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GENERALIZATION OF PRIMARY AND SEMIPRIMARY BI-IDEALS OF NON-COMMUTATIVE RINGS

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Abstract. We introduce 1-primary bi-ideals, 2-primary bi-ideals, and 3-primary bi-ideals of non-commutative rings. We also interact with several properties of the different semiprimary bi-ideals. We discuss the 2-primary bi-ideals and 3-primary bi-ideals, which are generalizations of 1-primary bi-ideals and 2-primary bi-ideals, respectively. We discuss the m_{p1}, m_{p2}, m_p3-systems and generators of bi-ideals. A generalization of the m_{p1}-system is the m_{p2}-system, and a generalization of the m_{p2}-system is the m_{p3}-system. Given that Φ is a primary bi-ideal of $\mathscr{R}$, it is proved that Φ is a primary bi-ideal of $\mathscr{R}$ if and only if $\mathscr{R} \setminus \Phi$ is an m_{p1}-system (m_{p2}-system, m_{p3}-system) of $\mathscr{R}$. To prove that if Δ is a primary bi-ideal in $\mathscr{R}$ and if $\mathscr{M}$ is an m_{p3}-system of $\mathscr{R}$ with $\Delta \cap \mathscr{M} = \emptyset$, then there exists a 3-primary bi-ideal Φ of $\mathscr{R}$ such that $\Delta \subseteq \Phi$ with $\Phi \cap \mathscr{M} = \emptyset$. Let H^{Δ} be a primary ideal of $\mathscr{R}$. To prove that a primary bi-ideal Δ is a 3-primary bi-ideal of $\mathscr{R}$ and the converse is also valid. Let H^{Δ} be a primary ideal of $\mathscr{R}$, a primary bi-ideal Δ is a 1-primary bi-ideal (2-primary bi-ideal) in $\mathscr{R}$. Δ doesn't need to be a 1-primary bi-ideal. A 3-primary bi-ideal Φ with a primary bi-ideal Δ that fails to satisfy the m_{p3}-system is guaranteed. Examples are provided to illustrate our results.

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1. Introduction

Non-commutative rings started to be systematically studied in the 20th century. Another naturally occurring non-commutative entity is a matrix. Cayley introduced them, along with their addition and multiplication principles. Pierce claims that square matrices adhere to the well-known ring axioms.

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With his Wedderburns Theorem, which states that every finite division ring is commutative, Scottish mathematician Wedderburn made fundamental contributions to the science of non-commutative rings. Commutative and non-commutative ring theories came together and impacted one another in the 18th century. The basic findings for primary ideals and prime radicals in commutative rings are expanded in this study to non-commutative rings. Basic findings on non-commutative rings by Lam are [7] along with a few findings on radicals [21]. Numerous studies have examined various sorts of ideals in mathematical structures like rings and semirings [5,9], respectively. Dedekind established the concept of ideals, which comprised associative rings, in the theory of algebraic numbers. Algebraic numbers were added to the concept in this way. Additionally, it is a specific instance of Lajos' (m, n)-ideal. Lajos could analyse regular and intra-regular semigroups using quasi-ideals and generalized bi-ideals. A bi-ideal to quote [6] while describing various classes of semigroups. In some ways, it is arbitrary, but it is specified in terms of bi-ideals and associative rings. An expansion of LIs and RIs, particularly examples of bi-ideals, is a quasi-ideal. Steinfeld introduced semigroups and rings, which are now known as quasi-ideals. Semirings offer a variety of techniques to explain prime ideals, according to quote [5]. The prime ideal theory has been widely applied to commutative ring theory. Compared to commutative rings, it has not been applied to non-commutative rings as much. In non-commutative partial rings, distinct prime partial BIss were studied by Palanikumar et al. [12].

Van der Walt [20] investigated the prime and semiprimate BIs of associative rings with unity. Roux [8] extended associative rings devoid of unity to prime and semiprime BIs. Flaska et al. [3] described BIs in basic semirings. A few findings in the ideal theory of commutative semirings with non-zero identities were also described by Atani [1]. In general rings, McCoy provides some information regarding prime ideals [9]. In [2,5,17] provided information on the PID for rings and semirings. The terms prime bi-ideals and semiprime bi-ideals were established by Van der Walt [20]. The subsets X_1 and X_2 of $\mathscr{R}$ and the product $X_1 \cdot X_2$, what we mean is that the subring of $\mathscr{R}$ is generated by the set of all products $x_1 \cdot x_2$, where $x_1 \in X_1$ and $x_2 \in X_2$. By a bi-ideal Δ_1 of a ring $\mathscr{R}$, we mean a subring Δ_1 of $\mathscr{R}$ satisfying $\Delta_1 \mathscr{R} \Delta_1 \subseteq \Delta_1$. An ID Φ of a ring $\mathscr{R}$ is PID if and only if whenever $\Delta\Delta_1 \subseteq \Phi$, for ideals Δ and Δ_1 of $\mathscr{R}$ implies $\Delta \subseteq \Phi$ or $\Delta_1 \subseteq \Phi$ [9]. Recently, Palanikumar et al. discussed the new algebraic structures [10,11,13]. This paper is divided into five sections, each organized differently. In Section 2, basic definitions will be briefly described. We discuss the different types of primary BIDs and their extensions in Section 3. The semiprimary BIDs are discussed in Section 4. The conclusion is drawn in Section 5. This study hopes to accomplish a number of fundamental objectives, including:

(1) A 1-primary bi-ideal implies a 2-primary bi-ideal implies a 3-primary bi-ideal and an opposite direction does not hold.

(2) An m_{pb1}-system implies an m_{pb2}-system implies an m_{pb3}-system and opposite direction does not hold with Example.

(3) A 1-semi primary bi-ideal implies a 2-semi primary bi-ideal implies a 3-semi primary bi-ideal, and the reverse implication does not match.

(4) An n_{pb1}-system implies an n_{pb2}-system implies an n_{pb3}-system and opposite direction does not hold with example.

List of Abbreviations

RID	right ideal	primary BID	primary bi-ideal
LID	left ideal	primary ID	primary ideal
ID	ideal	TID	two sided ideal
BID	bi-ideal	semi primary BID	semi primary bi-ideal
PID	prime ideal	semi primary ID	semi primary ideal

2. Basic concepts

Here are a few definitions necessary for the remainder of our study.

Definition 2.1. (i) A non-empty subsets Γ of a ring $(\mathscr{R}, +, \cdot)$ is said to be an LID (RID) of $\mathscr{R}$ if Γ is a subring of $\mathscr{R}$ and $\mathscr{R}\Gamma \subseteq \Gamma$ (respectively, $\Gamma\mathscr{R} \subseteq \Gamma$). If Γ is an LID and RID of $\mathscr{R}$, then Γ is called an ID of $\mathscr{R}$.
(ii) A subring Γ of $\mathscr{R}$ is said to be a BID if $\Gamma\mathscr{R}\Gamma \subseteq \Gamma$.
(iii) A subring Γ of $\mathscr{R}$ is said to be a QID if $\Gamma\mathscr{R} \cap \mathscr{R}\Gamma \subseteq \Gamma$.

Definition 2.2. [8] (i) The BID Γ of $\mathscr{R}$ is a prime BID if $\delta_1\mathscr{R}\delta_2 \subseteq \Gamma$ implies $\delta_1 \in \Gamma$ or $\delta_2 \in \Gamma$.
(ii) The BID Γ of $\mathscr{R}$ is a semiprime BID if $\delta_1\mathscr{R}\delta_1 \subseteq \Gamma$ implies $\delta_1 \in \Gamma$.

Theorem 2.3. [8] *(i) The BID Γ of $\mathscr{R}$ is prime BID if and only if $\Gamma_1\Gamma_2 \subseteq \Gamma$, with Γ_1 is an RID of $\mathscr{R}$ and Γ_2 is an LID of $\mathscr{R}$ implies $\Gamma_1 \subseteq \Gamma$ or $\Gamma_2 \subseteq \Gamma$.*
(ii) The BID Γ of $\mathscr{R}$ is semiprime BID if and only if $\Gamma_1^2 \subseteq \Gamma$ (or $\Gamma_2^2 \subseteq \Gamma$) implies $\Gamma_1 \subseteq \Gamma$ (or $\Gamma_2 \subseteq \Gamma$) for any LID Γ_1 (or RID Γ_2) of $\mathscr{R}$.

Lemma 2.4. [9] *A non-empty subset Γ of $\mathscr{R}$ $\beta \in \Gamma$. Then*
$(\beta)_r = \{n\beta + \beta\mathscr{R} | n \in \mathbb{Z}^+\}$ *is an RID generated by* β.
$(\beta)_l = \{n\beta + \mathscr{R}\beta | n \in \mathbb{Z}^+\}$ *is an LID generated by* β.
$(\beta) = \{n\beta + \mathscr{R}\beta + \beta\mathscr{R} + \mathscr{R}\beta\mathscr{R} | n \in \mathbb{Z}^+\}$ *is a ID generated by* β.
$(\beta)_b = \{n\beta + m\beta^2 + \beta\mathscr{R}\beta | n \in \mathbb{Z}^+\}$ *is a BID generated by* β.

Definition 2.5. An ID Φ of $\mathscr{R}$ is said to be primary if for any IDs Γ_1 and Γ_2 of $\mathscr{R}$, $\Gamma_1\Gamma_2 \subseteq \Phi$ implies that $\Gamma_1 \subseteq \Phi$ or $\Gamma_2 \subseteq \sqrt{\Phi}$.

Definition 2.6. [4] Let $\mathscr{R}$ be a non-commutative ring. For an ID Γ of $\mathscr{R}$, the radical of Γ is defined as follows: $\sqrt{\Gamma} = \{\psi \in \mathscr{R}|$ every m-system containing ψ intersects $\Gamma\} \subseteq$ radical Γ. That is, $\sqrt{\Gamma} = \{\psi \in \mathscr{R}|M(\psi) \cap \Gamma \neq \emptyset\}$.

Lemma 2.7. [4, 8] *(i) Let Γ and Γ_1 be two IDs of $\mathscr{R}$. If $\Gamma \subseteq \Gamma_1$, then $\sqrt{\Gamma} \subseteq \sqrt{\Gamma_1}$.*
(ii) If $\psi \in \sqrt{\Gamma}$, then there exists a positive integer n such that $\psi^n \in \Gamma$.
(iii) Let Γ be any BID of a ring $\mathscr{R}$ and let $L^{\Gamma} = \{\tau \in \Gamma|\mathscr{R}\tau \subseteq \Gamma\}$ and $H^{\Gamma} = \{\sigma \in L^{\Gamma}|\sigma\mathscr{R} \subseteq L^{\Gamma}\}$.

3. Characterization of primary BIDs

Here, we introduce three types of primary BIDs.

Definition 3.1. (i) A BID Φ of $\mathscr{R}$ is called 1-primary if $\Delta_1\Delta_2 \subseteq \Phi$ implies $\Delta_1 \subseteq \Phi$ or $\Delta_2 \subseteq \sqrt{\Phi}$ for any BIDs Δ_1 and Δ_2 of $\mathscr{R}$.
(ii) A BID Φ of $\mathscr{R}$ is called 2-primary if $\beta\mathscr{R}\delta \subseteq \Phi$ implies $\beta \in \Phi$ or $\delta \in \sqrt{\Phi}$.
(iii) A BID Φ of $\mathscr{R}$ is called 3-primary if $\Gamma_1\Gamma_2 \subseteq \Phi$ implies $\Gamma_1 \subseteq \Phi$ or $\Gamma_2 \subseteq \sqrt{\Phi}$ for any IDs Γ_1 and Γ_2 of $\mathscr{R}$.

Theorem 3.2. *Every 1-primary BID is a 2-primary BID.*

Proof. Let Φ be a 1-prime BID of $\mathscr{R}$. Let $\beta, \delta \in \mathscr{R}$ and $\beta\mathscr{R}\delta \subseteq \Phi$. Now, $(\beta\mathscr{R}) \cdot (\mathscr{R}\delta) \subseteq \beta\mathscr{R}\delta \subseteq \Phi$, since $\beta\mathscr{R}$ and $\mathscr{R}\delta$ are BIDs. Hence, $\beta\mathscr{R} \subseteq \Phi$ or $\mathscr{R}\delta \subseteq \sqrt{\Phi}$. Suppose that $\beta\mathscr{R} \subseteq \Phi$. Consider $<\beta>_b \cdot <\beta>_b \subseteq \beta\mathscr{R} \subseteq \Phi$. Then $\beta \in \Phi$. Similarly, if $\mathscr{R}\delta \subseteq \sqrt{\Phi}$, then $\delta \in \sqrt{\Phi}$. Thus, Φ is a 2-primary BID of $\mathscr{R}$. □

The converse of the Theorem 3.2 does not hold.

Example 3.3. Consider the ring $\mathscr{R} = \mathscr{M}_2(\mathbb{Z}_2)$ and $\Phi = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ is a 2-primary BID, but not a 1-primary BID. Now, $\Delta_1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}$, $\Delta_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ and $\sqrt{\Phi} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$. Since $\Delta_1 \cdot \Delta_2 \subseteq \Phi$, but $\Delta_1 \nsubseteq \Phi$ and $\Delta_2 \nsubseteq \sqrt{\Phi}$.

Theorem 3.4. *Every 2-primary BID is a 3-primary BID.*

Proof. Let Φ be an 2-prime BID of $\mathscr{R}$. For the IDs Γ_1 and Γ_2 of $\mathscr{R}$ such that $\Gamma_1 \cdot \Gamma_2 \subseteq \Phi$. Assume $\Gamma_1 \nsubseteq \Phi$, let $\beta \in \Gamma_1 \setminus \Phi$. For any $\delta \in \Gamma_2$, $\beta\mathscr{R}\delta \subseteq <\beta> \cdot <\delta> \subseteq \Gamma_1 \cdot \Gamma_2 \subseteq \Phi$. Hence, $\delta \in \sqrt{\Phi}$. Then $\Gamma_2 \subseteq \sqrt{\Phi}$. Thus, Φ is a 3-primary BID of $\mathscr{R}$. □

The converse of Theorem 3.4 does not hold.

Example 3.5. Consider the ring $\mathscr{R} = \mathscr{M}_2(\mathbb{Z}_2)$, $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right\}$, $\delta = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$,
$\Phi = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$, $\sqrt{\Phi} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$. Here Φ is a 3-primary BID but not a 2-primary BID. Now, $\beta \notin \Phi, \delta \notin \sqrt{\Phi}$ and $\beta\mathscr{R}\delta \subseteq \Phi$ imply Φ is not a 2-primary BID of $\mathscr{R}$.

Definition 3.6. (i) A subset $\mathscr{M}$ of $\mathscr{R}$ is called an m_{p_1}-system if for any $\beta, \delta \in \mathscr{M}$, there exists $\beta_1 \in< \beta >_b$ and $\delta_1 < \delta >_b$ such that $\beta_1\delta_1 \in \mathscr{M}$.
(ii) A subset $\mathscr{M}$ of $\mathscr{R}$ is called an m_{p_2}-system if for any $\beta, \delta \in \mathscr{M}$, there exists $\beta_1 \in< \beta >_r$ and $\delta_1 \in< \delta >_l$ such that $\beta_1\delta_1 \in \mathscr{M}$.
(iii) A subset $\mathscr{M}$ of $\mathscr{R}$ is called an m_{p_3}-system if for any $\beta, \delta \in \mathscr{M}$, there exists $\beta_1 \in< \beta >$ and $\delta_1 \in< \delta >$ such that $\beta_1\delta_1 \in \mathscr{M}$.

Theorem 3.7. *If Φ is a BID of $\mathscr{R}$, then Φ is a 1-primary (2-primary, 3-primary) BID if and only if $\mathscr{R} \setminus \Phi$ is an m_{p_1}-system (m_{p_2}-system, m_{p_3}-system) of $\mathscr{R}$.*

Proof. Let Φ be a 1-primary BID of $\mathscr{R}$. To show that $\mathscr{R} \setminus \Phi$ is an m_{p_1}-system. Let $\tau, \sigma \in \mathscr{R} \setminus \Phi$. Hence, $\tau, \sigma \in \mathscr{R}$ but $\tau, \sigma \notin \Phi$. So $< \tau >_b \cdot < \sigma >_b \nsubseteq \Phi$. There exists $\tau' \in< \tau >_b$ and $\sigma' \in< \sigma >_b$ such that $\tau' \cdot \sigma' \notin \Phi$. Hence, $\tau' \cdot \sigma' \in \mathscr{R} \setminus \Phi$. So we have proved that for $\tau, \sigma \in \mathscr{R} \setminus \Phi$, there exists $\tau' \in< \tau >_b$ and $\sigma' \in< \sigma >_b$ such that $\tau' \cdot \sigma' \in \mathscr{R} \setminus \Phi$. So $\mathscr{R} \setminus \Phi$ is an m_{p_1}-system.

Conversely, let $\mathscr{R} \setminus \Phi$ be an m_{p_1}-system. We show that Φ is a 1-primary BID of $\mathscr{R}$. Let $\Delta_1 \cdot \Delta_2 \subseteq \Phi$ for the BIDs Δ_1 and Δ_2 of $\mathscr{R}$. Let us shows that $\Delta_1 \subseteq \Phi$ or $\Delta_2 \subseteq \sqrt{\Phi}$. Let us arrive at a contradiction. If $\Delta_1 \nsubseteq \Phi$ and $\Delta_2 \nsubseteq \sqrt{\Phi}$, let $\delta_1 \in \Delta_1 \setminus \Phi$ and let $\delta_2 \in \Delta_2 \setminus \sqrt{\Phi}$. Since $\delta_2 \notin \sqrt{\Phi}$, so there exists an m_{p_1}-system $\mathscr{R} \setminus \Phi$ in $\mathscr{R}$ such that $\delta_2 \in \mathscr{R} \setminus \Phi$ and $(\mathscr{R} \setminus \Phi) \cap \Phi = \emptyset$. Thus, $\delta_1, \delta_2 \in \mathscr{R} \setminus \Phi$ implies $< \delta_1 >_b \cdot < \delta_2 >_b \nsubseteq \Phi$, which is a contradiction. Thus, $\Delta_1 \subseteq \Phi$ or $\Delta_2 \subseteq \sqrt{\Phi}$. Hence, Φ is a 1-primary BID of $\mathscr{R}$. □

Lemma 3.8. *Every m_{p_1}-system is an m_{p_2}-system.*

Proof. Given that $\mathscr{M}$ be an m_{p_1}-system of $\mathscr{R}$. For any $\beta, \delta \in \mathscr{M}$, there exists $\beta_1 \in< \beta >_b$ and $\delta_1 \in< \delta >_b$ such that $\beta_1 \cdot \delta_1 \in \mathscr{M}$. Let us shows that $\mathscr{M}$ is an m_{p_2}-system. For $\beta, \delta \in \mathscr{M}$, there exists $\beta_1 \in< \beta >_r$ and $\delta_1 \in< \delta >_l$. Since RIDs and LIDs are BIDs, we have $\beta_1 \cdot \delta_1 \in \mathscr{M}$. Hence, $\mathscr{M}$ is an m_{p_2}-system of $\mathscr{R}$. □

As shown in the following example, the converse is need not be true.

Example 3.9. Consider the ring $\mathscr{R} = \mathscr{M}_2(\mathbb{Z}_2)$ and $\mathscr{M} = \mathscr{R} \setminus \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ is an m_{p_2}- system, but not m_{p_1}-system. For $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \in \mathscr{M}$, but there is no $\beta_1 \in \left\langle \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\rangle_b$ and no $\delta_1 \in \left\langle \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle_b$ such that $\beta_1\delta_1 \in \mathscr{M}$ because $\left\langle \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\rangle_b \cdot \left\langle \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle_b = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$.

Lemma 3.10. *Every m_{p_2}-system is an m_{p_3}-system.*

Proof. Given that $\mathscr{M}$ be an m_{p_2}-system of $\mathscr{R}$. For any $\beta, \delta \in \mathscr{M}$, there exists $\beta_1 \in <\beta>_r$ and $\delta_1 \in <\delta>_l$ such that $\beta_1\delta_1 \in \mathscr{M}$. Let us shows that $\mathscr{M}$ is an m_{p_3}-system. For $\beta, \delta \in \mathscr{M}$, there exists $\beta_1 \in <\beta>$ and $\delta_1 \in <\delta>$. Since IDs are RIDs and LIDs, we have $\beta_1\delta_1 \in \mathscr{M}$. Hence, $\mathscr{M}$ is an m_{p_3}-system of $\mathscr{R}$. □

However, the converse is not hold by the example.

Example 3.11. $\mathscr{R} = \mathscr{M}_2(\mathbb{Z}_2)$ and $\mathscr{M} = \mathscr{R} \setminus \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$ is an m_{p_3}- system, but not m_{p_2}-system. For $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \mathscr{M}$, but there is no $\beta_1 \in \left\langle \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\rangle_r$ and no $\delta_1 \in \left\langle \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle_l$ such that $\beta_1 \cdot \delta_1 \in \mathscr{M}$ because $\left\langle \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\rangle_r \cdot \left\langle \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle_l = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$.

Remark 3.12. Let $\sqrt{\Delta}$ be any BID of a ring $\mathscr{R}$. Then $\sqrt{L^\Delta} = \{\tau \in \sqrt{\Delta} | \mathscr{R}\tau \subseteq \sqrt{\Delta}\}$ and $\sqrt{H^\Delta} = \{\sigma \in \sqrt{L^\Delta} | \sigma\mathscr{R} \subseteq \sqrt{L^\Delta}\}$.

Lemma 3.13. *Let $\sqrt{\Delta}$ be a BID of $\mathscr{R}$. Then $\sqrt{L^\Delta}$ is an LID of $\mathscr{R}$ such that $\sqrt{L^\Delta} \subseteq \sqrt{\Delta}$.*

Proof. Let $\tau, \sigma \in \sqrt{L^\Delta}$. Then $\tau, \sigma \in \sqrt{\Delta}$ and $\mathscr{R}\tau \subseteq \sqrt{\Delta}$ and $\mathscr{R}\sigma \subseteq \sqrt{\Delta}$. Since $\sqrt{\Delta}$ is a BID of $\mathscr{R}$, $\tau - \sigma \in \sqrt{\Delta}$ and $\tau\sigma \in \sqrt{\Delta}$. Now, $\mathscr{R}(\tau - \sigma) \subseteq \mathscr{R}\tau - \mathscr{R}\sigma \subseteq \sqrt{\Delta}$. Thus, $\tau - \sigma \in \sqrt{L^\Delta}$. Now, $\mathscr{R}(\tau\sigma) \subseteq (\mathscr{R}\tau)(\mathscr{R}\sigma) \subseteq \sqrt{\Delta}$. Thus, $\tau\sigma \in \sqrt{L^\Delta}$. Hence, $\sqrt{L^\Delta}$ is a subring of $\mathscr{R}$. Let $\tau \in \sqrt{L^\Delta}$ and $\psi \in \mathscr{R}$. Since $\psi\tau \in \mathscr{R}\tau \subseteq \sqrt{\Delta}$, we have $\psi\tau \in \sqrt{\Delta}$ and $\mathscr{R}\psi\tau \subseteq \mathscr{R}\mathscr{R}\tau \subseteq \mathscr{R}\tau \subseteq \sqrt{\Delta}$. Thus, $\psi\tau \in \sqrt{L^\Delta}$. Hence, $\sqrt{L^\Delta}$ is an LID of $\mathscr{R}$ and $\sqrt{L^\Delta} \subseteq \sqrt{\Delta}$. □

Lemma 3.14. *Let $\sqrt{\Delta}$ be a BID of $\mathscr{R}$. Then $\sqrt{H^\Delta}$ is a subring of $\mathscr{R}$.*

Proof. Let $\tau, \sigma \in \sqrt{H^\Delta}$. Then $\tau, \sigma \in \sqrt{L^\Delta}$ and $\tau\mathscr{R} \subseteq \sqrt{L^\Delta}$ and $\sigma\mathscr{R} \subseteq \sqrt{L^\Delta}$. Since $\tau \in \sqrt{L^\Delta}$, $\tau \in \sqrt{\Delta}$ and $\mathscr{R}\tau \subseteq \sqrt{\Delta}$. Since $\sigma \in \sqrt{L^\Delta}$, $\sigma \in \sqrt{\Delta}$ and $\mathscr{R}\sigma \subseteq \sqrt{\Delta}$. Since $\tau, \sigma \in \sqrt{\Delta}$ and $\sqrt{\Delta}$ is a subring of $\mathscr{R}$. We have $\tau - \sigma \in \sqrt{\Delta}$ and $\tau\sigma \in \sqrt{\Delta}$. Now, $\mathscr{R}(\tau - \sigma) \subseteq \mathscr{R}\tau - \mathscr{R}\sigma \subseteq \sqrt{\Delta}$ implies $\tau - \sigma \in \sqrt{L^\Delta}$. Now, $(\tau - \sigma)\mathscr{R} \subseteq \tau\mathscr{R} - \sigma\mathscr{R} \subseteq \sqrt{L^\Delta}$. Hence, $\tau - \sigma \in \sqrt{H^\Delta}$. Now, $\mathscr{R}(\tau\sigma) \subseteq (\mathscr{R}\tau)(\mathscr{R}\sigma) \subseteq \sqrt{\Delta}$ implies $\tau\sigma \in \sqrt{L^\Delta}$ and $(\tau\sigma)\mathscr{R} \subseteq (\tau\mathscr{R})(\sigma\mathscr{R}) \subseteq \sqrt{L^\Delta}$. That is $\tau\sigma \in \sqrt{H^\Delta}$. Hence, $\sqrt{H^\Delta}$ is a subring of $\mathscr{R}$. □

Lemma 3.15. *Let $\sqrt{\Delta}$ be an LID of $\mathscr{R}$. Then $\sqrt{L^{\Delta}} = \sqrt{\Delta}$.*

Proof. Clearly, $\sqrt{L^{\Delta}} \subseteq \sqrt{\Delta}$. Let $\tau \in \sqrt{\Delta}$, since $\sqrt{\Delta}$ is an LID of $\mathscr{R}$. We have $\mathscr{R}\tau \subseteq \sqrt{\Delta}$ implies $\tau \in \sqrt{L^{\Delta}}$. Thus, $\sqrt{\Delta} \subseteq \sqrt{L^{\Delta}}$. Hence, $\sqrt{L^{\Delta}} = \sqrt{\Delta}$. □

Theorem 3.16. *Let $\sqrt{\Delta}$ is a BID of $\mathscr{R}$. Then $\sqrt{H^{\Delta}}$ is the unique largest TID of $\mathscr{R}$ contained in $\sqrt{\Delta}$.*

Proof. Let $\sqrt{\Delta}$ is any BID of $\mathscr{R}$. To prove that $\sqrt{H^{\Delta}}$ is the TID of $\mathscr{R}$. Since $\sqrt{L^{\Delta}} \subseteq \sqrt{\Delta}$ and $\sqrt{H^{\Delta}} \subseteq \sqrt{L^{\Delta}}$. Therefore, $\sqrt{H^{\Delta}} \subseteq \sqrt{L^{\Delta}} \subseteq \sqrt{\Delta}$. Let $\tau \in \sqrt{H^{\Delta}}$ and $\chi \in \mathscr{R}$. Then $\tau \in \sqrt{H^{\Delta}} \subseteq \sqrt{\Delta}$ implies that $\tau \in \sqrt{\Delta}$. Since τ is an element of $\sqrt{L^{\Delta}}$. We have $\mathscr{R}\tau \subseteq \sqrt{\Delta}$ and $\tau\mathscr{R} \subseteq \sqrt{L^{\Delta}}$. Then $\chi\tau \in \mathscr{R}\tau \subseteq \sqrt{\Delta}$ implies $\chi\tau \in \sqrt{\Delta}$ and $\mathscr{R}\chi\tau \subseteq \mathscr{R}\mathscr{R}\tau \subseteq \mathscr{R}\tau \subseteq \sqrt{\Delta}$ implies that $\chi\tau \in \sqrt{L^{\Delta}}$. Now, $\tau\chi \in \tau\mathscr{R} \subseteq \sqrt{L^{\Delta}}$. Hence, $\tau\chi \in \sqrt{L^{\Delta}}$ and $\chi\tau \in \sqrt{L^{\Delta}}$. First to prove that $\tau\chi \in \sqrt{H^{\Delta}}$ and $\chi\tau \in \sqrt{H^{\Delta}}$. Now, $\tau\chi\mathscr{R} \subseteq \tau\mathscr{R}\mathscr{R} \subseteq \tau\mathscr{R} \subseteq \sqrt{L^{\Delta}}$. Hence, $\tau\chi\mathscr{R} \subseteq \sqrt{L^{\Delta}}$ implies $\tau\chi \in \sqrt{H^{\Delta}}$. Now, $\chi\tau\mathscr{R} \subseteq \mathscr{R}\tau\mathscr{R} \subseteq \mathscr{R}\sqrt{L^{\Delta}} \subseteq \sqrt{L^{\Delta}}$. Since $\sqrt{L^{\Delta}}$ is an LID of $\mathscr{R}$, $\chi\tau \in \sqrt{H^{\Delta}}$. Hence, $\sqrt{H^{\Delta}}$ is a TID of $\mathscr{R}$. It enough to prove $\sqrt{H^{\Delta}}$ is a largest two sided ID of $\mathscr{R}$. Let $\sqrt{\mathscr{S}}$ be any ID of $\mathscr{R}$ and $\sqrt{\mathscr{S}} \subseteq \sqrt{\Delta}$. Let $\varrho \in \sqrt{\mathscr{S}}$. Then $\varrho \in \sqrt{\Delta}$ and $\mathscr{R}\varrho \subseteq \sqrt{\mathscr{S}} \subseteq \sqrt{\Delta}$. Hence, $\mathscr{R}\varrho \subseteq \sqrt{\Delta}$ implies $\varrho \in \sqrt{L^{\Delta}}$. Hence, $\sqrt{\mathscr{S}} \subseteq \sqrt{L^{\Delta}}$. Next, $\varrho \in \sqrt{L^{\Delta}}$ and $\varrho\mathscr{R} \subseteq \sqrt{\mathscr{S}} \subseteq \sqrt{L^{\Delta}}$. Therefore, $\varrho\mathscr{R} \subseteq \sqrt{L^{\Delta}}$. Thus, $\varrho \in \sqrt{H^{\Delta}}$. Hence, $\sqrt{\mathscr{S}} \subseteq \sqrt{H^{\Delta}}$. □

Theorem 3.17. *A BID Δ of a ring $\mathscr{R}$ is 2-primary BID if and only if $\Gamma_1\Gamma_2 \subseteq \Delta$, with Γ_1 is an RID of $\mathscr{R}$ and Γ_2 is an LID of $\mathscr{R}$ implies $\Gamma_1 \subseteq \Delta$ or $\Gamma_2 \subseteq \sqrt{\Delta}$.*

Proof. Let Δ be a 2-primary BID and $\Gamma_1\Gamma_2 \subseteq \Delta$. Suppose $\Gamma_1 \nsubseteq \Delta$. For all $\delta \in \Gamma_2$ and $\beta \in \Gamma_1 \setminus \Delta$, we have $\beta\mathscr{R}\delta \subseteq \Gamma_1\Gamma_2 \subseteq \Delta$. Since Δ is primary and $\beta \notin \Delta$, we have $\delta \in \sqrt{\Delta}$ for all $\delta \in \Gamma_2$. So $\Gamma_2 \subseteq \sqrt{\Delta}$.

Conversely, suppose that $\beta\mathscr{R}\delta \subseteq \Delta$. Now, $(\beta\mathscr{R})(\mathscr{R}\delta) \subseteq \beta\mathscr{R}\delta$ implies $\beta\mathscr{R} \subseteq \Delta$ or $\mathscr{R}\delta \subseteq \sqrt{\Delta}$. If $\beta\mathscr{R} \subseteq \Delta$, then $< \beta >_r< \delta >_l= \{n\beta + \beta\mathscr{R} | n \in \mathbb{Z}^+\} \cdot \{m\delta + \mathscr{R}\delta | m \in \mathbb{Z}^+\} = n\beta m\delta + n\beta\mathscr{R}\delta + \beta\mathscr{R}m\delta + \beta\mathscr{R}\mathscr{R}\delta \subseteq \beta\mathscr{R} \subseteq \Delta$. Thus, $\beta \in \Delta$ or $\delta \in \sqrt{\Delta}$. Similarly, suppose that $\mathscr{R}\delta \subseteq \sqrt{\Delta}$ implies that $< \beta >_r< \delta >_l\subseteq \mathscr{R}\delta \subseteq \sqrt{\Delta}$. Thus, $\beta \in \Delta$ or $\delta \in \sqrt{\Delta}$. □

Theorem 3.18. *A BID Δ is a 3-primary BID of $\mathscr{R}$ if and only if H^{Δ} is a primary ID of $\mathscr{R}$.*

Proof. Let Δ be an 3-primary BID of $\mathscr{R}$. To show that H^{Δ} is a primary ID of $\mathscr{R}$. Let Γ_1 and Γ_2 be the IDs of $\mathscr{R}$ such that $\Gamma_1 \cdot \Gamma_2 \subseteq H^{\Delta}$. By Theorems 3.16 and 3.17 and Proposition 6 [8], H^{Δ} and $\sqrt{H^{\Delta}}$ are unique largest TID contained in Δ and $\sqrt{\Delta}$ respectively. Thus, $\Gamma_1 \subseteq H^{\Delta}$ or $\Gamma_2 \subseteq \sqrt{H^{\Delta}}$.

Conversely, suppose that Γ_1 and Γ_2 are IDs of $\mathscr{R}$ such that $\Gamma_1 \cdot \Gamma_2 \subseteq \Delta$. Then $\Gamma_1 \cdot \Gamma_2 \subseteq H^{\Delta}$ implies $\Gamma_1 \subseteq H^{\Delta} \subseteq \Delta$ or $\Gamma_2 \subseteq \sqrt{H^{\Delta}} \subseteq \sqrt{\Delta}$. Hence, Δ is a 3-primary BIDs of $\mathscr{R}$. □

Corollary 3.19. *If Δ is a 1-primary BID of $\mathscr{R}$, then H^{Δ} is a primary ID of $\mathscr{R}$.*

Proof. Let Δ be a 1-primary BID of $\mathscr{R}$. Let us show that H^{Δ} is a primary ID of $\mathscr{R}$. Let Γ_1 and Γ_2 be an IDs of $\mathscr{R}$ such that $\Gamma_1\Gamma_2 \subseteq H^{\Delta}$. To show that $\Gamma_1 \subseteq H^{\Delta}$ or $\Gamma_2 \subseteq \sqrt{H^{\Delta}}$. Since $H^{\Delta} \subseteq \Delta$ and $\sqrt{H^{\Delta}} \subseteq \sqrt{\Delta}$.

Hence, $\Gamma_1\Gamma_2 \subseteq \Delta$. Since Γ_1 and Γ_2 are IDs of $\mathscr{R}$, it is a BIDs and Δ is a 1-primary BID of $\mathscr{R}$. Hence, $\Gamma_1 \subseteq \Delta$ or $\Gamma_2 \subseteq \sqrt{\Delta}$. By Proposition 6 [8], H^{Δ} is the largest ID of $\mathscr{R}$ such that $H^{\Delta} \subseteq \Delta$ and by Theorem 3.16, $\sqrt{H^{\Delta}}$ is the largest ID of $\mathscr{R}$ such that $\sqrt{H^{\Delta}} \subseteq \sqrt{\Delta}$. Thus, $\Gamma_1 \subseteq H^{\Delta}$ or $\Gamma_2 \subseteq \sqrt{H^{\Delta}}$. Hence, H^{Δ} is a primary ID of $\mathscr{R}$. □

Based on the following example, it is evident that the converse of the corollary 3.19 cannot hold.

Example 3.20. Let $\mathscr{R} = \mathscr{M}_2(\mathbb{Z}_2)$, $\Delta = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$ is a BID and $H^{\Delta} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ is a primary ID, but Δ is not a 1-primary BID of $\mathscr{R}$. For the BIDs $\Delta_1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ and $\Delta_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$. Since $\Delta_1 \cdot \Delta_2 \subseteq \Delta$, but $\Delta_1 \nsubseteq \Delta$ and $\Delta_2 \nsubseteq \sqrt{\Delta}$.

Corollary 3.21. *If Δ is a 2-primary BID of $\mathscr{R}$, then H^{Δ} is a primary ID of $\mathscr{R}$.*

Proof. Let Δ be an 2-primary BID of $\mathscr{R}$. Let us show that H^{Δ} is a primary ID of $\mathscr{R}$. Let Γ_1 and Γ_2 be an IDs of $\mathscr{R}$ such that $\Gamma_1\Gamma_2 \subseteq H^{\Delta}$. To show that $\Gamma_1 \subseteq H^{\Delta}$ or $\Gamma_2 \subseteq \sqrt{H^{\Delta}}$. Since $H^{\Delta} \subseteq \Delta$ and $\sqrt{H^{\Delta}} \subseteq \sqrt{\Delta}$. Hence, $\Gamma_1\Gamma_2 \subseteq \Delta$. Since Γ_1 is an ID of $\mathscr{R}$, it is an RID and since Γ_2 is an ID of $\mathscr{R}$, it is an LID. Since Δ is an 2-primary BID of $\mathscr{R}$, we have $\Gamma_1 \subseteq \Delta$ or $\Gamma_2 \subseteq \sqrt{\Delta}$. By Proposition 6 [8], H^{Δ} is the largest ID of $\mathscr{R}$ such that $H^{\Delta} \subseteq \Delta$ and by Theorem 3.16, $\sqrt{H^{\Delta}}$ is the largest ID of $\mathscr{R}$ such that $\sqrt{H^{\Delta}} \subseteq \sqrt{\Delta}$. Thus, $\Gamma_1 \subseteq H^{\Delta}$ or $\Gamma_2 \subseteq \sqrt{H^{\Delta}}$. Hence, H^{Δ} is a primary ID of $\mathscr{R}$. □

The converse of Corollary 3.21 does not hold by the example.

Example 3.22. Consider the ring $\mathscr{R} = \mathscr{M}_2(\mathbb{Z}_2)$. Let $\Delta = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$ is a BID and $H^{\Delta} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ is a primary ID. Now, $\beta\mathscr{R}\delta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathscr{R} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \subseteq \Delta$ but $a \notin \Delta$ and $b \notin \sqrt{\Delta}$. Thus, Δ is not a 2-primary BID of $\mathscr{R}$.

Theorem 3.23. *Let $\mathscr{M}$ be an m_{p_3}- system and Δ be a BID of $\mathscr{R}$ with $\Delta \cap \mathscr{M} = \emptyset$. Then there exists a 3-primary BID Φ of $\mathscr{R}$ containing Δ with $\Phi \cap \mathscr{M} = \emptyset$.*

Proof. Let $\mathcal{X} = \{\Gamma_2 | \Gamma_2$ is a BID with $\Delta \subseteq \Gamma_2$ and $\Gamma_2 \cap \mathscr{M} = \emptyset\}$. Clearly $\mathcal{X}$ is non-empty. By Zorn's lem, there exists a maximal element Φ in $\mathcal{X}$. We claim Φ is a 3-primary BID of $\mathscr{R}$. In view of the Theorem 3.18, it is enough if we show that H^{Φ} is a primary ID in $\mathscr{R}$. Since $H^{\Phi} \subseteq \Phi$ and $\Phi \cap \mathscr{M} = \emptyset$, this implies that $H^{\Phi} \cap \mathscr{M} = \emptyset$. Then H^{Φ} is a largest ID in $\mathscr{R}$ such that $H^{\Phi} \cap \mathscr{M} = \emptyset$. We claim that $<\beta><\delta> \subseteq H^{\Phi}$. Then $<\beta> \subseteq H(Q)$ or $<\delta> \subseteq H(Q)$. By proving a contradiction. If $<\beta> \nsubseteq H^{\Phi}$ and $<\delta> \nsubseteq \sqrt{H^{\Phi}}$, then $\tau \in <\beta> \setminus H^{\Phi}$ and $\sigma \in <\delta> \setminus \sqrt{H^{\Phi}}$. Then $<\tau> \subseteq <\beta>$ and $<\sigma> \subseteq <\delta>$.

If $<\beta><\delta>\subseteq H^{\Phi}$, then $<\tau><\sigma>\subseteq<\beta><\delta>\subseteq H^{\Phi}$. Since $<\delta>\nsubseteq\sqrt{H^{\Phi}}$ and hence $(<\delta>)^n\nsubseteq H^{\Phi}$ implies that $<\delta>\nsubseteq H^{\Phi}$. Then $(H^{\Phi}+<\tau>)\cap\mathscr{M}\neq\emptyset$ and $(H^{\Phi}+<\sigma>)\cap\mathscr{M}\neq\emptyset$. Thus, $(H^{\Phi}+<\tau>)(H^{\Phi}+<\sigma>)\subseteq H^{\Phi}$. Then the BID $(H(Q)+\tau)$ contains an element m_{p_1} of $\mathscr{M}$. Then there exist $\varpi_1\in(H^{\Phi}+<\tau>)\cap\mathscr{M}$. Similarly the BID $(H(Q)+\sigma)$ contains an element m_{p_2} of $\mathscr{M}$. Then there exists $\varpi_2\in(H^{\Phi}+<\sigma>)\cap\mathscr{M}$. Since $\mathscr{M}$ is m_{p_3}-system of A, $\varpi_1{}'\in<\varpi_1>$ and $\varpi_2{}'\in<\varpi_2>$ $\varpi_1{}'\varpi_2{}'\in\mathscr{M}$ for some $\varpi_1{}'\in<\varpi_1>\subseteq(H^{\Phi}+<\tau>)$ and $\varpi_2{}'\in<\varpi_2>\subseteq(H^{\Phi}+<\sigma>)$. Hence, $\varpi_1{}'\varpi_2{}'\in(H^{\Phi}+<\tau>)(H^{\Phi}+<\sigma>)\subseteq H^{\Phi}$. Which is a contradiction. Thus, $<\beta><\delta>\nsubseteq H^{\Phi}$. Hence, H^{Φ} is a primary ID of $\mathscr{R}$. By Theorem 3.18, Φ is a 3-primary BID of $\mathscr{R}$. If H^{Φ} is not largest element in $\mathcal{X}$, then there is an maximal ID Φ' in $\mathscr{R}$ such that $H^{\Phi}\subseteq\Phi'$ and $\Phi'\cap\mathscr{M}=\emptyset$. It can be easily seen that Φ' is a primary ID; hence, Φ' is the required BID of $\mathscr{R}$. □

4. Characterization of semiprimary BIDs

Here, we introduce three types of semiprimary BIDs.

Definition 4.1. (i) A BID Φ of $\mathscr{R}$ is called a 1-semiprimary if $\Delta^2\subseteq\Phi$ implies $\Delta\subseteq\Phi$ or $\Delta\subseteq\sqrt{\Phi}$ for any BID Δ of $\mathscr{R}$.
(ii) A BID Φ of $\mathscr{R}$ is called a 2-semiprimary if $\beta\mathscr{R}\beta\subseteq\Phi$ implies $\beta\in\Phi$ or $\beta\in\sqrt{\Phi}$.
(iii) A BID Φ of $\mathscr{R}$ is called a 3-semiprimary if $\Gamma_1^2\subseteq\Phi$ implies $\Gamma_1\subseteq\Phi$ or $\Gamma_1\subseteq\sqrt{\Phi}$ for any ID Γ_1 of $\mathscr{R}$.

Theorem 4.2. *Every 1-semiprimary BID is a 2-semiprimary BID of $\mathscr{R}$.*

Proof. Let Φ is a 1-semiprimary BID of $\mathscr{R}$. Let $\beta\in\mathscr{R}$ and $\beta\mathscr{R}\beta\subseteq\Phi$. Now, $(\beta\mathscr{R})\cdot(\mathscr{R}\beta)\subseteq\beta\mathscr{R}\beta\subseteq\Phi$, since $\beta\mathscr{R}$ and $\mathscr{R}\beta$ are BIDs. Hence, $\beta\mathscr{R}\subseteq\Phi$ or $\mathscr{R}\beta\subseteq\sqrt{\Phi}$. Suppose that $\beta\mathscr{R}\subseteq\Phi$. Consider $<\beta>_b\cdot<\beta>_b\subseteq\beta\mathscr{R}\subseteq\Phi$. Then $\beta\in\Phi$. Similarly, if $\mathscr{R}\beta\subseteq\sqrt{\Phi}$, then $\beta\in\sqrt{\Phi}$. Thus, Φ is a 2-semiprimary BID of $\mathscr{R}$. □

The converse of Theorem 4.2 cannot be true.

Example 4.3. Consider the ring $\mathscr{R}=\mathscr{M}_2(\mathbb{Z}_2)$. Let $\Phi=\left\{\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix}\right\}$ is a 2-semiprimary BID, but not a 1-semiprimary BID. For the BID $\Delta=\left\{\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&1\end{pmatrix}\right\}$. Since $\Delta^2\subseteq\Phi$, but $\Delta\nsubseteq\Phi$ or $\Delta\nsubseteq\sqrt{\Phi}$.

Theorem 4.4. *Every 2-semiprimary (2-primary) BID is a 3-semiprimary BID of $\mathscr{R}$.*

Proof. Suppose that Φ is a 2-semiprimary BID and $\Gamma^2\subseteq\Phi$ for an ID Γ of $\mathscr{R}$. To show that $\Gamma\subseteq\Phi$ or $\Gamma\subseteq\sqrt{\Phi}$. If $\Gamma\nsubseteq\Phi$ and $\Gamma\nsubseteq\sqrt{\Phi}$. For $\beta\in\Gamma$, but $\beta\notin\Phi$ and $\beta\notin\sqrt{\Phi}$. Now $\beta\mathscr{R}\beta\subseteq<\beta>\cdot<\beta>\subseteq\Gamma^2\subseteq\Phi$. Since Φ is a 2-semiprimary BID of $\mathscr{R}$, then $\beta\in\Phi$ or $\beta\in\sqrt{\Phi}$. Which is contradiction, hence $\Gamma\subseteq\Phi$ or $\Gamma\subseteq\sqrt{\Phi}$. Thus, Φ is a 3-semiprimary BID of $\mathscr{R}$. □

The converse of Theorem 4.4 is false.

Example 4.5. Consider the ring $\mathscr{R} = \mathscr{M}_2(\mathbb{Z}_2)$ and $\Phi = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$ is a 3-semiprimary BID of $\mathscr{R}$. Now $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \notin \Phi$, $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \notin \sqrt{\Phi}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathscr{R} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \subseteq \Phi$ imply Φ is not a 2-primary BID of $\mathscr{R}$.

Definition 4.6. (i) A subset N of $\mathscr{R}$ is called n_{p_1}-system if for any $\beta \in N$, there exist $\beta_1, \beta_2 \in < \beta >_b$ such that $\beta_1\beta_2 \in N$.
(ii) A subset N of $\mathscr{R}$ is called n_{p_2}-system if for any $\beta \in N$, there exist $\beta_1, \beta_2 \in < \beta >_r$ ($\beta_1, \beta_2 \in < \beta >_l$) such that $\beta_1\beta_2 \in N$.
(iii) A subset N of $\mathscr{R}$ is called n_{p_3}-system if for any $\beta \in N$, there exist $\beta_1, \beta_2 \in < \beta >$ such that $\beta_1\beta_2 \in N$.

Theorem 4.7. *If Φ is a BID of $\mathscr{R}$, then Φ is a 1-semiprimary BID (2-semiprimary, 3-semiprimary) if and only if $\mathscr{R} \setminus \Phi$ is an n_{p_1}-system (n_{p_2}-system, n_{p_3}-system).*

Proof. Let Φ be a 1-semiprimary BID of $\mathscr{R}$. To show that $\mathscr{R} \setminus \Phi$ is an n_{p_1}-system. Let $\beta \in \mathscr{R} \setminus \Phi$. Hence, $\beta \in \mathscr{R}$ but $\beta \notin \Phi$. So $< \beta >_b \cdot < \beta >_b \nsubseteq \Phi$. There exists $\beta', \beta'' \in < \beta >_b$ such that $\beta' \cdot \beta'' \notin \Phi$. Hence, $\beta' \cdot \beta'' \in \mathscr{R} \setminus \Phi$. So we have proved that for $\beta \in \mathscr{R} \setminus \Phi$ there exists $\beta', \beta'' \in < \beta >_b$ such that $\beta' \cdot \beta'' \in \mathscr{R} \setminus \Phi$. So $\mathscr{R} \setminus \Phi$ is an n_{p_1}-system.

Conversely, let $\mathscr{R} \setminus \Phi$ is an n_{p_1}-system. We show that Φ is a 1-semiprimary BID of $\mathscr{R}$. Let $\Delta^2 \subseteq \Phi$ for the BID Δ of $\mathscr{R}$. Let us shows that $\Delta \subseteq \Phi$ or $\Delta \subseteq \sqrt{\Phi}$. Let us arrive at a contradiction. If $\Delta \nsubseteq \Phi$ and $\Delta \nsubseteq \sqrt{\Phi}$, let $\delta_1 \in \Delta \setminus \Phi$ and $\delta_1 \in \Delta \setminus \sqrt{\Phi}$. Since $\delta_1 \notin \sqrt{\Phi}$, so there exists an n_{p_1}-system $\mathscr{R} \setminus \Phi$ in $\mathscr{R}$ such that $\delta_1 \in \mathscr{R} \setminus \Phi$ and $(\mathscr{R} \setminus \Phi) \cap \Phi = \emptyset$. Thus, $\delta_1 \in \mathscr{R} \setminus \Phi$ implies $< \delta_1 >_b \cdot < \delta_1 >_b \nsubseteq \Phi$, which is a contradiction. Thus, $\Delta \subseteq \Phi$ or $\Delta \subseteq \sqrt{\Phi}$. Hence, Φ is a 1-semiprimary BID of $\mathscr{R}$. Similarly, we can prove the other two cases. □

Lemma 4.8. *Every n_{p_1}-system is an n_{p_2}-system.*

Proof. Given that N be an n_{p_1}-system of $\mathscr{R}$. For any $\beta \in N$, there exists $\beta_1, \beta_2 \in < \beta >_b$ such that $\beta_1 \cdot \beta_2 \in N$. Let us shows that N is an n_{p_2}-system. For $\beta \in N$, there exists $\beta_1, \ \beta_2 \in < \beta >_r$ ($\beta_1, \ \beta_2 \in < \beta >_l$). Since RIDs and LIDs are BIDs, we have $\beta_1 \cdot \beta_2 \in N$. Hence, N is an n_{p_2}-system of $\mathscr{R}$. □

Here is an example demonstrating that the converse of the above Lemma is false.

Example 4.9. Consider the ring $\mathscr{R} = \mathscr{M}_2(\mathbb{Z}_2)$. Let $N = \mathscr{R} \setminus \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$ is an n_{p_2}- system, but not n_{p_1}- system. For $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in N$, but there is no $\beta_1, \beta_2 \in \left\langle \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\rangle_b$ such that $\beta_1 \cdot \beta_2 \in N$.

Since $\left\langle \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\rangle_b . \left\langle \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\rangle_b = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \notin N.$

Theorem 4.10. *Let Δ be a 2-semiprimary BID of a ring $\mathscr{R}$. Then $\Gamma^2 \subseteq \Delta$ implies $\Gamma \subseteq \Delta$ or $\Gamma \subseteq \sqrt{\Delta}$ for any LID $(RID)\Gamma$ of $\mathscr{R}$.*

Proof. Let Δ be a 2-semiprimary and $\Gamma^2 \subseteq \Delta$ for any LID Γ of $\mathscr{R}$. To show that $\Gamma \subseteq \Delta$ or $\Gamma \subseteq \sqrt{\Delta}$. If not, $\Gamma \nsubseteq \Delta$ and $\Gamma \nsubseteq \sqrt{\Delta}$. then there exists $\beta \in \Gamma$, but $\beta \notin \Delta$ and $\beta \notin \sqrt{\Delta}$. Now, $\beta\mathscr{R}\beta \subseteq \Gamma\mathscr{R}\Gamma \subseteq \Gamma^2 \subseteq \Delta$. Since Δ is 2-semiprimary implies that $\beta \in \Delta$ or $\beta \in \sqrt{\Delta}$. Which is a contradiction hence $\Gamma \subseteq \Delta$ or $\Gamma \subseteq \sqrt{\Delta}$.

Conversely, suppose that $\beta\mathscr{R}\beta \subseteq \Delta$. Now, $(\beta\mathscr{R})(\mathscr{R}\beta) \subseteq \beta\mathscr{R}\beta$ implies $\beta\mathscr{R} \subseteq \Delta$ or $\mathscr{R}\beta \subseteq \sqrt{\Delta}$. If $\beta\mathscr{R} \subseteq \Delta$, then $< \beta >_r < \beta >_l = \{n\beta + \beta\mathscr{R} | n \in \mathbb{Z}^+\} \cdot \{\beta + \mathscr{R}\beta | m \in \mathbb{Z}^+\} \subseteq \beta\mathscr{R} \subseteq \Delta$. Thus, $\beta \in \Delta$ or $\beta \in \sqrt{\Delta}$. Similarly, suppose that $\mathscr{R}\beta \subseteq \sqrt{\Delta}$ implies that $< \beta >_r < \beta >_l \subseteq \mathscr{R}\beta \subseteq \sqrt{\Delta}$. Thus, $\beta \in \Delta$ or $\beta \in \sqrt{\Delta}$. □

Theorem 4.11. *A BID Δ is a 3-semiprimary BID of $\mathscr{R}$ if and only if H^Δ is a semiprimary ID of $\mathscr{R}$.*

Proof. Let Δ be an 3-semiprimary BID of $\mathscr{R}$. To show that H^Δ is a semiprimary ID of $\mathscr{R}$. Let Γ be a ID of $\mathscr{R}$ such that $\Gamma^2 \subseteq H^\Delta$. By Theorems 3.16 and 4.10 H^Δ and $\sqrt{H^\Delta}$ are unique largest TID contained in Δ and $\sqrt{\Delta}$ respectively. Thus, $\Gamma \subseteq H^\Delta$ or $\Gamma \subseteq \sqrt{H^\Delta}$.

Conversely, suppose that H^Δ is a semiprimary ID of $\mathscr{R}$ and Γ is a ID of $\mathscr{R}$ such that $\Gamma^2 \subseteq \Delta$. To show that $\Gamma \subseteq \Delta$ or $\Gamma \subseteq \sqrt{\Delta}$. Now, $\Gamma^2 \subseteq H^\Delta$ implies $\Gamma \subseteq H^\Delta \subseteq \Delta$ or $\Gamma \subseteq \sqrt{H^\Delta} \subseteq \sqrt{\Delta}$. Hence, Δ is a 3-semiprimary BID of $\mathscr{R}$. □

Corollary 4.12. *If Δ is a 1-semiprimary (2-semiprimary) BID of $\mathscr{R}$, then H^Δ is a semiprimary ID of $\mathscr{R}$.*

Here is an example demonstrating that the converse of Corollary 4.12 is false.

Example 4.13. Let $\mathscr{R} = \mathscr{M}_2(\mathbb{Z}_2)$, $H^\Delta = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ and $\Delta = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$. Now, H^Δ is a semiprimary ID, but Δ is not a 1-semiprimary BID of $\mathscr{R}$. Since $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \subseteq \Delta$, but $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \nsubseteq \Delta$.

5. Conclusions

This paper introduces various primary BIDs of non-commutative rings and identifies prime BIDs and semiprime BIDs. In addition to this, we introduced three m-systems and characterized them. Also, 1-primary BID implies 2-primary BID implies 3-primary BID. When it comes to exist, the reverse does not apply. Using semirings, ternary semirings, partial semirings and ordered semirings as the basis for the extension of various ideals, such as quasi-ideals, tri-ideals, and bi-quasi-ideals, will be the next direction of the work.

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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