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RINGS CONVEX DOMINATIONS IN GRAPHS

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Abstract. A subset $S \subseteq V(G)$ is said to be a rings convex dominating set of a graph G if it is both a convex dominating set of G and a rings dominating set of G. The minimum cardinality of a rings convex dominating set is called a rings convex domination number of G and is denoted by $\gamma_{ricon}(G)$. In this paper, the authors give characterizations of a rings convex dominating set of some graphs and graphs obtained from the join, and corona product of two graphs. Furthermore, the rings convex domination numbers of these graphs are determined, and the graphs with no rings convex dominating sets are investigated.

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1. Introduction

What is known today as domination in graphs is believed to have started back in 1962 when Berge in [1] introduced the concept of external stability. This notion has opened a lot of opportunities for studies and research, and many mathematicians are exploring many variants of domination in graphs. This evidence is shown in many papers such as [7], [8], [9], [10], [11], and [16].

The concept of convexity in graphs was introduced in the book by Buckley and Harary [2] and was further developed and investigated by Chartrand and Zhang [4]. This interesting concept led to the development of studies on domination in graphs. In 2004, Lemańska [12] introduced the notion of convex domination in graphs. Later, Canoy [15] discussed aspects of convexity and the convex dominating set in the context of the composition of a graph into a complete graph. This intriguing notion has inspired many researchers to conduct various studies on convex domination. One such study was conducted by J.A. Hassan et al. [6], who introduced the concept of convex hop domination in graphs.

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One of the newest variants of domination in graphs is the concept of rings domination, introduced by Saja Abed and M.N. Al-Harere in 2022 [14]. Since it is a recent concept, there have been no further studies on it until the following year, when Caay [7] studied equitable rings domination in graphs.

In this study we introduce the notion of rings convex domination in graphs. This means that for a given a graph G, the dominating set of G must have to be a convex dominating set and a rings domінting set.

2. Preliminaries and the Working Definitions

The graph we consider in this study is a simple connected graph $G = (V(G), E(G))$. This means, the graphs have no loops and multiple edges. We denote the set of vertices of G as $V(G)$ and the set of edges of G as $E(G)$.

Given $v \in V(G)$, the neighborhood of v is the set $N_G(v) = \{u \in V(G) : uv \in E(G).\}$. Given $D \subseteq V$, the set $N_G(D) = N(D) = \bigcup_{v \in D} N_G(v)$ and the set $N_G[D] = N[D] = D \bigcup N(D)$ are the *open neighborhood* and the *closed neighborhood* of D, respectively. A *spanning subgraph* of a graph G is a subgraph obtained by deleting some edges of G with the same vertex set.

Example 2.1. [4] A cycle C_n is a spanning subgraph of a complete graph K_n.

Theorem 2.2. [4] *A graph G is a cycle graph if and only if every vertex of G is adjacent to two other vertices.*

The following are the definitions of the binary operations in graphs used in this study: join, and corona.

Definition 2.3. [3] The **join** $G + H$ of the two graphs G and H is the graph with vertex set

$$V(G + H) = V(G) + V(H)$$

and the edge set

$$E(G + H) = E(G) \cup E(H) \cup \{uv \ : \ u \in V(G), \ v \in V(H)\}.$$

Definition 2.4. [13] The **corona** $G \circ H$ of two graphs G and H is the graph obtained by taking one copy of G of order n and n copies of H, and then joining the ith vertex of G to every vertex in the ith copy of H.

In this paper, we mean $\deg_G(v)$ to be the degree of a vertex v of a graph G. If no confusion arises, we denote $\deg(v)$. We also denote $\Delta(G)$ and $\delta(G)$ to be the maximum and minimum degree of G, respectively. However in binary operation in graphs, we mean $\deg_G(v)$ and $\deg_H(u)$ to be the degree of v and u in the graph G and H, respectively. This means that $\deg_G(v)$ may not be equal to $\deg_{G*H}(v)$, as $\deg_{G*H}(v)$ denotes the degree of v in some binary operation G and H.

In [1], a subset S of $V(G)$ is a *dominating set* of G if for every $v \in V(G)\backslash S$, there exists $u \in S$ such that $uv \in E(G)$. That is, $N[S] = V(G)$. The minimum cardinality of the dominating set S of G is called a *domination number* of G and is denoted by $\gamma(G)$. If a dominating set S equals the $\gamma(G)$, we say that S is a γ-set of G.

In this paper, if u is in a dominating set S, and $v \in V(G) \setminus S$ such that $uv \in E(G)$, then we say u *dominates* v or v is *dominated* by u.

In [14], a dominating S of G is a *rings dominating set* of G if every $v_1 \in V(G) \setminus S$ is adjacent to atleast other vertices in $V(G) \setminus S$. The minimum cardinality of the rings dominating set S of G is called a *rings domination number* of G and is denoted by $\gamma_{ri}(G)$. If S is a rings dominating set of the smallest cardinality, then S is called $\boldsymbol{\gamma}$*-set* of G. In this paper, if u is in a rings dominating set S, and $v \in V(G) \setminus S$ such that $uv \in E(G)$, then we say u *rings dominates* v, or v is *rings dominated* by u.

Remark 2.5. [14] For a γ_{ri}-set S of G of order n, we have

i. the order of G is $n \geq 4$.
ii. for each $v \in V(G) \setminus S$, the $\deg(v) \geq 3$.
iii. $1 \leq |S| \leq n - 3$.
iv. $3 \leq |V(G) \setminus S| \leq n - 1$.
v. $1 \leq \gamma_{ri}(G) \leq |S| \leq n - 3$.

In [15], for any vertices $u, v \in V(G)$, a $u - v$ *geodesic* is the shortest path in G having u and v as the end-vertices. In this paper, we denote the length of a $u - v$ geodesic by $d_G(u, v)$. We define $\mathcal{G}_G[u, v]$ to be the set of vertices that lie in the $u - v$ geodesic. A set $C \subseteq V(G)$ is *convex in* G if for every two vertices $u, v \in C$, the vertex set of every $u - v$ geodesic is in C. A subset $S \subseteq V(G)$ that is both a convex and a dominating set is said to be *convex dominating set* and the smallest cardinality of a convex dominating set of G is said to be a *convex domination number of* G denoted by $\gamma_{con}(G)$. We say that a set S is said to be γ_{con}-set if its cardinality is equal to $\gamma_{con}(G)$. In this study, if S is a convex dominating set such that $u \in S$ and $v \in V(G) \setminus S$ with $uv \in E(G)$, then we say u *convex dominates* v, or v is *convex dominated* by u.

The following theorem, although seemingly obvious to prove, serves as the motivation for the working definition in this paper.

Theorem 2.6. *Let $S \subseteq V(G)$ be a convex dominating set of G. Then S is a rings dominating set if and only $V(G) \setminus S$ contains a subgraph isomorphic to C_k for some $k \leq |V(G) \setminus S|$.*

Following Theorem 2.6, we now have our working definition of this paper.

Definition 2.7. A convex dominating set $S \subseteq V(G)$ is said to be rings convex dominating set of G if every vertex $v \in V(G) \setminus S$ is adjacent to atleast two vertices in $V(G) \setminus S$. The minimum cardinality of a rings convex dominating set of G is called rings convex domination number of G and is denoted by $\boldsymbol{\gamma_{ricon}(G)}$. A rings convex dominating set S of G is said to be $\boldsymbol{\gamma_{ricon}}$**-set** of G whenever $|S| = \gamma_{ricon}(G)$.

Remark 2.8. If $S \subseteq V(G)$ is convex dominating set of G, and $u \in S$ and $v \in V(G) \setminus S$ such that $uv \in E(G)$, then u convex dominates v, or v is convex dominated by u.

Example 2.9. Consider a complete graph K_5 of order 5 shown in Figure 1. Note that $S = \{u_1, u_2\} \subset V(K_5)$ is already a convex dominating set of K_5 since S convex dominates other vertices of K_5. Now $V(K_5) \setminus S = \{u_3, u_4, u_5\}$. Observe that u_3 is adjacent to u_4 and u_5 which are not in S, and the same case with u_4 and u_5 which are adjacent to vertices not in S. This means, S is also a rings dominating set. Therefore, $\gamma_{ricon}(K_5) = 2$.

Remark 2.10. The generalization of Example 2.9 will be shown in Proposition 3.2.

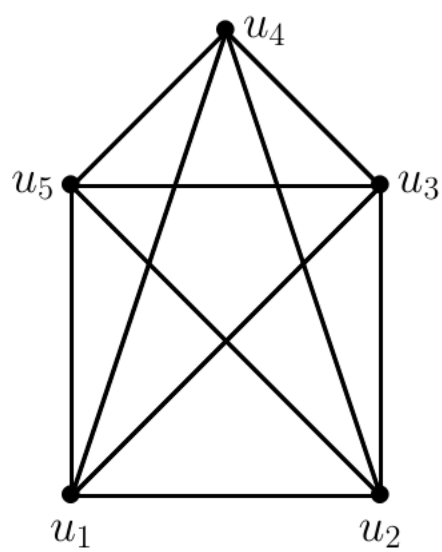

FIGURE 1. Complete graph of order 5.

3. Rings Convex Domination in Graphs

We first present the main condition for which the rings convex dominating set in G exists.

Theorem 3.1. *Let G be any graph. If G has an induced subgraph G' such that $G' \cong C_k$, $k \geq 3$, and $V(G) \setminus G'$ forms a convex dominating set of G with cardinality atleast 2, then there exists a rings convex dominating set of G.*

Proof. The proof is obvious and directly follows from the definition. $\square$

We will generalize Example 2.9.

Proposition 3.2. *Let $n \geq 5$. Then $\gamma_{ricon}(K_n) = 2$.*

Proof. Since every vertices of K_n are adjacent to all other vertices, every pair of vertices are shortest path, and forms a dominating set. In particular, let $u_1, u_2 \in V(K_n)$. Then $\{u_1, u_2\}$ is a convex dominating set. Since $n \geq 3$, $|V(K_n) \setminus \{u_1, u_2\}| \geq 3$, and these the vertices of $V(K_n) \setminus \{u_1, u_2\}$ are adjacent to every other vertices of $V(K_n) \setminus \{u_1, u_2\}$. This means that $\{u_1, u_2\}$ is also a rings dominating set. Therefore, $\gamma_{ricon}(K_n) = 2$. □

The following proposition follows directly from the results in [14].

Proposition 3.3. *Any graph tree T_n has no rings convex dominating set. In particular, P_n, C_n and S_n have no rings convex dominating set.*

Proposition 3.4. *Let $G = K_{P_1,\cdots,P_k}$ be a complete k-partite graph where $k \geq 4$. Then $\gamma_{ricon}(G) = 2$.*

Proof. Let u_{ij} be the jth vertex of the ith partition P_i of G. Then $u_{ij}v_{st} \in E(G)$, for some $v_{sp} \in P_s$, $s = 1, \cdots, k$. Thus, $d(u_{ij}, v_{st}) = 2$. Since $u_{ij}v_{st} \in E(G)$, u_{ij} dominates all vertices of G except u_{ik}, $k \neq j$. But $u_{ij}v_{st} \in E(G)$, implying v_{st} dominates u_{ik}. Hence, u_{ij}, v_{st} is a convex dominating set. Since $k \geq 4$, and u_{ik} is adjacent to at least 2 vertices not in u_{ij}, v_{st} and w_{ab} is adjacent to at least 2 vertuces not in u_{ij}, v_{st}, $a \neq i \neq s$, $b \in \{1, \cdots, |P_a|\}$, it follows that u_{ij}, v_{st} forms a rings dominating set, and u_{ij}, v_{st} is a rings convex dominating set. Therefore, $\gamma_{ricon}(G) = 2$. □

Corollary 3.5. *Let $G = K_{m,n}$ be a complete bipartite graph where $m, n \geq 2$. Then $\gamma_{ricon}(G) = 2$.*

Theorem 3.6. *Let S and S' be the γ_{ricon}-sets of G and H, respectively. Then*

$$\gamma_{ricon}(G + H) = \min\big\{|S|, |S'|\big\}.$$

Proof. Let S and S' be the γ_{ricon}-sets of G and H, respectively. If $u \in S \subseteq V(G)$, then $uv \in E(G+H)$, for all $v \in V(H)$. Thus, S is a convex dominating set of $G + H$. Similarly, S' is a convex dominating set of $G+H$. Since every $x \in V(G)$ and $y \in V(H)$ are adjacent to at least two vertices in G and H, respectively, every $x' \in V(G) \setminus S$ and $y' \in V(H) \setminus S'$ are adjacent to at least 2 vertices of $V(G) \setminus S$ and $V(H) \setminus S'$, respectively. Hence S and S' are γ_{ricon}-sets of $G + H$. Therefore, $\gamma_{ricon}(G + H) = \min\{|S|, |S'|\}$. □

Corollary 3.7. *Let S be a γ_{ricon}-set of G. Then S is a γ_{ricon}-set of $G + H$ if and only if $\delta(G) \geq 2$.*

Theorem 3.8. *Let G be a graph of order $n \geq 2$. If H is a graph such that $\delta(H) \geq 2$. Then $\gamma_{ricon}(G \circ H) = n$.*

Proof. Let $u_i \in V(G)$, for some $i \in \{1, \cdots, n\}$. Then u_i dominates all vertices of the ith copy of H. Thus, $V(G)$ dominates all n copies of H. If $u_i, u_j \in V(G)$, $i \neq j$, then $V(G)$ is a convex set for which all vertices dominate. Hence, $V(G)$ is a convex dominating set. Since $\deg(v) \geq 2$, for all $v \in V(H)$, it follows that $V(G)$ is also a rings dominating set. Therefore, $V(G)$ is a rings convex dominating set of $G \circ H$. Consequently, $\gamma_{ricon}(G \circ H) = n$. □

Theorem 3.9. *If S is a rings convex dominating set of G, then S is a rings convex dominating set of $G+H$.*

Proof. Let S be a rings convex dominating set of G. Then S is a convex dominating set of G. Since every vertex in G is adjacent to all vertices in H in $G+H$, it follows that S is a convex dominating set of G. Since S is a rings convex dominating set in G, every vertices of $V(G)\setminus S$ has degree at least 2. This means thta every vertex of $V(H)$ is also adjacent to at least 2 vertices in $V(G)\setminus S$. Hence, every vertices of $V(G+H)\setminus S$ is adjacent to at least vertices of $V(G+H)$. Therefore, S is also a rings convex dominating set of $G+H$. $\square$

Theorem 3.10. *Let G and H be graphs of orders $n\geq 3$ and $m\geq 3$, respectively. Then $S=\{u\}\cup\{v\}$ is a rings convex dominating set of $G+H$ for any $u\in V(G)$, and $v\in V(H)$. That is, $\gamma_{ricon}(G+H)=2$.*

Proof. Let $S=\{u\}\cup\{v\}\subset V(G)$ such that $u\in V(G)$ and $v\in V(H)$. Since $uv_i\in E(G+H)$, for some $v_i\in V(H)$, $i=1,\cdots,|V(H)|$, this means that u dominates $V(H)$. Also, since $vw_j\in E(G+H)$, for some $w_j\in V(G)$, $j=1,\cdots,|V(G)|$, this means that v dominates $V(H)$. Thus, S is a dominating set of $G+H$. Since $uv\in E(G+H)$, $d(u,v)=2$ implying S is a γ_{con}-set of $G+H$. Since $n\geq 3$, every $v_i\in V(H)$, $v_i\neq v$, $i=1,\cdots,|V(H)|-1$, v_i is adjacent to all $w_j\in V(G)\setminus\{u\}$. Similarly, since $m\geq 3$, every $u_k\in V(G)$, $u_k\neq u$, $u=1,\cdots,|V(G)|-1$, u_k is adjacent to all $z_l\in V(H)\setminus\{v\}$. Hence, S is a rings convex dominating set of $G+H$. $\square$

Example 3.11. Consider graphs G_1 and G_2 in Figure 2. Then G_1+G_2 is shown in Figure 3 with $S=\{u_1,v_1\}$ as the rings convex dominating set of G_1+G_2.

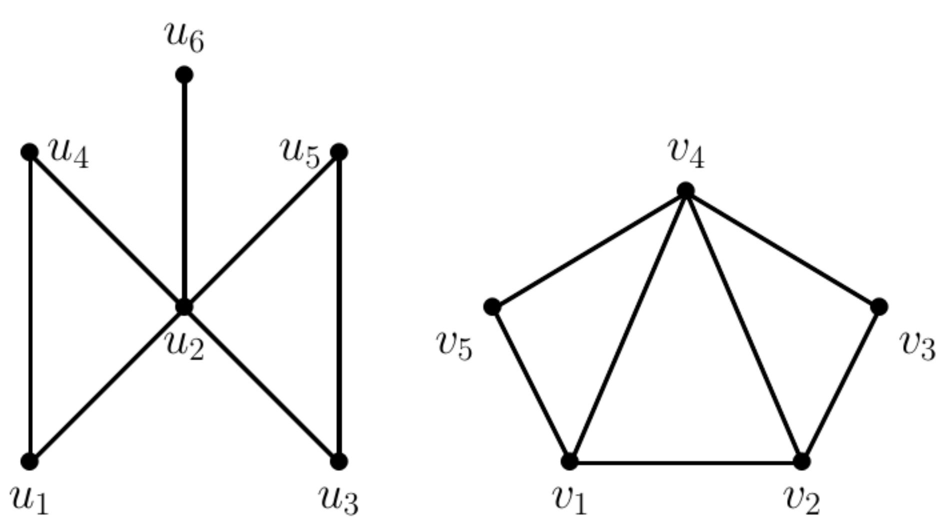

FIGURE 2. Graphs G_1 and G_2.

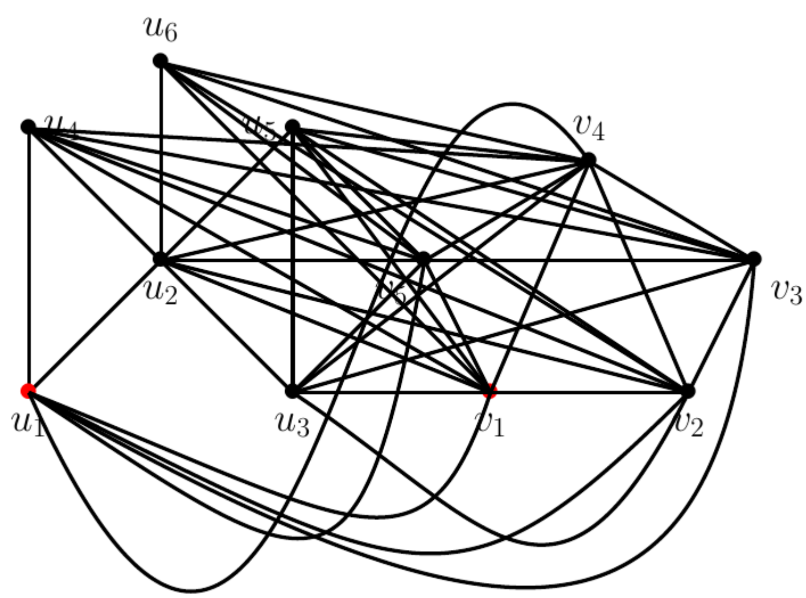

Figure 3. Graph $G_1 + G_2$ with $|u_1, v_1$ as rings convex dominating set.

4. Conclusions and Recommendations

The paper has introduced the concept of rings convex dominating sets of some graphs and results on some binary operations such as join and corona of two graphs. The existence of the rings convex dominating sets of some graphs and some binary operations are examined because not all graphs have this set. For future investigation, the authors recommend to explore this parameter to determine the exact values of some graphs and some binary operations that have not been discussed in the study. Moreover, we suggest to look for the relationship with other parameters of domination which are related to this parameter.

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Authors' Contributions

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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