SPECIAL CASES AND APPLICATIONS OF THE CAUCHY COMPANION OPERATOR

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Abstract. In this paper, proved the famous Cauchy theorem in q-series, also we give some special roles of the Cauchy companion operator $E(a, b; \theta)$ and apply these roles to represent the Cauchy polynomials $P_n(x, y)$ and the finite q-shifted factorial $(a; q)_n$ to derive generating function, Mehler's formula and three Rogers formulas for $P_n(x, y)$ and $(a; q)_n$.

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1. Introduction, Definitions and Preliminaries

In 1998, Chen and Liu [16] have developped a method named "*parameter augmentation*" of deriving hypergeometric identities. Recently, Fang [19] introduced the q-exponential operator ${}_1\Phi_0\left[\begin{matrix} b; \\ & q; -c\theta \\ 0; \end{matrix}\right]$ and give some properties of q-series. This method has more realizations as in [2,3,6,9,10,12–16,24,31,32]. In this paper, we use this method and give easy proofs of results on q-series.

Let us review some common notation and terminology in [20] for basic hypergeometric series. Assume that q is a fixed nonzero real or complex number and $0 < q < 1$. The q-shifted factorial [17,20] is defined for any real or complex parameter a by:

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1}(1 - aq^k), \quad (a; q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k), \tag{1.1}$$

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where

$$(a;q)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} a^k. \tag{1.2}$$

The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} & \text{if} \quad k \leq n, \\ \\ 0 & \text{if} \quad k > n. \end{cases} \tag{1.3}$$

We also adopt the following notation for multiple q-shifted factorial

$$(a_1, a_2, \cdots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n \quad (m \in \mathbb{N} := \{1, 2, 3, \cdots\};\ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}).$$

and

$$(a_1, a_2, \ldots, a_m; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \ldots (a_m; q)_\infty.$$

The basic (or q-) hypergeometric function of the variable z is defined as follows [15,22,30,34] (see also the monographs by Slater [29, Chapter 3] and by Srivastava and Karlsson [33, p. 347, Eq. (272)]):

$${}_{\mathfrak{r}}\Phi_{\mathfrak{s}} \left[\begin{matrix} a_1, a_2, \cdots, a_{\mathfrak{r}}; \\ \\ b_1, b_2, \cdots, b_{\mathfrak{s}}; \end{matrix} \; q; z \right] := \sum_{n=0}^{\infty} \left[(-1)^n \, q^{\binom{n}{2}} \right]^{1+\mathfrak{s}-\mathfrak{r}} \frac{(a_1, a_2, \cdots, a_{\mathfrak{r}}; q)_n}{(b_1, b_2, \cdots, b_{\mathfrak{s}}; q)_n} \frac{z^n}{(q;q)_n}. \tag{1.4}$$

The series in (1.4) converges absolutely for all z if $\mathfrak{r} \leqq \mathfrak{s}$. For $\mathfrak{s} = \mathfrak{r} + 1$, we also note that

$${}_{\mathfrak{r}+1}\Phi_{\mathfrak{r}} \left[- \begin{matrix} a_1, a_2, \cdots, a_{\mathfrak{r}+1}; \\ \\ b_1, b_2, \cdots, b_{\mathfrak{r}}; \end{matrix} \; q; z \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \cdots, a_{\mathfrak{r}+1}; q)_n}{(b_1, b_2, \cdots, b_{\mathfrak{r}}; q)_n} \frac{z^n}{(q;q)_n}.$$

The Cauchy theorem [20] is given as

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z| < 1, \tag{1.5}$$

which can derive the following two identities

$$\sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}, \quad |z| < 1. \tag{1.6}$$

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q;q)_n} z^n = (-z;q)_\infty. \tag{1.7}$$

The Cauchy polynomials is defined by [25,26,28]:

$$P_n(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} y^k x^{n-k} = (y/x;q)_n x^n \tag{1.8}$$

which has the following generating function [7,8,13]

$$\sum_{n=0}^{\infty} P_n(x,y)\frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(xt;q)_\infty}, \quad |xt|<1. \tag{1.9}$$

The Cauchy companion operator [11,19] is defined as follows

$$E(a,b;\theta) := \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n}(-b\theta)^n, \tag{1.10}$$

where [21,23,27]

$$\theta\{f(x)\} = \frac{f(xq^{-1})-f(x)}{q^{-1}x}. \tag{1.11}$$

Lemma 1.1. *Suppose that θ acts on the variable x. For $m \in \{0,1,2,\cdots\}$, the following operational formula holds true*:

$$\theta^k\{x^n\} = \frac{(q,q)_n}{(q,q)_{n-k}} q^{-nk+\binom{k+1}{2}} x^{n-k}, \quad n \geqq k. \tag{1.12}$$

Assume that the operator $E(a,b;\theta)$ acts on the parameter c, the following identities are given in [19]:

Proposition 1.1. [19]. *We have:*

$$E(a,b;\theta)\{(ct;q)_\infty\} = \frac{(abt,ct;q)_\infty}{(bt;q)_\infty}; \tag{1.13}$$

$$E(a,b;\theta)\left\{\frac{(ct;q)_\infty}{(cv;q)_\infty}\right\} = \frac{(ct;q)_\infty}{(cv;q)_\infty}\, {}_2\Phi_1\left[\begin{matrix} a, t/v; \\ & q; bq/c \\ q/cv; \end{matrix}\right], \quad |bq/c|<1; \tag{1.14}$$

$$E(a,b;\theta)\left\{\frac{(cs,ct;q)_\infty}{(cv;q)_\infty}\right\} = \frac{(cs,ct,abt;q)_\infty}{(cv,bt;q)_\infty}\, {}_3\Phi_2\left[\begin{matrix} a, s/v, q/ct; \\ & q; q \\ q/bt, q/cv; \end{matrix}\right], \tag{1.15}$$

provided that $s/v = q^{-n}$ and $|bst/v|<1$.

Also, the following operator identity was derived in [1] by using the q-difference method:

Proposition 1.2.

$$E(a,b;\theta)\{(cs,ct;q)_\infty\} = \frac{(cs,ct,abt;q)_\infty}{(bt;q)_\infty}\, {}_2\Phi_1\left[\begin{matrix} a, q/ct; \\ & q; cs \\ q/bt; \end{matrix}\right], \tag{1.16}$$

provided that $a = q^{-n}$ and $\max\{|bt|,|abt|\}<1$.

In 2011, Abdlhussein [1] gave the following q-exponential operator as a special case of the Cauchy companion operator (1.10) when $a = 0$

$$T(b\theta) = \sum_{n=0}^{\infty} \frac{(b\theta)^n}{(q;q)_n} \tag{1.17}$$

and derive the following operator identity [1] by setting $a = 0$ in (1.10):

Proposition 1.3. *We have:*

$$T(b\theta)\{(ct;q)_\infty\} = \frac{(ct;q)_\infty}{(bt;q)_\infty}, \quad |bt| < 1. \tag{1.18}$$

2. Operator Identities

In this section, we prove the identity (1.16) without using q-difference method and give some roles for the q-exponential operator $T(b\theta)$ by special substitutions in the previous identities of the Cauchy companion operator.

Theorem 2.1. *It is asserted that*

$$E(a,b;\theta)\{(cs,ct;q)_\infty\} = \frac{(cs,ct,abt;q)_\infty}{(bt;q)_\infty}\, {}_2\Phi_1\left[\begin{matrix} a, q/ct; & \\ & q; cs \\ q/bt; & \end{matrix}\right], \tag{2.1}$$

provided that $a = q^{-n}$ *and* $\max\{|bt|, |cs|\} < 1$.

Proof. Rewrite (1.15) as

$$E(a,b;\theta)\left\{\frac{(cs,ct;q)_\infty}{(cv;q)_\infty}\right\} = \frac{(cs,ct,abt;q)_\infty}{(cv,bt;q)_\infty} \sum_{k=0}^{\infty} \frac{(a,s/v,q/ct;q)_k\, q^k}{(q/bt,q/cv;q)_k}. \tag{2.2}$$

Since

$$(s/v;q)_k = \frac{P_k(v,s)}{v^k}, \tag{2.3}$$

we have

$$(q/cv;q)_k = \frac{P_k(cv,q)}{(cv)^k}. \tag{2.4}$$

Next, substituting (2.3) and (2.4) in (2.2), we obtain:

$$E(a,b;\theta)\left\{\frac{(cs,ct;q)_\infty}{(cv;q)_\infty}\right\} = \frac{(cs,ct,abt;q)_\infty}{(cv,bt;q)_\infty} \sum_{k=0}^{\infty} \frac{(a,q/ct;q)_k\, q^k}{(q/bt;q)_k} \frac{c^k P_k(v,s)}{P_k(cv,q)}. \tag{2.5}$$

Taking $v = 0$, we get

$$E(a,b;\theta)\{(cs,ct;q)_\infty\} = \frac{(cs,ct,abt;q)_\infty}{(bt;q)_\infty} \sum_{k=0}^{\infty} \frac{(a,q/ct;q)_k\,(cs)^k}{(q/bt;q)_k}$$

$$= \frac{(cs, ct, abt; q)_\infty}{(cv, bt; q)_\infty} \, {}_2\Phi_1 \left[\begin{array}{c} a, q/ct; \\ \qquad q; cs \\ q/bt; \end{array} \right] \tag{2.6}$$

which achieves the proof. □

In the following theorems, we give some new roles for the q-exponential operator $T(b\theta)$ as follows:

Theorem 2.2. *Suppose that $T(b\theta)$ acts on the variable c, then we have:*

$$T(b\theta) \left\{ \frac{(cs; q)_\infty}{(cv; q)_\infty} \right\} = \frac{(cs; q)_\infty}{(cv; q)_\infty} \, {}_2\Phi_1 \left[\begin{array}{c} s/v, 0; \\ \qquad q; bq/c \\ q/cv; \end{array} \right], \tag{2.7}$$

where $|bq/c| < 1$.

Proof. Taking $a = 0$ in (1.14), we get the result. □

Theorem 2.3. *Suppose that θ acts on the variable c, then the following assertion holds true:*

$$T(b\theta) \left\{ \frac{(cs, ct; q)_\infty}{(cv; q)_\infty} \right\} = \frac{(cs, ct; q)_\infty}{(cv; q)_\infty} \, {}_3\Phi_2 \left[\begin{array}{c} s/v, q/ct, 0; \\ \qquad q; q \\ q/bt, q/cv; \end{array} \right], \tag{2.8}$$

provided that $s/v = q^{-n}$ and $|bst/v| < 1$.

Proof. Taking $a = 0$ in (1.15), we get the result. □

Theorem 2.4. *Suppose that θ acts on the variable c, then the following assertion holds true:*

$$T(b\theta) \{(cs, ct; q)_\infty\} = \frac{(cs, ct; q)_\infty}{(bt; q)_\infty} \, {}_2\Phi_1 \left[\begin{array}{c} q/ct, 0; \\ \qquad q; cs \\ q/b; \end{array} \right], \quad \max\{|bt|, |cs|\} < 1. \tag{2.9}$$

Proof. Taking $a = 0$ in (2.1), we get the result. □

3. Operator Applications

In this section, we use the q-exponential operator $T(b\theta)$ to represent the Cauchy polynomials $P_n(x, y)$ and the finite q-shifted factorial $(a; q)_n$, we use this representations and the operator roles in the previous section to derive the generating function, Mehler's formula and three types of Rogers formula for $P_n(x, y)$ and $(a; q)_n$, also we give an operator proof of the famous Cauchy theorem.

In the following proposition, we represent the Cauchy polynomials (1.8) by the q-exponential operator.

Theorem 3.1. *Suppose that the q-exponential operator $T(b\theta)$ acts on the variable y, then we have:*

$$T_y(x\theta)\left\{(-1)^n q^{\binom{n}{2}} y^n\right\} = P_n(x,y). \tag{3.1}$$

Proof. By the definition of the q-exponential operator $T_y(x\theta)$ and identity (1.12), we get:

$$\begin{aligned} T_y(x\theta)\left\{(-1)^n q^{\binom{n}{2}} y^n\right\} &= \sum_{k=0}^{\infty} \frac{(-x\theta)^k}{(q;q)_k}\theta^k\{y^n\} \\ &= \sum_{k=0}^{n}(-1)^n q^{\binom{n}{2}} \frac{q^{-nk+\binom{k+1}{2}}}{(q;q)_k}\frac{(q,q)_n}{(q,q)_{n-k}}(-x)^k y^{n-k}. \end{aligned}$$

Changing the order of summation $n-k$ by k, we get the desire result. □

Taking $x=1$ and $y=a$, the q-exponential operator acts on a, we can give the following representation for the finite q-shifted factorial or finite q-binomial

Proposition 3.2. *Suppose that the q-exponential operator $T(b\theta)$ acts on the variable y, then we have:*

$$T_a(\theta)\left\{(-1)^n q^{\binom{n}{2}} a^n\right\} = (a;q)_n. \tag{3.2}$$

Now, we derive the generating function for the Cauchy polynomials by using our representation (3.1) and identity (1.18) as follows

Theorem 3.3 (Generating function for $P_n(x,y)$)**.** *We have:*

$$\sum_{n=0}^{\infty} P_n(x,y)\frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(xt;q)_\infty}, \quad |xt|<1. \tag{3.3}$$

Proof. Upon using (3.1), we have

$$\begin{aligned} \sum_{n=0}^{\infty} P_n(x,y)\frac{t^n}{(q;q)_n} &= \sum_{n=0}^{\infty} T_y(x\theta)\left\{(-1)^n q^{\binom{n}{2}} y^n\right\}\frac{t^n}{(q;q)_n} \\ &= T_y(x\theta)\left\{\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}} y^n \frac{t^n}{(q;q)_n}\right\} \\ &= T_y(x\theta)\left\{(yt;q)_\infty\right\} = \frac{(yt;q)_\infty}{(xt;q)_\infty}, \end{aligned}$$

where we have used the identity (1.18). □

So that, the generating function for the finite q-shifted factorial can be introduce by using our representation (3.2) and identity (1.18), or by setting $y=a$ and $x=1$ in (3.4). The following proof can be set as a new proof for the famous Cauchy theorem (1.5).

Theorem 3.4 (Cauchy Theorem)**.** *We have:*

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} t^n = \frac{(at;q)_\infty}{(t;q)_\infty}, \quad |t|<1. \tag{3.4}$$

Proof. Using (3.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\,t^n &= \sum_{n=0}^{\infty} T_a(\theta)\left\{(-1)^n q^{\binom{n}{2}} a^n\right\}\frac{t^n}{(q;q)_n}\\ &= T_a(\theta)\left\{\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}} a^n \frac{t^n}{(q;q)_n}\right\}\\ &= T_a(\theta)\left\{(at;q)_\infty\right\}\\ &= \frac{(at;q)_\infty}{(t;q)_\infty},\end{aligned}$$

where we have used the identity (1.18). □

Remark 3.1. *Taking $a=0$, the equation (3.4) reduce to the Euler's identity (1.6).*

We derive Mehler's formula for the Cauchy polynomials by using the operator representation (3.1) and identity (2.7) of the q-exponential operator

Theorem 3.5 (Mehler's formula for $P_n(x,y)$)**.**

$$\sum_{n=0}^{\infty} P_n(x,y)P_n(z,w)\frac{(-1)^n q^{-\binom{n}{2}} t^n}{(q;q)_n} = \frac{(ywt;q)_\infty}{(yzt;q)_\infty}\,{}_2\Phi_1\left[\begin{matrix} w/z, 0; & \\ & q; xq/y \\ q/yzt; & \end{matrix}\right], \quad |xq/z|<1. \tag{3.5}$$

Proof.

$$\begin{aligned}\sum_{n=0}^{\infty} P_n(x,y)P_n(z,w)\frac{(-1)^n q^{-\binom{n}{2}} t^n}{(q;q)_n} &= \sum_{n=0}^{\infty} T_y(x\theta)\left\{(-1)^n q^{\binom{n}{2}} y^n\right\} P_n(z,w)\frac{(-1)^n q^{-\binom{n}{2}} t^n}{(q;q)_n}\\ &= T_y(x\theta)\left\{\sum_{n=0}^{\infty} P_n(z,w)\frac{(yt)^n}{(q;q)_n}\right\}\\ &= T_y(x\theta)\left\{\frac{(ywt;q)_\infty}{(yzt;q)_\infty}\right\}\\ &= \frac{(ywt;q)_\infty}{(yzt;q)_\infty}\,{}_2\Phi_1\left[\begin{matrix} w/z, 0; & \\ & q; xq/y \\ q/yzt; & \end{matrix}\right].\end{aligned}$$

□

The following Mehler's formula for the finite q-shifted factorial can be derived by the same way of (3.12) when we use representation (3.2) and identity (2.7), or directly by setting $y=a$, $w=b$ and $x=z=1$ in (3.12).

Theorem 3.6 (Mehler's formula for $(a;q)_n$)**.**

$$\sum_{n=0}^{\infty}(a,b;q)_n\frac{(-1)^n q^{-\binom{n}{2}}t^n}{(q;q)_n}=\frac{(abt;q)_\infty}{(at;q)_\infty}\,{}_2\Phi_1\left[\begin{matrix}b,0;\\ & q;q/a\\ q/at;\end{matrix}\right],\quad |q/a|<1. \tag{3.6}$$

Proposition 3.7. *We have the following identity*

$${}_2\Phi_0\left[\begin{matrix}a,b;\\ & q;z\\ -;\end{matrix}\right]=\frac{(abz;q)_\infty}{(az;q)_\infty}\,{}_2\Phi_1\left[\begin{matrix}b,0;\\ & q;\frac{q}{a}\\ q/az;\end{matrix}\right]. \tag{3.7}$$

Proof. Comparing both sides of (3.6) and taking $t=z$, we get the desire result. □

Next, we derive extended Mehler's formula for the Cauchy polynomials by using the operator representation (3.1) and identity (2.7) of the q-exponential operator.

Theorem 3.8 (Extended Mehler's formula for $P_n(x,y)$)**.** *For $m,\ n\in\mathbb{N}$, we have*

$$\sum_{k=0}^{\infty}(-1)^k q^{-\binom{k}{2}-nk-mk}P_{m+k}(x,y)P_{n+k}(z,w)\frac{t^k}{(q;q)_k}$$
$$=P_m(x,y)P_n(z,w)\frac{(ywt;q)_\infty}{(yztq^{-n};q)_\infty}\,{}_2\Phi_1\left[\begin{matrix}wq^n/z,0;\\ & q;xq^{1-m}/y\\ q^{1+n}/yzt;\end{matrix}\right], \tag{3.8}$$

provided both sides of (3.8) *are convergent.*

Remark 3.2. *Taking $n=m=0$ in Theorem 3.8, we get Theorem 3.6.*

Proof. Upon using the fact that $P_{k+j}(a,b)=P_j(a,b)P_k(a,q^jb)$, we have

$$\sum_{k=0}^{\infty}(-1)^k q^{-\binom{k}{2}-nk-mk}P_{m+k}(x,y)P_{n+k}(z,w)\frac{t^k}{(q;q)_k}$$
$$=P_m(x,y)P_n(z,w)\left\{\sum_{k=0}^{\infty}(-1)^k q^{-\binom{k}{2}}P_k(x,q^m y)P_k(z,q^n w)\frac{(tq^{-n-m})^k}{(q;q)_k}\right\}. \tag{3.9}$$

Setting $y=yq^m$, $w=wq^n$ and $t=tq^{-n-m}$ in Theorem 3.6, the right hand side of (3.9) reads

$$P_m(x,y)P_n(z,w)\frac{(ywt;q)_\infty}{(yztq^{-n};q)_\infty}\,{}_2\Phi_1\left[\begin{matrix}wq^n/z,0;\\ & q;xq^{1-m}/y\\ q^{1+n}/yzt;\end{matrix}\right].$$

Summarizing the above calculations, we get the desire result. □

The following Mehler's formula for the finite q-shifted factorial can be derived directly by setting $y=a$, $w=b$ and $x=z=1$ in (3.8).

Theorem 3.9 (Extended Mehler's formula for $(a;q)_n$)**.** *For $m,\ n \in \mathbb{N}$, we have*

$$\sum_{k=0}^{\infty}(-1)^k q^{-\binom{k}{2}-nk-mk}(a;q)_{m+k}(b;q)_{n+k}\frac{t^k}{(q;q)_k}$$
$$= (a;q)_m(b;q)_n\frac{(abt;q)_\infty}{(atq^{-n};q)_\infty}\,{}_2\Phi_1\left[\begin{matrix} bq^n, 0; & \\ & q; q^{1-m}/b \\ q^{1+n}/bt; & \end{matrix}\right], \tag{3.10}$$

provided both sides of (3.10) *are convergent.*

Proposition 3.10. *For $m,\ n \in \mathbb{N}$, the following identity holds true:*

$${}_2\Phi_0\left[\begin{matrix} aq^m, bq^n; & \\ & q; cq^{-m-n} \\ -; & \end{matrix}\right] = \frac{(abc;q)_\infty}{(acq^{-n};q)_\infty}\,{}_2\Phi_1\left[\begin{matrix} bq^n, 0; & \\ & q; q^{1-m}/b \\ q^{1+n}/bt; & \end{matrix}\right]. \tag{3.11}$$

Remark 3.3. *Setting $m = n = 0$ in* (3.11)*, we get* (3.7)*.*

Proof. Comparing both sides of equation (3.10), we get equation (3.11). □

Now, we give the Rogers formula for the Cauchy polynomials by using representation (3.1) and identity (2.7) of the q-exponential operator

Theorem 3.11 (Rogers formula for $P_n(x,y)$)**.**

$$\sum_{m,n=0}^{\infty} P_{n+m}(x,y)(-1)^m q^{-\binom{m}{2}}\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} = \frac{(yt;q)_\infty}{(ys;q)_\infty}\,{}_2\Phi_1\left[\begin{matrix} t/s, 0; & \\ & q; xq/y \\ q/ys; & \end{matrix}\right], \tag{3.12}$$

where $\max\{|ys|, |xq/y|\} < 1$.

Proof.

$$\sum_{m,n=0}^{\infty} P_{n+m}(x,y)(-1)^m q^{-\binom{m}{2}-nm}\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$
$$= \sum_{m,n=0}^{\infty} T_y(x\theta)\left\{(-1)^{n+m}q^{\binom{n+m}{2}}y^{n+m}\right\}(-1)^m q^{-\binom{m}{2}-nm}\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$
$$= T_y(x\theta)\left\{\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(yt)^n}{(q;q)_n}\sum_{m=0}^{\infty}\frac{(ys)^m}{(q;q)_m}\right\}$$
$$= T_y(x\theta)\left\{\frac{(yt;q)_\infty}{(ys;q)_\infty}\right\}$$
$$= \frac{(yt;q)_\infty}{(ys;q)_\infty}\,{}_2\Phi_1\left[\begin{matrix} t/s, 0; & \\ & q; xq/y \\ q/ys; & \end{matrix}\right].$$

□

The Rogers formula for the finite q-shifted factorial can be easily obtained from the above theorem by setting $y = a$ and $x = 1$, or by using representation (3.2) and identity (2.7) of the q-exponential operator

Theorem 3.12 (Rogers formula for $(a;q)_n$)**.**

$$\sum_{m,n=0}^{\infty}(a;q)_{n+m}(-1)^m q^{-\binom{m}{2}-nm}\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}=\frac{(at;q)_\infty}{(as;q)_\infty}\,{}_2\Phi_1\left[\begin{matrix}t/s,0; & \\ & q;q/a \\ q/as; & \end{matrix}\right], \tag{3.13}$$

where $\max\{|as|,|q/a|\}<1$.

Here, we derive another Rogers formula for the Cauchy polynomials by using representation (3.1) and identity (2.9) as follows:

Theorem 3.13 (The second Rogers formula for $P_n(x,y)$)**.**

$$\sum_{m,n=0}^{\infty}P_{n+m}(x,y)q^{-nm}\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}=\frac{(yt;ys;q)_\infty}{(xt;q)_\infty}\,{}_2\Phi_1\left[\begin{matrix}q/yt,0; & \\ & q;ys \\ q/xt; & \end{matrix}\right], \tag{3.14}$$

where $\max\{|ys|,|xt|\}<1$.

Proof.

$$\begin{aligned}
&\sum_{m,n=0}^{\infty}P_{n+m}(x,y)q^{-nm}\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\\
&=\sum_{m,n=0}^{\infty}T_y(x\theta)\left\{(-1)^{n+m}q^{\binom{n+m}{2}}y^{n+m}\right\}q^{-nm}\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\\
&=T_y(x\theta)\left\{\sum_{n=0}^{\infty}\frac{(-1)^nq^{\binom{n}{2}}(yt)^n}{(q;q)_n}\sum_{m=0}^{\infty}\frac{(-1)^mq^{\binom{m}{2}}(ys)^m}{(q;q)_m}\right\}\\
&=T_y(x\theta)\left\{(yt,ys;q)_\infty\right\}\\
&=\frac{(yt,ys;q)_\infty}{(xt;q)_\infty}\,{}_2\Phi_1\left[\begin{matrix}q/yt,0; & \\ & q;ys \\ q/xt; & \end{matrix}\right].
\end{aligned}$$

□

Also, the following second Rogers formula for the finite q-shifted factorial can be proved from the above theorem by setting $y = a$ and $x = 1$ directly, or by using representation (3.2) and identity (2.9).

Theorem 3.14 (The second Rogers formula for $(a;q)_n$)**.**

$$\sum_{m,n=0}^{\infty}(a;q)_{n+m}q^{-nm}\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}=\frac{(at;as;q)_\infty}{(t;q)_\infty}\,{}_2\Phi_1\left[\begin{matrix}q/at,0; & \\ & q;as\\ q/t; & \end{matrix}\right], \tag{3.15}$$

where $\max\{|as|,|t|\}<1$.

Finally, we give the third Rogers formula or the extended Rogers formula for the Cauchy polynomials $P_n(x,y)$ by using our representation (3.1) and identity (2.8), as follows:

Theorem 3.15 (The third Rogers formula for $P_n(x,y)$)**.**

$$\begin{aligned}\sum_{m,n,k=0}^{\infty}&P_{n+m+k}(x,y)(-1)^k q^{-\binom{k}{2}-nm-nk-mk}\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\frac{v^k}{(q;q)_k}\\ &=\frac{(yt;ys;q)_\infty}{(yv,xt;q)_\infty}\,{}_3\Phi_2\left[\begin{matrix}q/yt,s/v,0; & \\ & q;q\\ q/xt,q/yv; & \end{matrix}\right],\end{aligned} \tag{3.16}$$

provided that $s/v=q^n$ *and* $|xst/v|<1$.

Proof.

$$\begin{aligned}\sum_{m,n,k=0}^{\infty}&P_{n+m+k}(x,y)(-1)^k q^{-\binom{k}{2}-nm-nk-mk}\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\frac{v^k}{(q;q)_k}\\ &=\sum_{m,n,k=0}^{\infty}T_y(x\theta)\left\{(-1)^{n+m+k}q^{\binom{n+m+k}{2}}y^{n+m+k}\right\}(-1)^k q^{-\binom{k}{2}-nm-nk-mk}\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\frac{v^k}{(q;q)_k}\\ &=T_y(x\theta)\left\{\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(yt)^n}{(q;q)_n}\sum_{m=0}^{\infty}\frac{(-1)^m q^{\binom{m}{2}}(ys)^m}{(q;q)_m}\sum_{k=0}^{\infty}\frac{(yv)^k}{(q;q)_k}\right\}\\ &=T_y(x\theta)\left\{\frac{(yt,ys;q)_\infty}{(yv;q)_\infty}\right\}\\ &=\frac{(yt,ys;q)_\infty}{(yv,xt;q)_\infty}\,{}_3\Phi_2\left[\begin{matrix}q/yt,s/v,0; & \\ & q;q\\ q/xt,q/yv; & \end{matrix}\right],\end{aligned}$$

which achieves the proof. □

So that, the third Rogers formula for the finite q-shifted factorial can be given from the above theorem by setting $y=a$ and $x=1$, or by using our representation (3.2) and identity (2.8).

Theorem 3.16 (The third Rogers formula for $(a;q)_n$)**.**

$$\sum_{m,n,k=0}^{\infty}(a;q)_{n+m+k}\,(-1)^k q^{-\binom{k}{2}-nm-nk-mk}\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\frac{v^k}{(q;q)_k}$$

$$= \frac{(at; as; q)_\infty}{(av, t; q)_\infty} \, {}_3\Phi_2 \left[\begin{array}{c} q/at, s/v, 0; \\ \hfill q; q \\ q/t, q/av; \end{array} \right], \tag{3.17}$$

provided that $s/v = q^n$ *and* $|st/v| < 1$.

4. Conclusions

In this paper, we have given some special roles of Cauchy companion operator $E(a, b; \theta)$ and apply these roles to represent the Cauchy polynomials $P_n(x, y)$ and the finite q-shifted factorial $(a; q)_n$. Finally, we have derived generating function, Mehler's formula and three Rogers formulas for $P_n(x, y)$ and $(a; q)_n$.

Authors' Contributions

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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