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PARTIALLY NONEXPANSIVE MAPPINGS IN GEODESIC SPACES

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Abstract. This paper investigates fixed point properties of partially nonexpansive mappings in geodesic spaces. The class of mappings considered herein is independent of the class of quasi-nonexpansive mappings. We obtain certain theorems regarding Δ and strong convergence. Additionally, we obtain a common fixed point theorem of a countable family of commuting partially nonexpansive self-mappings under certain conditions.

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1. Introduction

Let $(\mathcal{X}, \|.\|)$ be a Banach space and $\mathcal{Z}$ a nonempty subset of $\mathcal{X}$. A mapping $\Phi : \mathcal{Z} \to \mathcal{Z}$ is considered nonexpansive if it satisfies the condition $\|\Phi(\zeta) - \Phi(\varrho)\| \leq \|\zeta - \varrho\|$ for every $\zeta, \varrho \in \mathcal{Z}$. A point $z \in \mathcal{Z}$ is considered a fixed point of Φ if $\Phi(z) = z$. It is well-known that in a general Banach space, a nonexpansive mapping may not necessarily possess a fixed point. However, in 1965, Browder [3], Göhde [9], and Kirk [10] independently established fixed point theorems for nonexpansive mappings that satisfy specific geometric conditions, such as uniform convexity or normal structure. This category of mappings holds significant relevance in various mathematical contexts, including transition operators for initial value problems (of differential inclusion), accretive operators, monotone operators, variational inequality problems and equilibrium problems.

In 2008, Suzuki [26] introduced a condition on mappings, called condition (C),

Definition 1.1. [26]. Let $\mathcal{X}$ be a Banach space and $\mathcal{Z}$ a nonempty subset of $\mathcal{X}$. A mapping $\Phi : \mathcal{Z} \to \mathcal{Z}$ is said to satisfy condition (C) if

$$\frac{1}{2}\|\zeta - \Phi(\zeta)\| \leq \|\zeta - \vartheta\| \text{ implies } \|\Phi(\zeta) - \Phi(\vartheta)\| \leq \|\zeta - \vartheta\| \; \forall\, \zeta, \vartheta \in \mathcal{Z}.$$

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This condition is less stringent than nonexpansiveness but stronger than quasi-nonexpansiveness. Furthermore, he derived some intriguing fixed point theorems and convergence results for such mappings. Dhompongsa et al. [5] quickly observed that the same conclusion is reached if the domain $\mathcal{Z}$ of Φ is a bounded closed and convex subset of a Banach space, and every asymptotic center of a bounded sequence relative to $\mathcal{Z}$ is nonempty and compact. Additionally, Dhompongsa and Kaewcharoen [6] expanded Betiuk-Pilarska and Prus's result [2] on the weak fixed point property to continuous mappings satisfying condition (C) on an order uniformly noncreasy (OUNC) Banach lattice.

In 2011, García-Falset *et al.* [7] further generalized condition (C) into the following class of mappings.

Definition 1.2. [7]. Let $\mathcal{Z}$ be a nonempty subset of a Banach space $\mathcal{X}$. A mapping $\Phi : \mathcal{Z} \to \mathcal{Z}$ is said to fulfill condition (E_μ) if there exists $\mu \geq 1$ such that

$$\|\zeta - \Phi(\vartheta)\| \leq \mu\|\zeta - \Phi(\zeta)\| + \|\zeta - \vartheta\| \; \forall \, \zeta, \vartheta \in \mathcal{Z}.$$

We say that Φ satisfies condition (E) if it satisfies (E_μ) for some $\mu \geq 1$.

Takahashi [27] introduced convex structure to the metric space and derived theorems regarding the existence of fixed points for nonexpansive mappings. Goebel and Kirk [8] employed the Krasnosel'skiĭ-Mann iterative method to approximate fixed points of nonexpansive mappings in nonlinear spaces. In recent years, several papers have been published focusing on significant fixed point results within the framework of geodesic spaces, as documented in [1,13,14,16–18,21–24]. For instance, Ariza-Ruiz et al. [1] extended well-known theorems regarding firmly nonexpansive mappings, including the asymptotic behavior of the Picard iterative method, from linear spaces to geodesic spaces. Leuştean [14] broadened celebrated fixed point theory results in geodesic spaces, such as the monotone modulus of uniform convexity and asymptotic regularity for the Ishikawa iterative method. Nanjaras et al. [16] expanded Suzuki's findings on fixed point theorems and convergence theorems to a specific type of metric spaces, known as CAT(0) spaces.

Inspired by the aforementioned advancements, we aim to approximate fixed points of partially nonexpansive mappings within nonlinear spaces, specifically in geodesic spaces. We broaden the scope of partially nonexpansive mappings from Banach spaces to geodesic spaces and establish Δ and strong convergence theorems under specific conditions. Our findings serve to generalize, extend, and complement various results outlined in [7,15,16].

2. Preliminaries

Let $(\mathcal{M}, \Omega)$ denote a metric space and $[0, l] \subset \mathbb{R}$. Considering a pair of points $\zeta, \vartheta \in \mathcal{M}$, a path $\Theta : [0, 1] \to \mathcal{M}$ joins ζ and ϑ if it satisfies

$$\Theta(0) = \zeta \text{ and } \Theta(1) = \vartheta.$$

Such a path Θ is termed a geodesic if it adheres to the condition:

$$\Omega(\Theta(s), \Theta(t)) = \Omega(\Theta(0), \Theta(1))|s - t|, \text{ for all } s, t \in [0, 1].$$

A metric space $(\mathcal{M}, \Omega)$ is deemed a geodesic space if every pair of points $\zeta, \vartheta \in \mathcal{M}$ can be connected by a geodesic. It's important to note that the geodesic segment linking ζ and ϑ may not be unique. The precise formulation of hyperbolic spaces, as described by Kohlenbach [11], adheres to this framework.

Definition 2.1. [11]. A triplet $(\mathcal{M}, \Omega, W)$ is called a hyperbolic metric space (or W-hyperbolic space) if $(\mathcal{M}, \Omega)$ is a metric space and function $W : \mathcal{M} \times \mathcal{M} \times [0, 1] \to \mathcal{M}$ satisfies the following conditions for all $\zeta, \vartheta, z, w \in \mathcal{M}$ and $s, t \in [0, 1]$

(W1) $\Omega(z, W(\zeta, \vartheta, s)) \leq (1 - s)\Omega(z, \zeta) + s\Omega(z, \vartheta)$;
(W2) $\Omega(W(\zeta, \vartheta, s), W(\zeta, \vartheta, t)) = |s - t|\Omega(\zeta, \vartheta)$;
(W3) $W(\zeta, \vartheta, s) = W(\vartheta, \zeta, 1 - s)$;
(W4) $\Omega(W(\zeta, z, s), W(\vartheta, w, s)) \leq (1 - s)\Omega(\zeta, \vartheta) + s\Omega(z, w)$.

Every Busemann space is uniquely geodesic, meaning that for any pair of points $\zeta, \vartheta \in \mathcal{M}$, there exists precisely one geodesic segment connecting ζ and ϑ, as outlined in [4]. In other words, for all $\zeta, \vartheta \in \mathcal{M}$ and any $s \in [0, 1]$, there is an element $w \in \mathcal{M}$ which is unique (say $w = W(\zeta, \vartheta, s)$) such that

$$\Omega(\zeta, w) = s\Omega(\zeta, \vartheta) \text{ and } \Omega(\vartheta, w) = (1 - s)\Omega(\zeta, \vartheta). \tag{2.1}$$

Notably, the following spaces serve as prominent examples of W-hyperbolic spaces: all normed spaces, Hadamard manifolds, CAT(0)-spaces, and the Hilbert open unit ball equipped with the hyperbolic metric (cf. [1,11]).

Remark 2.2. Given that $W(\zeta, \vartheta, s) = (1 - s)\zeta + s\vartheta$ for all $\zeta, \vartheta \in \mathcal{M}$ and $s \in [0, 1]$, it consequently implies that all normed linear spaces exhibit W-hyperbolic characteristics.

We will employ the notation

$$W(\zeta, \vartheta, s) := (1 - s)\zeta \oplus s\vartheta$$

to represent a point $W(\zeta, \vartheta, s)$ within a W-hyperbolic space. For any $\zeta, \vartheta \in \mathcal{M}$, we define

$$[\zeta, \vartheta] = \{(1 - s)\zeta \oplus s\vartheta : s \in [0, 1]\}$$

as a geodesic segment. A nonempty subset $\mathcal{Z}$ of the W-hyperbolic space $(\mathcal{M}, \Omega, W)$ is termed convex if $[\zeta, \vartheta] \subset \mathcal{Z}$ for all $\zeta, \vartheta \in \mathcal{Z}$.

Definition 2.3. [13]. A W-hyperbolic space $(\mathcal{M}, \Omega)$ is considered uniformly convex (UCW-hyperbolic space) if, for every $\varepsilon \in (0, 2]$ and any $b > 0$, there exists a $\delta \in (0, 1]$ such that

$$\left.\begin{array}{l} \Omega(\zeta, z) \le b \\ \Omega(\vartheta, z) \le b \\ \Omega(\zeta, \vartheta) \ge \varepsilon b \end{array}\right\} \Rightarrow \Omega\left(\frac{1}{2}\zeta \oplus \frac{1}{2}\vartheta, z\right) \le (1-\delta)b$$

for all $\zeta, \vartheta, z \in \mathcal{M}$.

Remark 2.4. Leuştean [14] demonstrated that complete CAT(0) spaces are complete uniformly convex hyperbolic spaces, denoted as UCW-hyperbolic spaces.

Let $\{\zeta_n\}$ be a bounded sequence in a hyperbolic space $(\mathcal{M}, \Omega, W)$ and $\mathcal{Z}$ a nonempty subset of $\mathcal{M}$. A functional $r(\ .\ , \{\zeta_n\}) : \mathcal{M} \to [0, +\infty)$ can be defined as follows:

$$r(\vartheta, \{\zeta_n\}) = \limsup_{n \to +\infty} \Omega(\vartheta, \zeta_n).$$

The asymptotic radius of $\{\zeta_n\}$ with respect to (in short, wrt) $\mathcal{Z}$ is described as

$$r(\mathcal{Z}, \{\zeta_n\}) = \inf\{r(\vartheta, \{\zeta_n\}) : \vartheta \in \mathcal{Z}\}.$$

A point ζ in $\mathcal{Z}$ is called as an asymptotic center of $\{\zeta_n\}$ wrt $\mathcal{Z}$ if

$$r(\zeta, \{\zeta_n\}) = r(\mathcal{Z}, \{\zeta_n\}).$$

$A(\mathcal{Z}, \{\zeta_n\})$ is denoted as set of all asymptotic centers of $\{\zeta_n\}$ wrt $\mathcal{Z}$.

Definition 2.5. [25]. Consider a bounded sequence $\{\zeta_n\}$ within a W-hyperbolic space $(\mathcal{M}, \Omega)$. The sequence $\{\zeta_n\}$ Δ-converges to ζ if ζ serves as the unique asymptotic center for every subsequence $\{\rho_n\}$ derived from $\{\zeta_n\}$.

Consider a W-hyperbolic space $(\mathcal{M}, \Omega)$ and let $\mathcal{Z} \subset \mathcal{M}$ with $\mathcal{Z} \neq \emptyset$. A sequence $\{\zeta_n\}$ in $\mathcal{M}$ is termed Fejér monotone with respect to $\mathcal{Z}$ if

$$\Omega(\zeta^\dagger, \zeta_{n+1}) \le \Omega(\zeta^\dagger, \zeta_n), \text{ for all } n \ge 0, \text{ for all } \zeta^\dagger \in \mathcal{Z}.$$

Let $\Phi : \mathcal{M} \to \mathcal{M}$ be a mapping, we denote $F(\Phi) := \{\zeta \in \mathcal{M} : \Phi(\zeta) = \zeta\}$.

Definition 2.6. [19]. A mapping $\Phi : \mathcal{Z} \to \mathcal{Z}$, where $F(\Phi) \neq \emptyset$, adheres to Condition (I) if there exists another function $f : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:

(1) $f(r) > 0$ for $r \in (0, +\infty)$ and $f(0) = 0$.

(2) $\Omega(\zeta, \Phi(\zeta)) \ge f(\Omega(\zeta, F(\Phi)))$ for all $\zeta \in \mathcal{Z}$,

where $\Omega(\zeta, F(\Phi)) = \inf\{\Omega(\zeta, \vartheta) : \vartheta \in F(\Phi)\}$.

Definition 2.7. Consider a metric space $(\mathcal{M}, \Omega)$ and let $\mathcal{Z} \subset \mathcal{M}$ where $\mathcal{Z} \neq \emptyset$. A mapping $\Phi : \mathcal{Z} \to \mathcal{Z}$ is deemed compact if the closure of $\Phi(\mathcal{Z})$ is compact.

Proposition 2.8. [14]. *Let $(\mathcal{M}, \Omega, W)$ be a complete UCW-hyperbolic space, $\mathcal{Z} \subset \mathcal{M}$ such that $\mathcal{Z} \neq \emptyset$ and $\mathcal{Z}$ be closed convex. If a sequence $\{\zeta_n\}$ in $\mathcal{M}$ is bounded. Then $\{\zeta_n\}$ has a unique asymptotic center wrt $\mathcal{Z}$.*

Lemma 2.9. [14]. *Consider $\{\zeta_n\}$ as a bounded sequence in $(\mathcal{M}, \Omega, W)$, where $A(\mathcal{Z}, \zeta_n) = \{z\}$. Let $\{r_n\}$ and $\{s_n\}$ be two sequences in $\mathbb{R}$, with $r_n \in [0, +\infty)$ for all $n \in \mathbb{N}$, $\limsup r_n \leq 1$, and $\limsup s_n \leq 0$. Assuming $\vartheta \in \mathcal{Z}$ and the existence of $m, N \in \mathbb{N}$ such that*

$$\Omega(\vartheta, \zeta_{n+m}) \leq r_n \Omega(z, \zeta_n) + s_n, \text{ for all } n \geq N.$$

Then $\vartheta = z$.

Lemma 2.10. [1]. *Let $(\mathcal{M}, \Omega, W)$ be a W-hyperbolic space, $\mathcal{Z} \subset \mathcal{M}$ such that $\mathcal{Z} \neq \emptyset$. If $\{\zeta_n\}$ is Fejér monotone wrt $\mathcal{Z}$, $A(\mathcal{Z}, \{\zeta_n\}) = \{\zeta\}$ and $A(\mathcal{M}, \{\rho_n\}) \subseteq \mathcal{Z}$ for every subsequence $\{\rho_n\}$ of $\{\zeta_n\}$. Then the sequence $\{\zeta_n\}$ Δ-converges to $\zeta \in \mathcal{Z}$.*

Motivated by Definition 1.2, the following definition can be defined:

Definition 2.11. Let $(\mathcal{M}, \Omega)$ be a metric space and $\mathcal{Z} \subset \mathcal{M}$ such that $\mathcal{Z} \neq \emptyset$. A mapping $\Phi : \mathcal{Z} \to \mathcal{Z}$ is said to satisfy condition (E) if there exists $\mu \in [1, \infty)$ such that

$$\Omega(\zeta, \Phi(\vartheta)) \leq \mu \Omega(\zeta, \Phi(\zeta)) + \Omega(\zeta, \vartheta) \; \forall \, \zeta, \vartheta \in \mathcal{Z}.$$

Proposition 2.12. *Let $(\mathcal{M}, \Omega)$ be a metric space and $\mathcal{Z} \subset \mathcal{M}$ such that $\mathcal{Z} \neq \emptyset$. Let $\Phi : \mathcal{Z} \to \mathcal{Z}$ be a mapping satisfying condition (E). Then Φ is a quasi-nonexpansive mapping.*

Lemma 2.13. [12, 20] *Let X be a complete UCW-hyperbolic space, then the intersection of any decreasing sequence of nonempty bounded closed convex subsets of X is nonempty.*

Lemma 2.14. [8] *Let X be a UCW-hyperbolic space. Let $\{\zeta_n\}$ and $\{\vartheta_n\}$ be two bounded sequences in X and $\lambda \in (0, 1)$. Assume that $\zeta_{n+1} = \lambda \vartheta_n \oplus (1 - \lambda)\zeta_n$ and $d(\vartheta_{n+1}, \vartheta_n) \leq d(\zeta_{n+1}, \zeta_n)$, for any $n \in \mathbb{N}$. Then $\lim_{n\to\infty} d(\zeta_n, \vartheta_n) = 0$ holds.*

3. Main Results

Llorens-Fuster [15] explored a novel class of mappings as described below:

Definition 3.1. Let $\Phi : \mathcal{Z} \to \mathcal{Z}$ be a mapping. A mapping Φ is called as partially nonexpansive, (in short, PNE), if

$$\left\| \Phi\left(\frac{1}{2}(\zeta + \Phi(\zeta))\right) - \Phi(\zeta) \right\| \leq \frac{1}{2} \|\zeta - \Phi(\zeta)\|$$

for all $\zeta \in \mathcal{Z}$.

Remark 3.2.
- Having fixed points for a mapping is not necessarily implied by either condition PNE or condition (E).
- Every mapping $\Phi : \mathcal{Z} \to \mathcal{Z}$ satisfying condition (C) is partially nonexpansive. However, it should be noted that the converse of above Proposition is not necessarily true.
- The class of partially nonexpansive mappings with fixed point and the class of quasi-nonexpansive mappings are independent in nature.

The concept of partially nonexpansive (PNE) mappings can be extended to nonlinear spaces in the following manner:

Definition 3.3. Let $(\mathcal{M}, \Omega, W)$ be a UCW-hyperbolic space and $\mathcal{Z} \subset \mathcal{M}$ such that $\mathcal{Z} \neq \emptyset$, $\mathcal{Z}$ be convex. A mapping $\Phi : \mathcal{Z} \to \mathcal{Z}$ is said to be partially nonexpansive (PNE) if

$$\Omega\left(\Phi\left(\frac{1}{2}\zeta \oplus \frac{1}{2}\Phi(\zeta)\right), \Phi(\zeta)\right) \leq \frac{1}{2}\Omega(\zeta, \Phi(\zeta))$$

for all $\zeta \in \mathcal{Z}$.

Partially nonexpansive mappings enjoy an approximate fixed point property in Banach spaces. We have a similar conclusion for partially nonexpansive mappings in geodesic spaces.

Lemma 3.4. *Let $(\mathcal{M}, \Omega, W)$ be a UCW-hyperbolic space. Let $\mathcal{Z} \subset \mathcal{M}$ such that $\mathcal{Z} \neq \emptyset$, $\mathcal{Z}$ be a bounded, convex and $\Phi : \mathcal{Z} \to \mathcal{Z}$ be a PNE mapping. Let $\zeta_0 \in \mathcal{Z}$ and define the sequence $\{\zeta_n\}$ by the successive iteration*

$$\zeta_{n+1} = \frac{1}{2}\zeta_n \oplus \frac{1}{2}\Phi(\zeta_n) \tag{3.1}$$

for any $n \in \mathbb{N} \cup \{0\}\}$. Then $\lim_{n\to\infty} \Omega(\zeta_n, \Phi(\zeta_n)) = 0$ holds, i.e., $\{\zeta_n\}$ is an approximate fixed point sequence of Φ.

Proof. From (2.1) and (3.1), we have

$$\frac{1}{2}\Omega(\zeta_n, \Phi(\zeta_n)) = \Omega(\zeta_n, \zeta_{n+1}),$$

Using the fact that, Φ is PNE mapping implies that

$$\begin{aligned} \Omega(\Phi(\zeta_{n+1}), \Phi(\zeta_n)) &= \Omega\left(\Phi\left(\frac{1}{2}\zeta_n \oplus \frac{1}{2}\Phi(\zeta_n)\right), \Phi(\zeta_n)\right) \\ &\leq \frac{1}{2}\Omega(\zeta_n, \Phi(\zeta_n)) \\ &= \Omega(\zeta_n, \zeta_{n+1}) \end{aligned}$$

for any $n \in \mathbb{N}$. Using Lemma 2.14, we conclude that $\lim_{n\to\infty} \Omega(\zeta_n, \Phi(\zeta_n)) = 0$. □

Theorem 3.5. *Let $(\mathcal{M}, \Omega, W)$ be a complete UCW-hyperbolic space and $\mathcal{Z} \subset \mathcal{M}$ such that $\mathcal{Z} \neq \emptyset$, $\mathcal{Z}$ be a bounded closed convex. Let $\Phi : \mathcal{Z} \to \mathcal{Z}$ be a PNE mapping satisfying condition (E). Then Φ has a fixed point in $\mathcal{Z}$.*

Proof. Let $\zeta_0 \in \mathcal{Z}$ and define a sequence $\{\zeta_n\}$ as follows:

$$\zeta_{n+1} = \frac{1}{2}\zeta_n \oplus \frac{1}{2}\Phi(\zeta_n)$$

for all $n \in \mathbb{N} \cup \{0\}\}$. By Proposition 2.8, the sequence $\{\zeta_n\}$ has a unique asymptotic center wrt $\mathcal{Z}$. Let $\{z\} = A(\mathcal{Z}, \{\zeta_n\})$ and by definition of $A(\mathcal{Z}, \{\zeta_n\})$, $z \in \mathcal{Z}$. By Lemma 3.4. we have

$$\lim_{n\to\infty} \Omega\left(\Phi(\zeta_n), \zeta_n\right) = 0 \tag{3.2}$$

and Φ satisfies condition (E)

$$\Omega\left(\zeta_n, \Phi(z)\right) \leq \mu\Omega\left(\Phi(\zeta_n), \zeta_n\right) + \Omega\left(\zeta_n, z\right).$$

From (3.2), we have

$$\limsup_{n\to\infty} \Omega\left(\zeta_n, \Phi(z)\right) \leq \limsup_{n\to\infty} \Omega\left(\zeta_n, z\right).$$

That is

$$r\left(\Phi(z), \{\zeta_n\}\right) \leq r\left(z, \{\zeta_n\}\right).$$

Since the asymptotic center of $\{\zeta_n\}$ is unique it follows that $z = \Phi(z)$. □

By leveraging Theorem 3.5 in conjunction with [1, Lemma 6.2], we derive the subsequent corollary.

Corollary 3.6. *Let $(\mathcal{M}, \Omega, W)$ be a complete UCW-hyperbolic space and $\mathcal{Z} \subset \mathcal{M}$ such that $\mathcal{Z} \neq \emptyset$, $\mathcal{Z}$ be bounded closed convex. Let $\Phi : \mathcal{Z} \to \mathcal{Z}$ be a PNE mapping satisfying condition (E). Then $F(\Phi)$ is nonempty closed and convex.*

Next, we present a common fixed-point theorem for a countable family of commuting partially nonexpansive self-mappings on a given set that satisfies condition (E).

Theorem 3.7. *Let $(\mathcal{M}, \Omega, W)$ be a complete UCW-hyperbolic space and $\mathcal{Z} \subset \mathcal{M}$ such that $\mathcal{Z} \neq \emptyset$, $\mathcal{Z}$ be a bounded closed convex. Let $\{\Phi_j\}_{j=1}^{\infty}$ be a countable family of commuting partially nonexpansive self-mappings on $\mathcal{Z}$ satisfying condition (E). Then $\{\Phi_j\}_{j=1}^{\infty}$ has a common fixed point.*

Proof. For each $n \in \mathbb{N}$, we define

$$\mathcal{Z}_n := \bigcap_{j=1}^{n} F\left(\Phi_j\right).$$

Thus, $\mathcal{Z}_1 = F\left(\Phi_1\right)$. By Corollary 3.6, $\mathcal{Z}_1$ is nonempty, closed and convex subset of $\mathcal{M}$. Since $\mathcal{Z}_1 \subset \mathcal{Z}$, $\mathcal{Z}_1$ is bounded. Let $k \in \mathbb{N}$ such that $k \geq 2$ and assume that $\mathcal{Z}_{k-1}$ is nonempty, closed, bounded and convex. We claim that the set $\mathcal{Z}_k$ is nonempty, closed, bounded and convex. Since $\mathcal{Z}_{k-1} \neq \emptyset$, take $\zeta \in \mathcal{Z}_{k-1}$ and $j \in \mathbb{N}$ with $1 \leq j < k$. Since $\{\Phi_j\}_{j=1}^{\infty}$ is a family of commuting mappings, Φ_k and Φ_j commute,

$$\Phi_k(\zeta) = \Phi_k \circ \Phi_j(\zeta) = \Phi_j \circ \Phi_k(\zeta).$$

Thus $\Phi_k(\zeta)$ is a fixed point of Φ_j, it follows that $\Phi_k(\zeta) \in \mathcal{Z}_{k-1}$. Therefore we obtain $\Phi_k(\mathcal{Z}_{k-1}) \subset \mathcal{Z}_{k-1}$. In view of Theorem 3.5, Φ_k has a fixed point in $\mathcal{Z}_{k-1}$,

$$\mathcal{Z}_k = \mathcal{Z}_{k-1} \cap F(\Phi_k) \neq \emptyset.$$

Using Corollary 3.6, $\mathcal{Z}_k$ is closed and convex. By induction, the set $\mathcal{Z}_n$ is nonempty closed bounded and convex $\forall\, n \in \mathbb{N}$. Since $\mathcal{Z}_n \subset \mathcal{Z}_{n-1}$ $\forall\, n \in \mathbb{N}$. Using Lemma 2.13, we obtain

$$\bigcap_{j=1}^{\infty} F(\Phi_j) = \bigcap_{n=1}^{\infty} \mathcal{Z}_n \neq \emptyset.$$

This completes the proof. □

Before delving into the proofs of the Δ and strong convergence theorems, we establish the following lemma.

Lemma 3.8. *Let $(\mathcal{M}, \Omega, W)$ be a complete UCW-hyperbolic space and $\mathcal{Z} \subset \mathcal{M}$ such that $\mathcal{Z} \neq \emptyset$, $\mathcal{Z}$ be bounded closed convex. Let $\zeta_0 \in \mathcal{Z}$ and define the sequence $\{\zeta_n\}$ by the successive iteration*

$$\zeta_{n+1} = \frac{1}{2}\zeta_n \oplus \frac{1}{2}\Phi(\zeta_n) \tag{3.3}$$

for any $n \in \mathbb{N} \cup \{0\}\}$. Then $\lim_{n\to\infty} \Omega(\zeta_{\text{n}}, p)$ exists $\forall\, p \in F(\Phi)$.

Proof. In view of Theorem 3.5, $F(\Phi) \neq \emptyset$. Let $p \in F(\Phi)$, from (W1) and Proposition 2.12, we have

$$\begin{aligned}\Omega(\zeta_{n+1}, p) &= \Omega\left(\frac{1}{2}\zeta_n \oplus \frac{1}{2}\Phi(\zeta_n), p\right)\\ &\leq \frac{1}{2}\Omega(\zeta_n, p) + \frac{1}{2}\Omega(\Phi(\zeta_n), p)\\ &\leq \Omega(\zeta_n, p).\end{aligned}$$

That is

$$\Omega(\zeta_{n+1}, p) \leq \Omega(\zeta_n, p).$$

Thus, sequence $\{\Omega(\zeta_n, p)\}$ is bounded and monotone $\forall\, p \in F(\Phi)$. Therefore we obtain the desired result. □

Now we prove our Δ convergence theorem.

Theorem 3.9. *Let $(\mathcal{M}, \Omega, W)$ be a complete UCW-hyperbolic space and $\mathcal{Z} \subset \mathcal{M}$ such that $\mathcal{Z} \neq \emptyset$, $\mathcal{Z}$ be bounded closed convex. Let $\Phi : \mathcal{Z} \to \mathcal{Z}$ be a PNE mapping satisfying condition (E). Let $\zeta_0 \in \mathcal{Z}$ and consider the sequence $\{\zeta_n\}$ by the successive iteration*

$$\zeta_{n+1} = \frac{1}{2}\zeta_n \oplus \frac{1}{2}\Phi(\zeta_n) \tag{3.4}$$

for any $n \in \mathbb{N} \cup \{0\}\}$. Then the sequence $\{\zeta_n\}$ Δ-converges to a point in $F(\Phi)$.

Proof. In view of Theorem 3.5, $F(\Phi) \neq \emptyset$, let $z^\dagger \in F(\Phi)$. By Lemma 3.8, we have the following property for sequence $\{\Omega(\zeta_n, z^\dagger)\}$,

$$\Omega\left(\zeta_{n+1}, z^\dagger\right) \leq \Omega\left(\zeta_n, z^\dagger\right)$$

for all $z^\dagger \in F(\Phi)$. Therefore, sequence $\{\zeta_n\}$ is Fejér monotone wrt $F(\Phi)$. From Corollary 3.6, $F(\Phi)$ is closed and convex. In view of Proposition 2.8, the sequence $\{\zeta_n\}$ has unique asymptotic center $w^\dagger$ wrt $F(\Phi)$. Assume that $\{\rho_n\}$ is a subsequence of $\{\zeta_n\}$, again from Proposition 2.8, $\{\rho_n\}$ has a unique asymptotic center $\rho^\dagger$ wrt $F(\Phi)$. Now, by the condition (E)

$$\Omega(\rho_n, \Phi(\rho^\dagger)) \leq \mu\Omega(\Phi(\rho_n), \rho_n) + \Omega(\rho_n, \rho^\dagger).$$

Using the fact that $\lim_{n\to+\infty} \Omega(\zeta_n, \Phi(\zeta_n)) = 0$, we have

$$\Omega(\rho_n, \Phi(\rho^\dagger)) \leq \Omega(\rho_n, \rho^\dagger).$$

In view of Lemma 2.9, we have $\Phi(\rho^\dagger) = \rho^\dagger$. Using Lemma 2.10, we can conclude that the sequence $\{\zeta_n\}$ Δ-converges to a point in $F(\Phi)$. □

Theorem 3.10. *Suppose $\mathcal{M}$, $\mathcal{Z}$, Φ, and $\{\zeta_n\}$ are as defined in Theorem 3.9. If Φ is a compact mapping, then the sequence $\{\zeta_n\}$ strongly converges to a point in $F(\Phi)$.*

Proof. Considering Lemma 3.8, it is evident that the sequence $\{\zeta_n\}$ is bounded. Furthermore, referring to Lemma 3.4, we have

$$\lim_{n\to\infty} \Omega(\zeta_n, \Phi(\zeta_n)) = 0. \tag{3.5}$$

Given the definition of a compact mapping, the range of $\mathcal{Z}$ under Φ is confined within a compact set. Consequently, there exists a subsequence $\{\Phi(\zeta_{n_j})\}$ of $\{\Phi(\zeta_n)\}$ that strongly converges to $\zeta^\dagger \in \mathcal{Z}$. In view of (3.5), it implies that the subsequence $\{\zeta_{n_j}\}$ strongly converges to $\zeta^\dagger$. Using the fact that mapping Φ satisfies condition (E) and (3.5)

$$\begin{aligned}\Omega(\zeta_{n_j}, \Phi(\zeta^\dagger)) &\leq \mu\Omega(\zeta_{n_j}, \Phi(\zeta_{n_j})) + \Omega(\zeta_{n_j}, \zeta^\dagger)\\ &\leq \Omega(\zeta_{n_j}, \zeta^\dagger).\end{aligned}$$

Therefore, subsequence $\{\zeta_{n_j}\}$ strongly converges to $\Phi(\zeta^\dagger)$, it implies that $\Phi(\zeta^\dagger) = \zeta^\dagger$. Since $\lim_{n\to\infty} \Omega(\zeta_n, \zeta^\dagger)$ exists, the sequence $\{\zeta_n\}$ strongly converges to a point in $F(\Phi)$. □

Corollary 3.11. *Let $(\mathcal{M}, \Omega, W)$ be a complete UCW-hyperbolic space and $\mathcal{Z} \subset \mathcal{M}$ such that $\mathcal{Z} \neq \emptyset$, $\mathcal{Z}$ be a compact convex. Let $\Phi : \mathcal{Z} \to \mathcal{Z}$ be a PNE mapping satisfying condition (E). Let $\zeta_0 \in \mathcal{Z}$ and define the sequence $\{\zeta_n\}$ by the successive iteration*

$$\zeta_{n+1} = \frac{1}{2}\zeta_n \oplus \frac{1}{2}\Phi\left(\zeta_n\right) \tag{3.6}$$

for any $n \in \mathbb{N} \cup \{0\}\}$. Then the sequence $\{\zeta_n\}$ strongly converges to a point in $F(\Phi)$.

Theorem 3.12. *Suppose $\mathcal{M}$, $\mathcal{Z}$, Φ, and $\{\zeta_n\}$ are as defined in Theorem 3.9. If the mapping Φ satisfies condition (I), then the sequence $\{\zeta_n\}$ strongly converges to a point in $F(\Phi)$.*

Proof. According to Lemma 3.8, the sequences $\{\Omega(\zeta_n, z^\dagger)\}$ are monotonically non-increasing for all $z^\dagger \in F(\Phi)$. Consequently, the sequence $\{\Omega(\zeta_n, F(\Phi))\}$ also exhibits monotonic non-increasing behavior. This ensures the existence of $\lim_{n\to\infty} \Omega(\zeta_n, F(\Phi))$. In view of Lemma 3.4, we have

$$\lim_{n\to\infty} \Omega(\zeta_n, \Phi(\zeta_n)) = 0. \tag{3.7}$$

Since Φ satisfies condition (I),

$$\Omega(\zeta_n, \Phi(\zeta_n)) \geq f(\Omega(\zeta_n, F(\Phi))).$$

From (3.7), $\lim_{n\to\infty} f(\Omega(\zeta_n, F(\Phi))) = 0$ and

$$\lim_{n\to\infty} \Omega(\zeta_n, F(\Phi)) = 0. \tag{3.8}$$

Now, it can be confirmed that the sequence $\{\zeta_n\}$ is Cauchy. For any given $\varepsilon > 0$, according to (3.8), there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\Omega(\zeta_n, F(\Phi)) < \frac{\varepsilon}{4}.$$

Therefore

$$\inf\{\Omega(\zeta_{n_0}, z^\dagger) : z^\dagger \in F(\Phi)\} < \frac{\varepsilon}{4},$$

and there exists $z^\dagger \in F(\Phi)$ such that

$$\Omega(\zeta_{n_0}, z^\dagger) < \frac{\varepsilon}{2}.$$

Thus, for all $m, n \geq n_0$,

$$\begin{aligned} \Omega(\zeta_{n+m}, \zeta_n) \leq \Omega(\zeta_{n+m}, z^\dagger) + \Omega(z^\dagger, \zeta_n) &\leq 2\Omega(\zeta_{n_0}, z^\dagger) \\ &< 2\frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

and the sequence $\{\zeta_n\}$ is Cauchy. Due to the closedness property of the set $\mathcal{Z}$ within $\mathcal{M}$, the sequence $\{\zeta_n\}$ converges to a point $\zeta^\dagger \in \mathcal{Z}$. Now, by condition (E), we have

$$\begin{aligned} \Omega(\zeta^\dagger, \Phi(\zeta^\dagger)) &\leq \Omega(\zeta^\dagger, \zeta_n) + \Omega(\zeta_n, \Phi(\zeta^\dagger)) \\ &\leq \Omega(\zeta^\dagger, \zeta_n) + \mu\Omega(\zeta_n, \Phi(\zeta_n)) + \Omega(\zeta^\dagger, \zeta_n) \\ &\leq 2\Omega(\zeta^\dagger, \zeta_n) + \mu\Omega(\zeta_n, \Phi(\zeta_n)) \end{aligned}$$

from (3.7), $\zeta^\dagger = \Phi(\zeta^\dagger)$. Thus, the sequence $\{\zeta_n\}$ strongly converges to a point in $F(\Phi)$. □

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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