u(x,y,t) = (u^2(x,y,t))_{xx} + (u^2(x,y,t))_{yy} + hu(x,y,t)(1 - ru(x,y,t)), \tag{3.9}$$

where h, r are constants, subjects to

$$u(x,y,0) = e^{\sqrt{\frac{hr}{8}}(x+y)}.$$

For $\alpha = 1$, the exact solution of this problem is $u(x,y,t) = e^{\sqrt{\frac{hr}{8}}(x+y)+ht}$.

Here $\sigma(u(x,y,t)) = hu(x,y,t)(1-ru(x,y,t))$ and $g(x,y) = e^{\sqrt{\frac{hr}{8}}(x+y)}$. By recurrence relation (3.5), one finds that

$$\begin{aligned}
u_0(x,y,t) &= e^{\sqrt{\frac{hr}{8}}(x+y)},\\
\mathrm{Res}_k(x,y,t) &= u_k(x,y,t) - e^{\sqrt{\frac{hr}{8}}(x+y)} - \mathcal{K}^{-1}\Big\{v^{\alpha}\mathcal{K}\{(u_{k-1}^2(x,y,t))_{xx}\}\Big\}\\
&\quad - \mathcal{K}^{-1}\Big\{v^{\alpha}\mathcal{K}\{(u_{k-1}^2(x,y,t))_{yy}\}\Big\} - \mathcal{K}^{-1}\Big\{v^{\alpha}h\mathcal{K}\{u_{k-1}(x,y,t)\}\Big\}\\
&\quad + \mathcal{K}^{-1}\Big\{v^{\alpha}hr\mathcal{K}\{(u_{k-1}^2(x,y,t))\}\Big\},\ k \geqslant 1.
\end{aligned} \tag{3.10}$$

Consider $k=1$. The 1-st truncated series solution is

$$u_1(x,y,t) = f_0(x,y) + f_1(x,y)\frac{t^{\alpha}}{\Gamma(\alpha+1)} = e^{\sqrt{\frac{hr}{8}}(x+y)} + \frac{f_1(x,y)t^{\alpha}}{\Gamma(\alpha+1)}$$

and the iterative relation is obtained by

$$\begin{aligned}
\mathrm{Res}_1(x,y,t) &= \frac{f_1(x,y)t^{\alpha}}{\Gamma(1+\alpha)} - \mathcal{K}^{-1}\Big\{v^{\alpha}\mathcal{K}\Big\{(e^{2\sqrt{\frac{hr}{8}}(x+y)})_{xx}\Big\}\Big\}\\
&\quad - \mathcal{K}^{-1}\Big\{v^{\alpha}\mathcal{K}\Big\{(e^{2\sqrt{\frac{hr}{8}}(x+y)})_{yy}\Big\}\Big\} - \mathcal{K}^{-1}\Big\{v^{1+\alpha}he^{\sqrt{\frac{hr}{8}}(x+y)}\Big\}\\
&\quad + \mathcal{K}^{-1}\Big\{v^{\alpha}hr\mathcal{K}\Big\{e^{2\sqrt{\frac{hr}{8}}(x+y)}\Big\}\Big\}\\
&= f_1(x,y)\frac{t^{\alpha}}{\Gamma(\alpha+1)} - he^{\sqrt{\frac{hr}{8}}(x+y)}\frac{t^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned}$$

To find the coefficient $f_1(x,y)$, we impose the condition $t^{-\alpha}\mathrm{Res}_1(x,y,t)\big|_{t=0} = 0$ which leads to $f_1(x,y) = he^{\sqrt{\frac{hr}{8}}(x+y)}$. Hence, the 1-st order approximate solution is

$$u_1(x,y,t) = e^{\sqrt{\frac{hr}{8}}(x+y)} + e^{\sqrt{\frac{hr}{8}}(x+y)}\frac{ht^{\alpha}}{\Gamma(\alpha+1)}.$$

Consider $k=2$. We define the 2-nd truncated series solution

$$u_2(x,y,t) = f_0(x,y) + \frac{f_1(x,y)t^{\alpha}}{\Gamma(\alpha+1)} + \frac{f_2(x,y)t^{2\alpha}}{\Gamma(2\alpha+1)}.$$

Substitute into (3.10), then

$$\begin{aligned}
\mathrm{Res}_2(x,y,t) &= \frac{he^{\sqrt{\frac{hr}{8}}(x+y)}t^{\alpha}}{\Gamma(\alpha+1)} + \frac{f_2(x,y)t^{2\alpha}}{\Gamma(2\alpha+1)}\\
&\quad - \mathcal{K}^{-1}\left\{v^{\alpha}\mathcal{K}\Big\{\left[f_0^2 + \frac{2f_0f_1t^{\alpha}}{\Gamma(\alpha+1)} + \frac{f_1^2t^{2\alpha}}{\Gamma^2(\alpha+1)}\right]_{xx}\Big\}\right\}\\
&\quad - \mathcal{K}^{-1}\left\{v^{\alpha}\mathcal{K}\Big\{\left[f_0^2 + \frac{2f_0f_1t^{\alpha}}{\Gamma(\alpha+1)} + \frac{f_1^2t^{2\alpha}}{\Gamma^2(\alpha+1)}\right]_{yy}\Big\}\right\}\\
&\quad - \mathcal{K}^{-1}\left\{v^{\alpha}h\mathcal{K}\Big\{f_0 + f_1\frac{t^{\alpha}}{\Gamma(\alpha+1)}\Big\}\right\}
\end{aligned}$$

$$+\mathcal{K}^{-1}\left\{v^{\alpha}hr\mathcal{K}\Big\{f_0^2+\frac{2f_0f_1t^{\alpha}}{\Gamma(\alpha+1)}+\frac{f_1^2t^{2\alpha}}{\Gamma^2(\alpha+1)}\Big\}\right\}$$
$$=\frac{f_2(x,y)t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{h^2t^{2\alpha}e^{\sqrt{\frac{hr}{8}}(x+y)}}{\Gamma(2\alpha+1)}.$$

Solving for the coefficient $f_2(x,y)$, the condition (3.6) for $k=2$ is taken, $t^{-2\alpha}\mathrm{Res}_2(x,y,t)\big|_{t=0}=0$. Hence, $f_2(x,y)=h^2e^{\sqrt{\frac{hr}{8}}(x+y)}$. The 2nd-order approximate solution is found

$$u_2(x,y,t)=e^{\sqrt{\frac{hr}{8}}(x+y)}+e^{\sqrt{\frac{hr}{8}}(x+y)}\frac{ht^{\alpha}}{\Gamma(\alpha+1)}+e^{\sqrt{\frac{hr}{8}}(x+y)}\frac{h^2t^{2\alpha}}{\Gamma(2\alpha+1)}.$$

Using precisely the same method, it can be demonstrated that

$$f_k(x,y)=h^ke^{\sqrt{\frac{hr}{8}}(x+y)},\ k=0,\ 1,\ 2,\ \ldots.$$

Therefore, the k-order approximate solution of this problem is

$$u_k(x,y,t)=e^{\sqrt{\frac{hr}{8}}(x+y)}+\frac{he^{\sqrt{\frac{hr}{8}}(x+y)}t^{\alpha}}{\Gamma(\alpha+1)}+\frac{h^2e^{\sqrt{\frac{hr}{8}}(x+y)}t^{2\alpha}}{\Gamma(2\alpha+1)}+\ldots+\frac{h^ke^{\sqrt{\frac{hr}{8}}(x+y)}t^{k\alpha}}{\Gamma(k\alpha+1)}$$
$$=e^{\sqrt{\frac{hr}{8}}(x+y)}\sum_{n=0}^{k}\frac{(ht^{\alpha})^n}{\Gamma(n\alpha+1)}.$$

Note that when $k\to\infty$, the exact solution is written

$$u(x,y,t)=\lim_{k\to\infty}u_k(x,y,t)=e^{\sqrt{\frac{hr}{8}}(x+y)}E_{\alpha}(ht^{\alpha})$$

and if $\alpha=1$ this solution is reduced to $u(x,y,t)=e^{\sqrt{\frac{hr}{8}}(x+y)+ht}$.

Example 3.3. [3] *Consider the fractional population diffusion equation as follows :*

$$D_t^{\alpha}u(x,y,t)=(u^2(x,y,t))_{xx}+(u^2(x,y,t))_{yy}+\frac{1}{96}u^{-1}(x,y,t)-\frac{1}{2},\tag{3.11}$$

with respect to the initial condition

$$u(x,y,0)=\frac{1}{4}\sqrt{2(x^2+y^2)+y+5}.$$

The exact solution of the initial value problem when $\alpha=1$ is

$$u(x,y,t)=\frac{1}{4}\sqrt{2(x^2+y^2)+y+\frac{t}{3}+5}.$$

t	x	y	$u(x,y,t)^{Exact}$	$u_2(x,y,t)^{KRPSM}$	Absolute Error	
					KRPSM	ERPSM
0.1	0.1	0.1	1.2214027582	1.2212138645	0.0001888937	0.0001888937
		0.5	1.4918246976	1.4915939824	0.0002307153	0.0002307153
		0.9	1.8221188004	1.8218370041	0.0002817963	0.0002817963
	0.5	0.1	1.4918246976	1.4915939824	0.0002307153	0.0002307153
		0.5	1.8221188004	1.8218370041	0.0002817963	0.0002817963
		0.9	2.2255409285	2.2251967418	0.0003441867	0.0003441867
	0.9	0.1	1.8221188004	1.8218370041	0.0002817963	0.0002817963
		0.5	2.2255409285	2.2251967418	0.0003441867	0.0003441867
		0.9	2.7182818285	2.7178614378	0.0004203906	0.0004203906
0.9	0.1	0.1	2.7182818285	2.5474189662	0.1708628623	0.1708628623
		0.5	3.3201169227	3.1114245515	0.2086923713	0.2086923713
		0.9	4.0551999668	3.8003025290	0.2548974379	0.2548974379
	0.5	0.1	3.3201169227	3.1114245515	0.2086923713	0.2086923713
		0.5	4.0551999668	3.8003025290	0.2548974379	0.2548974379
		0.9	4.9530324244	4.6416999907	0.3113324337	0.3113324337
	0.9	0.1	4.0551999668	3.8003025290	0.2548974379	0.2548974379
		0.5	4.9530324244	4.6416999907	0.3113324337	0.3113324337
		0.9	6.0496474644	5.6693851712	0.3802622932	0.3802622932

FIGURE 4. The table shows a comparison of the exact solution and an 2-order approximate solution obtained from KRPSM and ERPSM [26] when $r = 2, \alpha = 1$ and $h = 1$ for the example 3.2

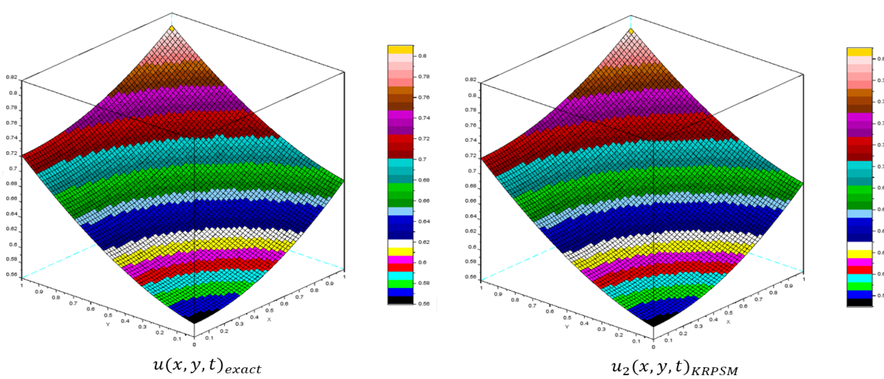


FIGURE 5. The 3D graph demonstrates a comparison of the exact solution and the approximate solution obtained from KRPSM ($u_2(x,y,t)_{KRPSM}$) when $x \in [0,1], y \in [0,1], t = 1$ and $\alpha = 1$ for example 3.3

Note that $\sigma(u(x,y,t)) = \dfrac{1}{96}u^{-1}(x,y,t) - \dfrac{1}{2}$ and $g(x,y) = \dfrac{1}{4}\sqrt{2(x^2+y^2)+y+5}$. The recurrence relation for this problem is

$$
\begin{aligned}
u_0(x,y,t) &= \frac{1}{4}\sqrt{2(x^2+y^2)+y+5},\\
\mathrm{Res}_k(x,y,t) &= u_k - \frac{1}{4}\sqrt{2(x^2+y^2)+y+5} - \mathcal{K}^{-1}\Big\{v^\alpha \mathcal{K}\{\big(u_{k-1}^2(x,y,t)\big)_{xx}\}\Big\}\\
&\quad - \mathcal{K}^{-1}\Big\{v^\alpha \mathcal{K}\{\big(u_{k-1}^2(x,y,t)\big)_{yy}\}\Big\} - \mathcal{K}^{-1}\left\{\frac{v^\alpha}{96}\mathcal{K}\{u_{k-1}^{-1}(x,y,t)\}\right\}\\
&\quad + \frac{t^\alpha}{2\Gamma(\alpha+1)},\ k \geqslant 1.
\end{aligned}
$$

For $k=1$, the 1-st truncated series solution is given by

$$u_1(x,y,t)=f_0(x,y)+\frac{f_1(x,y)t^\alpha}{\Gamma(\alpha+1)}=\frac{1}{4}\sqrt{2(x^2+y^2)+y+5}+\frac{f_1(x,y)t^\alpha}{\Gamma(\alpha+1)}$$

and the 1-st residual function is

$$\begin{aligned}\mathrm{Res}_1(x,y,t)&=u_1(x,y,t)-\frac{1}{4}\sqrt{2(x^2+y^2)+y+5}-\mathcal{K}^{-1}\Big\{v^\alpha\mathcal{K}\{\big(u_0^2(x,y,t)\big)_{xx}\}\Big\}\\&\quad-\mathcal{K}^{-1}\Big\{v^\alpha\mathcal{K}\{\big(u_0^2(x,y,t)\big)_{yy}\}\Big\}-\mathcal{K}^{-1}\left\{\frac{v^\alpha}{96}\mathcal{K}\{u_0^{-1}(x,y,t)\}\right\}\\&\quad+\frac{t^\alpha}{2\Gamma(\alpha+1)}\\&=\frac{f_1(x,y)t^\alpha}{\Gamma(\alpha+1)}-\frac{t^\alpha}{\Gamma(\alpha+1)24\sqrt{2(x^2+y^2)+y+5}}.\end{aligned}$$

By utilizing the condition (3.6) for $k=1$, $t^{-\alpha}\mathrm{Res}_1(x,y,t)\big|_{t=0}=0$ to get the coefficient

$$f_1(x,y)=\frac{1}{24\sqrt{2(x^2+y^2)+y+5}}.$$

Hence, the 1-st order approximate solution is

$$u_1(x,y,t)=\frac{\sqrt{2(x^2+y^2)+y+5}}{4}+\frac{t^\alpha}{24\sqrt{2(x^2+y^2)+y+5}\,\Gamma(\alpha+1)}.$$

Consider $k=2$. The 2-nd truncated series solution is found

$$\begin{aligned}u_2(x,y,t)&=f_0(x,y)+\frac{f_1(x,y)t^\alpha}{\Gamma(\alpha+1)}+\frac{f_2(x,y)t^{2\alpha}}{\Gamma(2\alpha+1)}\\&=\frac{\sqrt{2(x^2+y^2)+y+5}}{4}+\frac{t^\alpha}{24\sqrt{2(x^2+y^2)+y+5}\Gamma(\alpha+1)}\\&\quad+\frac{f_2(x,y)t^{2\alpha}}{\Gamma(2\alpha+1)},\end{aligned}$$

and the residual function is displayed

$$\begin{aligned}\mathrm{Res}_2(x,y,t)&=\frac{t^\alpha}{24\sqrt{2(x^2+y^2)+y+5}\,\Gamma(\alpha+1)}+\frac{f_2(x,y)t^{2\alpha}}{\Gamma(2\alpha+1)}\\&\quad+\frac{1}{144}\Big[\frac{1}{(2(x^2+y^2)+y+5)^2}-\frac{8x^2}{(2(x^2+y^2)+y+5)^3}\Big]\cdot\frac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\\&\quad+\frac{1}{576}\Big[\frac{4}{(2(x^2+y^2)+y+5)^2}-\frac{2(4y+1)^2}{(2(x^2+y^2)+y+5)^3}\Big]\cdot\frac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\\&\quad-\mathcal{K}^{-1}\left\{\frac{v^\alpha}{96f_0}\mathcal{K}\Big\{1-\frac{f_1t^\alpha}{f_0\Gamma(\alpha+1)}+\left(\frac{f_1t^\alpha}{f_0\Gamma(\alpha+1)}\right)^2-\dots\Big\}\right\}\\&=\frac{f_2(x,y)t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{1}{144}\Big[\frac{1}{(2(x^2+y^2)+y+5)^2}-\frac{8x^2}{(2(x^2+y^2)+y+5)^3}\Big]\\&\quad\times\frac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}+\frac{1}{576}\Big[\frac{4}{(2(x^2+y^2)+y+5)^2}-\frac{2(4y+1)^2}{(2(x^2+y^2)+y+5)^3}\Big]\end{aligned}$$

$$\times \frac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)} + \frac{1}{96} \cdot \frac{2}{3(2(x^2+y^2)+y+5)^{3/2}} \cdot \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \dots$$

By complying the condition (3.6) with $k = 2$, $t^{-2\alpha} Res_2(x,y,t)\big|_{t=0} = 0$, the coefficient $f_2(x,y)$ is obtained

$$f_2(x,y) = -\frac{1}{144(\sqrt{2(x^2+y^2)+y+5})^3}.$$

Therefore, the 2-nd order approximate series solution is

$$u_2(x,y,t) = \frac{\sqrt{2(x^2+y^2)+y+5}}{4} + \frac{1}{24\sqrt{2(x^2+y^2)+y+5}} \cdot \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{1}{144(2(x^2+y^2)+y+5)^{3/2}} \cdot \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

t	x,y	$u(x,y,t)^{Exact}$	$u_2(x,y,t)^{\text{KRPSM}}$	$u_2(x,y,t)^{\text{ADM}}$	Absolute Error	
					KRPSM	ADM
10	-450	224.93864904310	224.93864904167	224.93865161648	1.430067×10^{-9}	2.573384×10^{-6}
	-400	199.93879271751	199.93879271547	199.93879597465	2.036359×10^{-9}	3.257144×10^{-6}
	-300	149.93922379862	149.93922379379	149.93922959017	4.828451×10^{-9}	5.791548×10^{-6}
	-250	124.93956872558	124.93956871723	124.93957706660	8.345623×10^{-9}	8.341020×10^{-6}
	0	0.72168783649	0.62112999375	−0.087783840521	1.005578×10^{-1}	8.094717×10^{-1}
	50	25.07281063888	25.07280960570	25.073017531231	1.033189×10^{-6}	2.068923×10^{-4}
	100	50.06766255113	50.06766242143	50.067714479152	1.296991×10^{-7}	5.192803×10^{-5}
	200	100.06508298769	100.06508297145	100.06509599067	1.624485×10^{-8}	1.300298×10^{-5}
	350	175.06397640101	175.06397639798	175.063980649510	3.033620×10^{-9}	4.248496×10^{-6}
	500	250.06353359363	250.06353359259	250.06353567588	1.040888×10^{-9}	2.082251×10^{-6}
20	-450	224.93911213185	224.93911212613	224.93912242539	5.720324×10^{-9}	1.029354×10^{-5}
	-400	199.93931370960	199.93931370146	199.93932673818	8.145577×10^{-9}	2.316619×10^{-5}
	-300	149.93991852294	149.93991850363	149.93994168913	1.931392×10^{-8}	2.316619×10^{-5}
	-250	124.94040245920	124.94040242582	124.94043582328	3.338255×10^{-8}	3.336408×10^{-5}
	0	0.85391256383	0.80746991852	−2.400864341458	4.644357×10^{-2}	3.254777
	50	25.07696486154	25.07696072855	25.077792430696	4.132983×10^{-6}	8.275692×10^{-4}
	100	50.06974302577	50.06974250697	50.069950737872	5.188036×10^{-7}	2.077121×10^{-4}
	200	100.06612397144	100.06612390646	100.06617598337	6.497967×10^{-8}	5.201194×10^{-5}
	350	175.06457142057	175.06457140843	175.06458841455	1.213456×10^{-8}	1.699398×10^{-5}
	500	250.06395015409	250.06395014993	250.06395848309	4.163496×10^{-9}	8.329005×10^{-6}

Figure 6. The table shows a comparison of the exact solution and an 2-order approximate solution obtained from KRPSM and ADM [21] when $\alpha = 1$ for the example 3.3

4. Discussion and conclusion

The Kamal RPSM is an efficient technique for solving scientific models. In this study, we have successfully applied the Kamal transform together with the RPSM to find the solution to the fractional population diffusion equation in the Caputo sense. This method is superior to the classical version because it relies on the simplicity and convenience of finding the coefficient of the solution series through condition (3.6). The proposed method presented the solution to the nonlinear problem in the form of a convergent series, most of which converges to the exact solution as shown in examples 3.1 and 3.2.

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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