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ON FRACTIONAL DIVERGENCE THEOREM

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Abstract. In this paper, we utilize the conformable derivative to introduce the fractional versions of some concepts related to vector analysis. The fractional normal vector to a given surface is formulated in order to define the fractional surface integrals. Moreover, we discuss and prove the fractional form of the divergence Theorem.

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1. Introduction

In vector field theory, the surface integral of vector field or namely, the flux of the vecor field across a surface is considered as a basic characteristic of vector fields. Problems such as the Gauss's Theorem for electric fiels, concept of heat flows, and calculating moments and masses of thin shells cannot be understood without the understanding of the nature of integration over surfaces. Moreover, the relationship between the surface integral of a vector field around a closed surface and the triple integral of the divergence of the vector field over the region bounded by the surface namely, the Gauss's divergence theorem, or simply known as divergence theorem, is one of the most important results in vector calculus theory. It is the primary building block of how we derive conservation laws such as the conservation of mass, momentum, and energy in general sciences and particularly in physics [1].

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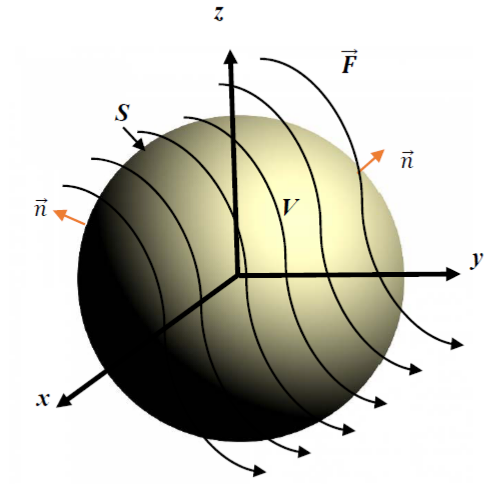

FIGURE 1. The divergence theorem relates the surface integral of around a closed sufface S and the triple integral of the $\vec{\nabla} \cdot \vec{F}$ over the region V bounded by that surface.

Definition 1.1. (*Surface Integral*) [1] *If* $\overrightarrow{F}(x,y,z) = P(x,y,z)\widehat{i} + Q(x,y,z)\widehat{j} + R(x,y,z)\widehat{k}$ *is a continuous vector field defined on an oriented surface* $S = \{(x,y,z)|\ z = g(x,y), (x,y) \in D = \underset{(x,y)}{\Pr} S\}$ *with a unit outward normal vector* $\overrightarrow{n}(x,y,z)$*, then the surface integral of* F *over* S *is*

$$\iint\limits_S \overrightarrow{F} \cdot ds = \iint\limits_S \left(\overrightarrow{F} \cdot \overrightarrow{n}\right) ds = \iint\limits_D \left(-P\frac{\partial z}{\partial x} - Q\frac{\partial z}{\partial y} + R\right) dA, \tag{1.1}$$

where $\overrightarrow{n}(x,y,z) = \frac{-\frac{\partial z}{\partial x}\widehat{i} - \frac{\partial z}{\partial y}\widehat{j} + \widehat{k}}{\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}}$ *and* $ds = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA.$

Theorem 1.1. (*Divergence Theorem*) [1] *Let* V *be a simple solid region with a positively (outward) oriented boundary surface* S*. Let* $\overrightarrow{F}(x,y,z) = P(x,y,z)\widehat{i} + Q(x,y,z)\widehat{j} + R(x,y,z)\widehat{k}$ *be a vector field with components function* P*,* Q*, and* R*, have continuous partial derivatives on an open region that contains* V *(Figure 1). Then*

$$\iint\limits_S \left(\overrightarrow{F} \cdot \overrightarrow{n}\right) ds = \iiint\limits_V \left(\overrightarrow{\nabla} \cdot \overrightarrow{F}\right) dV, \tag{1.2}$$

where the divergence of a vector field, $\overrightarrow{\nabla} \cdot \overrightarrow{F}$ *is given by*

$$\overrightarrow{\nabla} \cdot \overrightarrow{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}. \tag{1.3}$$

2. Fractional Vector Analysis

In [2] the definition of α-conformable fractional derivative was introduced as follows.

Definition 2.1. *Let* $\alpha \in (0,1)$*, and* $u : I \subseteq (0,\infty) \to \mathbb{R}$*. For* $x \in I$*, let*

$$D^\alpha u(x) = \lim_{\epsilon \to 0} \frac{u(x + \epsilon x^{1-\alpha}) - u(x)}{\epsilon}. \tag{2.1}$$

If the limit exists, then it is called the α-conformable fractional derivative of u at x. Moreover, u is said to be α-differentiable on $(0,r)$ for some $r>0$, if $\lim_{\epsilon\to 0^+} D^\alpha u(x)$ exists then we write

$$D^\alpha u(0) = \lim_{\varkappa\to 0} D^\alpha u(x). \tag{2.2}$$

For $\alpha \in (0,1]$ and u, υ *are* α-differentiable at a point $\varkappa$, one can easily see that the conformable derivative satisfies

$$\begin{aligned}
&(i)\ D^\alpha(c_1 u + c_2\upsilon) = c_1 D^\alpha(u) + c_2 D^\alpha(\upsilon), \text{ for all } c_1, c_2 \in \mathbb{R},\\
&(ii)\ D^\alpha(k) = 0, \text{ for all constant functions } f(x) = k,\\
&(iii)\ D^\alpha(u\upsilon) = uD^\alpha(\upsilon) + \upsilon D^\alpha(u),\\
&(iv)\ D^\alpha(\tfrac{u}{\upsilon}) = \tfrac{\upsilon D^\alpha(u) - uD^\alpha(\upsilon)}{\upsilon^2}, \upsilon(x) \neq 0.
\end{aligned} \tag{2.3}$$

In the following, we provide the α-conformable fractional derivatives of some basic functions,

$$\begin{aligned}
&(i)\ D^\alpha(x^p) = px^{p-\alpha},\\
&(ii)\ D^\alpha(\sin(\tfrac{1}{\alpha}x^\alpha)) = \cos(\tfrac{1}{\alpha}x^\alpha),\\
&(iii)\ D^\alpha(\cos(\tfrac{1}{\alpha}x^\alpha)) = -\sin(\tfrac{1}{\alpha}x^\alpha),\\
&(iv)\ D^\alpha(e^{\frac{1}{\alpha}x^\alpha}) = e^{\frac{1}{\alpha}x^\alpha}.
\end{aligned} \tag{2.4}$$

On letting $\alpha = 1$ in these derivatives, we get the corresponding classical rules for ordinary derivatives. For more on fractional calculus and its applications we refer to [4], [5], [6], [7], [8], and [9]. Many differential equations can be transformed to fractional form and can have many applications in many branches of science.

Let us write $D_s^\alpha u$ and $D_t^\alpha u$ to denote the partial α-conformable fractional derivative with respect to s and t respectively. Moreove we write $D_s^{2\alpha} u$ to denote $D_s^\alpha D_s^\alpha u$ and similarly, for $D_t^{2\alpha} u$.

Definition 2.2. [2] *The α-fractional integral of a function f starting from $\mathbf{a} \geq 0$ is denoted by $I_\alpha^{\mathbf{a}}(u)(x)$ such that*

$$I_\alpha^{\mathbf{a}}(u)(x) = I_1^{\mathbf{a}}\left(x^{\alpha-1}u\right) = \int_{\mathbf{a}}^x \frac{u(t)}{t^{1-\alpha}}\,dt,, \tag{2.5}$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0,1)$.

Recently, in 2020, utilizing the conformable derivative, the fractional form of some concepts related to vector analysis theory were proposed [3]. The fractional gradient, fractional divergence, the fractional Laplacian and fractional curl were introduced as follows.

Definition 2.3. [3] *The fractional gradient vector of a scalar field $f(x_1,x_2,x_3)$ is given by*

$$\overrightarrow{\nabla}^\alpha f = D_x^\alpha f\ \widehat{i} + D_y^\alpha f\ \widehat{j} + D_z^\alpha f\ \widehat{k}. \tag{2.6}$$

Definition 2.4. [3] *The fractional divergence of a vector field $\overrightarrow{F}(x,y,z) = P(x,y,z)\widehat{i} + Q(x,y,z)\widehat{j} + R(x,y,z)\widehat{k}$ is given by*

$$\overrightarrow{\nabla}^\alpha \cdot \overrightarrow{F} = D_x^\alpha P + D_y^\alpha Q + D_z^\alpha R. \tag{2.7}$$

Moreovre, the fractional line integral and the fractional version of Green's Theorem were also discussed.

3. Main Results

In this section, the goal is to provide the concept of both fractional surface integrals as well as the fractional version of the divergence Theorem.

Definition 3.1. (*Fractional Surface Integral of Vector Field*) *If* $\overrightarrow{F}(x,y,z) = P(x,y,z)\widehat{i} + Q(x,y,z)\widehat{j} + R(x,y,z)\widehat{k}$ *is a continuous vector field defined on an oriented surface* $S = \{(x,y,z) |\ z = g(x,y), (x,y) \in D = \Pr_{(x,y)} S\}$ *with a unit fractional outward normal vector* $\overrightarrow{n}_\alpha(x,y,z)$*, then the fractional surface integral of* F *over* S *is*

$$\iint\limits_S \overrightarrow{F} \cdot\ ds^\alpha = \iint\limits_S \left(\overrightarrow{F} \cdot \overrightarrow{n}_\alpha\right)\ ds^\alpha, \tag{3.1}$$

where

(*i*) $\overrightarrow{n}_\alpha(x,y,z) = \frac{\nabla^\alpha h(x,y,z)}{|\nabla^\alpha h(x,y,z)|} = \frac{-x^{1-\alpha} g_x\ \widehat{i} - y^{1-\alpha} g_y\ \widehat{j} + z^{1-\alpha}\ \widehat{k}}{\sqrt{(x^{1-\alpha}g_x)^2 + (y^{1-\alpha}g_y)^2 + (z^{1-\alpha})^2}}$,
such that the surface S *is the level surface* $h(x,y,z) = 0$ *of the function*

$$h(x,y,z) = z - g(x,y), \tag{3.2}$$

(*ii*) $ds^\alpha = \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} dx^\alpha dy^\alpha = \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} x^{1-\alpha} y^{1-\alpha} dx dy$,
is the fractional surface area element.

Now, we are interested in finding the fractional form of the divergence Theorem. To do so, let us consider the following theorem.

Theorem 3.1. (*Fractional Divergence Theorem*) *Let* V *be a simple solid region with a positively* (*outward*) *oriented boundary surface* S*. Let* $\overrightarrow{F}(x,y,z) = f_1(x,y,z)\widehat{i} + f_2(x,y,z)\widehat{j} + f_3(x,y,z)\widehat{k}$ *be a vector field with components function* f_1*,* f_2*, and* f_3*, have continuous partial derivatives on an open region that contains* V*. Then*

$$\iint\limits_S \left(\overrightarrow{F} \cdot \overrightarrow{n}_\alpha\right) ds^\alpha = \iiint\limits_V \left(\overrightarrow{\nabla}^\alpha \cdot \overrightarrow{F}\right) dV^\alpha, \tag{3.3}$$

where $\overrightarrow{n}_\alpha$ *is the unit fractional normal outward vector to the surface* S*,* $ds^\alpha = x^{1-\alpha} y^{1-\alpha} ds$ *is the fractional surface area element,* $\overrightarrow{\nabla}^\alpha$ *is the fractional divergence, and* $dV^\alpha = x^{1-\alpha} y^{1-\alpha} z^{1-\alpha} dV$ *is the fractional solid volume element.*

Proof.: Let S be a closed surface in $\mathbb{R}^3$. For simplicity, we will assume that $S = S_1 \cup S_2$, such that

$$S_1 \text{ has outward fractional normal } \overrightarrow{n}_\alpha \text{ that satisfies } \overrightarrow{n}_\alpha \cdot \widehat{k} \geq 0,$$
$$S_2 \text{ has outward fractional normal } -\overrightarrow{n}_\alpha \text{ that satisfies } -\overrightarrow{n}_\alpha \cdot \widehat{k} \leq 0.$$

Further, we will assume that $\underset{(x,y)}{\Pr} S_1 = \underset{(x,y)}{\Pr} S_2 = W$, where Pr is the projection on the $xy-$plane (Figure 2).

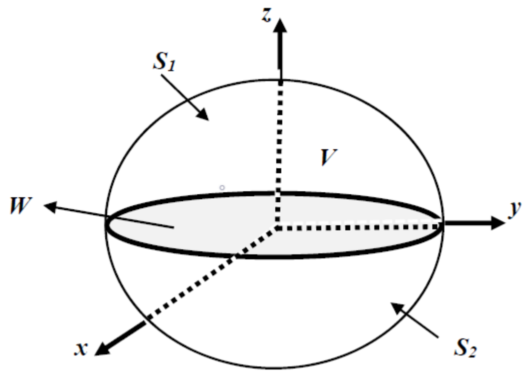

FIGURE 2. The projection of both surfaces S_1 and S_2 on the $xy-$plane.

Now, utilizing (2.7), the right hand side of (3.3) can be written as

$$\begin{aligned} &\iiint\limits_V \left(\overrightarrow{\nabla}^\alpha \cdot \overrightarrow{F}\right) dV^\alpha \\ &= \iiint\limits_V \frac{\partial^\alpha f_1}{\partial x^\alpha} dV^\alpha + \iiint\limits_V \frac{\partial^\alpha f_2}{\partial y^\alpha} dV^\alpha + \iiint\limits_V \frac{\partial^\alpha f_3}{\partial z^\alpha} dV^\alpha. \end{aligned} \tag{3.4}$$

Also, since $\overrightarrow{F}(x,y,z) = f_1(x,y,z)\widehat{i} + f_2(x,y,z)\widehat{j} + f_3(x,y,z)\widehat{k}$, we have

$$\overrightarrow{F} \cdot \overrightarrow{n}_\alpha = f_1\widehat{i} \cdot \overrightarrow{n}_\alpha + f_2\widehat{j} \cdot \overrightarrow{n}_\alpha + f_3\widehat{k} \cdot \overrightarrow{n}_\alpha. \tag{3.5}$$

Hence, the left hand side of (3.3) can be expanded as

$$\begin{aligned} &\iint\limits_S \left(\overrightarrow{F} \cdot \overrightarrow{n}_\alpha\right) ds^\alpha \qquad (3.6) \\ &= \iint\limits_S f_1 \left(\widehat{i} \cdot \overrightarrow{n}_\alpha\right) ds^\alpha + \iint\limits_S f_2 \left(\widehat{j} \cdot \overrightarrow{n}_\alpha\right) ds^\alpha + \iint\limits_S f_3 \left(\widehat{k} \cdot \overrightarrow{n}_\alpha\right) ds^\alpha. \end{aligned}$$

Therefore, if we can prove that

$$\begin{aligned} &\text{(i)} \iint\limits_S f_1 \left(\widehat{i} \cdot \overrightarrow{n}_\alpha\right) ds^\alpha = \iiint\limits_V \tfrac{\partial^\alpha f_1}{\partial x^\alpha} dV^\alpha, \\ &\text{(ii)} \iint\limits_S f_2 \left(\widehat{j} \cdot \overrightarrow{n}_\alpha\right) ds^\alpha = \iiint\limits_V \tfrac{\partial^\alpha f_2}{\partial y^\alpha} dV^\alpha, \\ &\text{(iii)} \iint\limits_S f_3 \left(\widehat{k} \cdot \overrightarrow{n}_\alpha\right) ds^\alpha = \iiint\limits_V \tfrac{\partial^\alpha f_3}{\partial z^\alpha} dV^\alpha, \end{aligned} \tag{3.7}$$

then the proof is complete.

In the following, we will prove (iii) and a similar argument can be established for both (i) and (ii).

Now, by setting that $dz^{\alpha} = z^{\alpha-1}dz$, the right hand side of (iii) can be reduced as follows

$$\begin{aligned}
&\iiint\limits_{V} \frac{\partial^{\alpha} f_3}{\partial z^{\alpha}} dV^{\alpha} \\
&= \iiint\limits_{V} \frac{\partial^{\alpha} f_3}{\partial z^{\alpha}} dx^{\alpha} dy^{\alpha} dz^{\alpha} \\
&= \iint\limits_{W} \left(\int \frac{\partial^{\alpha} f_3}{\partial z^{\alpha}} dz^{\alpha} \right) dA^{\alpha}
\end{aligned}$$

$$\begin{aligned}
&= \iint\limits_{W} \left(\int \left(z^{1-\alpha} \frac{df_3}{dz} \right) z^{\alpha-1} dz \right) dA^{\alpha} \\
&= \iint\limits_{W} \left(\int_{z_2}^{z_1} df_3 \right) dA^{\alpha} \\
&= \iint\limits_{W} \left[f_3(x,y,z_1) - f_3(x,y,z_2) \right] dA^{\alpha}, \qquad (3.8)
\end{aligned}$$

where $z_1(x,y) \equiv S_1$ and $z_2(x,y) \equiv S_2$.

Finally, the left hand side of ((iii), (3.7)) can be expanded as follows:

$$\iint\limits_{S} f_3 \left(\widehat{k} \cdot \overrightarrow{n}_{\alpha} \right) ds^{\alpha} = \iint\limits_{S_1} f_3 \left(\widehat{k} \cdot \overrightarrow{n}_{\alpha} \right) ds^{\alpha} + \iint\limits_{S_2} f_3 \left(\widehat{k} \cdot \overrightarrow{n}_{\alpha} \right) ds^{\alpha}, \qquad (3.9)$$

where on $z_1(x,y) \equiv S_1$ we have $\widehat{k} \cdot \overrightarrow{n}_{\alpha} = 1$ and on $z_2(x,y) \equiv S_2$ we have $\widehat{k} \cdot \overrightarrow{n}_{\alpha} = -1$.

So,

$$\begin{aligned}
&\iint\limits_{S} f_3 \left(\widehat{k} \cdot \overrightarrow{n}_{\alpha} \right) ds^{\alpha} \\
&= \iint\limits_{S_1} x^{1-\alpha} y^{1-\alpha} f_3 ds - \iint\limits_{S_2} x^{1-\alpha} y^{1-\alpha} f_3 ds \\
&= \iint\limits_{W} x^{1-\alpha} y^{1-\alpha} f_3(x,y,z_1) dA - \iint\limits_{W} x^{1-\alpha} y^{1-\alpha} f_3(x,y,z_2) dA \\
&= \iint\limits_{W} f_3(x,y,z_1) dA^{\alpha} - \iint\limits_{W} f_3(x,y,z_2) dA^{\alpha}, \qquad (3.10)
\end{aligned}$$

where $ds^{\alpha} = x^{1-\alpha} y^{1-\alpha} ds$ and $dA^{\alpha} = x^{1-\alpha} y^{1-\alpha} dA$. Therefore, by (3.8) and (3.10), we get ((iii), (3.7)). Hence, the proof is complete.

4. Conclusions

In this paper, utilizing the concept of conformable derivative, we establish fractional versions for both unit outward normal vector and the surface area element. This enable us to formulate the fractional vesion of the divergence theorem.

Authors' Contributions

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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