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MAXIMUM AND MINIMUM MODULUS PRINCIPLES FOR FRACTIONAL ANALYTIC FUNCTIONS

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Abstract. In this paper we utilize the concepts of $\alpha-$fractional axes and $\alpha-$fractional analytic function to formulate the fractional version of the maximum and munimum modulus principles.

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1. Introduction

The theory of complex analysis has so many interesting results that can be used in different scientfic disciplines. It is believed that complex analysis, has essential role not only in the progress of several areas in pure and applied mathematics but also in technology and engineering. It includes several significant aspects and theorems that can be implemented in order to handle some challenging problems encountered in both science and technology [14], [3]. For example, the well known Cauchy's residue theorem can be used to evaluate some real definite integrals as well as infinite series. Moreover, in electrical engineering, the voltage of the so called alternating current, that requires two parameters, can be presented as a complex number. In quantum physics, complex variables are used to present the particle state as well as the construction of the wave equation.

The key concept of complex theory is the analyticity of a complex valued function over a domain on the complex plane. Analytic functions are essentially well-behaved complex valued functions in the sense that the rules for computing derivatives of real valued functions can be adopted. However, compared

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to real differentiable functions, a far broader range of conclusions can be made about complex analytic functions. In the following we state some interesting theorems about analytic functions [9], [10], [2].

Theorem 1.1. [9] (*The Maximum Modulus Principile*) *If $f : D \to \mathbb{C}$ is a continuous function over a closed bounded region $D \subseteq \mathbb{C}$ such that f is analytic and non-costant in D°, the interior of D, then the maximum value of $|f(z)|$ over D if it exists, then it is somewhere on ∂D, the boundary of D.*

Theorem 1.2. [9] (*The Minimum Modulus Principile*) *If $f : D \to \mathbb{C}$ is continuous function over a closed bounded region $D \subseteq \mathbb{C}$ such that f is analytic, non-costant in D°, and $f(z) \neq 0$ for all $z \in D$, then the minimum value of $|f(z)|$ over D if it exists, it attained somewhere on ∂D.*

Utilizing the α-conformable fractional derivative where $\alpha \in (0, 1)$,the first attempt to formulate the fractional version of analytic functions was carried out in 2018 [16]. Concequently, Cauchy like theorem and a fractional Cauchy like formula for fractional analytic functions were established. However, the definition did not give Cauchy Riemann equations.

Recently, in [11], the concept of a fractional nalytic function was introduced again together with some related peoperties. Moreover, the fractional versions for both Cauchy Riemann equations and Cauchy integral formula were implemented.

In this paper, we are interested in developing a fractional versions of both the maximum and minimum modulus principles. Before we establish our main results, we commence with some required definitions and theorems.

2. Preliminaries

In [17] the definition of α-conformable fractional derivative was introduced as follows.

Definition 2.1. *Let $\alpha \in (0, 1)$, and $u : I \subseteq (0, \infty) \to \mathbb{R}$. For $x \in I$, let*

$$D^\alpha u(x) = \lim_{\epsilon \to 0} \frac{u(x + \epsilon x^{1-\alpha}) - u(x)}{\epsilon}. \tag{2.1}$$

If the limit exists, then it is called the α-conformable fractional derivative of u at x. Moreover, u is said to be α-differentiable on $(0, r)$ for some $r > 0$, and if $\lim_{\epsilon \to 0^+} D^\alpha u(x)$ exists then we write

$$D^\alpha u(0) = \lim_{\varkappa \to 0} D^\alpha u(x). \tag{2.2}$$

For $\alpha \in (0, 1]$ and u, v *are* α-differentiable at a point x, one can easily see that the conformable derivative satisfies

$$\begin{aligned}
&(i)\ D^\alpha(c_1 u + c_2 v) = c_1 D^\alpha(u) + c_2 D^\alpha(v), \text{ for all } c_1, c_2 \in \mathbb{R},\\
&(ii)\ D^\alpha(k) = 0, \text{ for all constant functions } f(x) = k,\\
&(iii)\ D^\alpha(uv) = uD^\alpha(v) + vD^\alpha(u),\\
&(iv)\ D^\alpha\left(\tfrac{u}{v}\right) = \tfrac{vD^\alpha(u) - uD^\alpha(v)}{v^2},\ v(x) \neq 0.
\end{aligned} \tag{2.3}$$

In the following, we provide the α-conformable fractional derivatives of some basic functions,

$$\begin{aligned} &(i)\ D^{\alpha}(x^{p}) = px^{p-\alpha},\\ &(ii)\ D^{\alpha}(\sin(\tfrac{1}{\alpha}x^{\alpha})) = \cos(\tfrac{1}{\alpha}x^{\alpha}),\\ &(iii)\ D^{\alpha}(\cos(\tfrac{1}{\alpha}x^{\alpha})) = -\sin(\tfrac{1}{\alpha}x^{\alpha}),\\ &(iv)\ D^{\alpha}(e^{\frac{1}{\alpha}x^{\alpha}}) = e^{\frac{1}{\alpha}x^{\alpha}}. \end{aligned} \tag{2.4}$$

On letting $\alpha = 1$ in these derivatives, we get the corresponding classical rules for ordinary derivatives.

Many differential equations can be transformed to fractional form and can have many applications in many branches of science [13], [4] , [1], [7], [8].

Let us write $D_s^{\alpha}u$ and $D_t^{\alpha}u$ to denote the partial α-conformable fractional derivative with respect to s and t respectively. Moreove we write $D_s^{2\alpha}u$ to denote $D_s^{\alpha}D_s^{\alpha}u$ and similarly, for $D_t^{2\alpha}u$.

Now, let $D \subseteq \mathbb{C}$ be a region in the complex plane such that for any $z \in D$, $z = x + iy$ for some $x; y \geq 0$. Then for any function $f : D \to \mathbb{C}$, one can write

$$f(z) = f(x, y) = u(x, y) + iv(x, y), \tag{2.5}$$

where $u(x, y)$ and $v(x, y)$ are the real and imaginary part of f, respectivly.

Definition 2.2. [11] *A function $f : D \to \mathbb{C}$ is said to be α-differentiable at $z_\circ = x_\circ + iy_\circ \in D - \{0\}$ and denoted by $f^{\alpha}(z_\circ)$, if*

$$\lim_{(\epsilon_1,\epsilon_2)\to(0,0)} \frac{f(x_\circ + \epsilon_1 x_\circ^{1-\alpha}, y_\circ + \epsilon_2 y_\circ^{1-\alpha}) - f(x, y)}{\epsilon_1 + i\epsilon_2}, \quad \alpha \in (0, 1) \tag{2.6}$$

exists. Moreover, if there exists $\delta > 0$ such that f is α-differentiable for all $z \in B(\delta, z_\circ)$ where $B(\delta, z_\circ)$ is an open disc centered at $z_\circ$, then f is said to be α-analytic at $z_\circ$.

Theorem 2.1. [11] (α-*Fractional Cauchy Integral Formula*) *Let f be α-analytic everywhere inside and on a simple closed α-contour C taken in the positive sense such that*

$$C := \{z = x^{\alpha} + iy^{\alpha} : |(x^{\alpha} + iy^{\alpha}) - (\xi_\circ^{\alpha} + i\xi_\circ^{\alpha})| = r\}.$$

Then for all $z_\circ \in C^\circ$, the interior of C, we have

$$f(z_\circ) = \frac{\alpha}{2\pi i}\oint_C \frac{f(z)}{z - z_\circ}dz^{\alpha}. \tag{2.7}$$

Now, we are ready to introduce our main results.

3. Main Results

3.1. $\alpha-$**Fractional Axes in the Eucleadian Space.** Let $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$. Then the set $E_1 = \{(x, 0) : x \in \mathbb{R}\}$, is just the x-axis, and the set $E_2 = \{(0, y) : y \in \mathbb{R}\}$, is the y-axis. Accordingly, let us introduce the following definition.

Definition 3.1.1. ($\alpha-$*fractional axes*) *Let*

$$\begin{aligned} P_1 &= \{x^\alpha : x \geq 0,\ \alpha \in (0,1)\ \} \equiv [0,\infty) \\ &\quad \textit{and} \\ P_2 &= \{-x^\alpha : x \geq 0,\ \alpha \in (0,1)\ \} \equiv (-\infty, 0)\,. \end{aligned} \tag{3.1.1}$$

Then $P = P_1 \cup P_2$ is called the $\alpha-$fractional x-axis. Similarly, let

$$\begin{aligned} Q_1 &= \{y^\alpha : y \geq 0,\ \alpha \in (0,1)\ \} \equiv [0,\infty) \\ &\quad \textit{and} \\ Q_2 &= \{-y^\alpha : y \geq 0,\ \alpha \in (0,1)\} \equiv (-\infty, 0)\,. \end{aligned} \tag{3.1.2}$$

Then $Q = Q_1 \cup Q_2$ is called the $\alpha-$fractional y-axis.

Noting that, utilizing (3.1.1) and (3.1.2), any point in the first quadrant is of the form (x^α, y^α), $x, y \geq 0$.

In order to see what is new in these fractional axes, for a, $b \in \mathbb{R}$ let us consider the equation $(x - x_\circ)^2 + (y - y_\circ)^2 = r^2$ which represents a circle in the classical $xy-$plane. However, in the new $\alpha-$fractional axes, the equation $(x^\alpha - x_\circ^\alpha)^2 + (y^\alpha - y_\circ^\alpha)^2 = r^2$ represents an $\alpha-$fractional circle which is not a circle in the classical $xy-$plane. This is the main ingrediant in this paper.

3.2. **Fractional Maximum / Minimum Modulus Principle.**

Theorem 3.2.1. (*The Fractional Maximum Modulus Principle*) *Let $f : G \subseteq \mathbb{R}^2 \to \mathbb{C}$ be an $\alpha-$fractional analytic function, where G is a simply connected region in the first quadrant. Then $|f(z)|$ cannot have maximum value in G° unless f is constant.*

Proof. If possible, assume $|f(z)|$ has a maximum value at some point $z_\circ = x_\circ^\alpha + iy_\circ^\alpha$ in G° and f is not constant in G. Then there exists $\alpha-$fractional circle γ in G with center $z_\circ$, namely,

$$(x^\alpha - x_\circ^\alpha)^2 + (y^\alpha - y_\circ^\alpha)^2 = r^2,$$

such that $|f(z_\circ)| > |f(\omega)|$ for some ω lies on such $\alpha-$fractional circle.

Since $|f(z)|$ is continuous, then there is an α-fractional arc, η around the point $\omega \in \gamma$ such that

$$|f(z_\circ)| > |f(z)| \quad \text{for all } z \in \eta. \tag{3.2.1}$$

Now, by the α-fractional Cauchy formula [11], we have

$$f(z_\circ) = \frac{\alpha}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_\circ} dz^\alpha, \tag{3.2.2}$$

where $z = x^\alpha + iy^\alpha$ and $z_\circ = x_\circ^\alpha + iy_\circ^\alpha$. Let ζ be the complement of α-fractional arc η in the α-fractional circle γ. Hence,

$$f(z_\circ) = \frac{\alpha}{2\pi i} \oint_\zeta \frac{f(z)}{z - z_\circ} dz^\alpha + \frac{\alpha}{2\pi i} \oint_\eta \frac{f(z)}{z - z_\circ} dz^\alpha. \tag{3.2.3}$$

So,

$$|f(z_\circ)| \leq \frac{\alpha}{2\pi} \oint_\zeta \frac{|f(z)|}{|z - z_\circ|} |dz^\alpha| + \frac{\alpha}{2\pi} \oint_\eta \frac{|f(z)|}{|z - z_\circ|} |dz^\alpha|. \tag{3.2.4}$$

But, on η, we have $|f(z)| < |f(z_\circ)|$ for all $z \in \eta$. Therefore,

$$|f(z_\circ)| \leq \frac{\alpha}{2\pi} \oint_\zeta \frac{|f(z_\circ)|}{|z - z_\circ|} |dz^\alpha| + \frac{\alpha}{2\pi} \oint_\eta \frac{|f(z_\circ)|}{|z - z_\circ|} |dz^\alpha|. \tag{3.2.5}$$

By (3.2.1), the inequality is strict, because it is strict on η. Therefore,

$$|f(z_\circ)| < \frac{\alpha}{2\pi} \oint_\zeta \frac{|f(z_\circ)|}{r} |dz^\alpha| + \frac{\alpha}{2\pi} \oint_\eta \frac{|f(z_\circ)|}{r} |dz^\alpha|. \tag{3.2.6}$$

Now, $z^\alpha = x^\alpha + iy^\alpha$, and lies on the $\alpha-$fractional circle $(x^\alpha - x_\circ^\alpha)^2 + (y^\alpha - y_\circ^\alpha)^2 = r^2$. Thus, $z^\alpha = re^{i\theta}$.

Consequently, $dz^\alpha = rie^{i\theta} d\theta$, and so, $|dz^\alpha| = r d\theta$, $0 \leq \theta \leq 2\pi$. Hence, (3.2.6) , can be reduced as

$$\begin{aligned} |f(z_\circ)| &< \frac{\alpha}{2\pi} |f(z_\circ)| \left(\oint_\zeta \frac{r}{r} d\theta + \frac{\alpha}{2\pi} \oint_\eta \frac{r}{r} d\theta \right) \\ &< \alpha |f(z_\circ)|, \end{aligned} \tag{3.2.7}$$

which is a contradiction since $0 < \alpha < 1$. Hence, $|f(z)|$ cannot have a maximum value at any interior point of G. This ends the proof.

Theorem 3.2.2. *Let $f : G \subseteq \mathbb{C} \to \mathbb{C}$ be an $\alpha-$fractional analytic function in G such that $f(z) \neq 0$ for all $z \in G$. Then, $\frac{1}{f}$ is $\alpha-$fractional analytic in G.*

Proof. By (2.6) , we have

$$\begin{aligned} D^\alpha \frac{1}{f(z)} &= \lim_{(\epsilon_1,\epsilon_2)\to(0,0)} \frac{\frac{1}{f(x+\epsilon_1 x^{1-\alpha}, y+\epsilon_2 y^{1-\alpha})} - \frac{1}{f(x,y)}}{\epsilon_1 + i\epsilon_2} \\ &= \lim_{(\epsilon_1,\epsilon_2)\to(0,0)} \frac{f(x,y) - f(x+\epsilon_1 x^{1-\alpha}, y+\epsilon_2 y^{1-\alpha})}{f(x,y) f(x+\epsilon_1 x^{1-\alpha}, y+\epsilon_2 y^{1-\alpha})} \cdot \frac{1}{\epsilon_1 + i\epsilon_2} \\ &= \lim_{(\epsilon_1,\epsilon_2)\to(0,0)} \left[\frac{f(x,y) - f(x+\epsilon_1 x^{1-\alpha}, y+\epsilon_2 y^{1-\alpha})}{\epsilon_1 + i\epsilon_2} \right. \\ &\qquad \left. \frac{1}{f(x,y) f(x+\epsilon_1 x^{1-\alpha}, y+\epsilon_2 y^{1-\alpha})} \right] \\ &= -\frac{f^\alpha(z)}{f^2(z)}. \end{aligned}$$

But, $f(z) \neq 0$ for all $z \in G$. Hence, $\frac{1}{f}$ is $\alpha-$fractional analytic.

Theorem 3.2.3. (*The Fractional Minimum Modulus Principle*) *Let $f : G \subseteq \mathbb{C} \to \mathbb{C}$ be an $\alpha-$fractional analytic function in G such that $f(z) \neq 0$ for all $z \in G$. Then, $|f|$ cannot have minimum in G.*

Proof. Let $g(z) = \frac{1}{f(z)}$. By Theorem (3.2.2), g is $\alpha-$fractional analytic function in G. Now, $|g(z)|$ has a maximum at z if $\frac{1}{|f(z)|}$ has a maximum at z. Moreover, $\frac{1}{|f(z)|}$ has a maximum at z if $|f(z)|$ has a minimum at z. But, by the fractional maximum modulus principle that is Theorem (3.2.1), $|g(z)|$ has no maximum in G. Hence, $|f(z)|$ cannot have minimum in G.

4. Conclusions

This paper has successfully introduced the fractional maximum modulus principle and the fractional minimum modulus principle. This was implemented by establishing the so called $\alpha-$fractional axes.

Authors' Contributions

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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