Asia Pac. J. Math. 2024 **11**:88

THE ERROR ANALYSIS OF WEAK GALERKIN FINITE ELEMENT METHOD FOR SOLVING TIME FRACTIONAL REACTION-ADVECTION-DIFFUSION EQUATION IN POROUS MEDIA

AHMED JABBAR HUSSEIN[1,*], JAWAD JAAYWEL SAADOON[2]

[1]College of Education for Pure Sciences, University of Thi Qar, Thi Qar, Iraq

[2]Engineering Technical College of Missan, Southern Technical University, Missan, Iraq

jawad.saadoon@stu.edu.iq

[*]Corresponding author: ahmedjabbar1974_math@utq.edu.iq

Received Jul. 5, 2024

Abstract. In this paper, we apply a weak Galerkin method for solving system of non linear time Fractional Reaction-advection-diffusion Equation in Porous Media(FRDEs). Based on a conservation form for nonlinear term and some of the technical derivational. Theorticly, the convergence for discrete time weak Galerkin finite element schemes is proved. Numerically, the accuracy and effectiveness of the weak Galerkin finite element method (WG-FEM) are illustrated by using Numerical example with the lower order Raviart-Thomas element RT_k for discrete weak derivative space.

2020 Mathematics Subject Classification. 65M60.

Key words and phrases. error analysis; weak Galerkin finite element method(WG-FEM); FRDEs.

1. Introduction

We consider the time fractional reaction-advection-diffusion equation in porous media as [1]:

$$ {}_0^C D_t^{\alpha_1} u - \epsilon_1 u_{xx} - \gamma_1 \, {}_0^C D_x^{\beta_1} u + \delta_1 u_x u^{m_1} - \zeta_1 (uv)_x = \Psi_1(u) + f_1(x,t), \tag{1.1} $$

$$ {}_0^C D_t^{\alpha_2} v - \epsilon_1 v_{xx} - \gamma_2 \, {}_0^C D_x^{\beta_2} v + \delta_2 v_x v^{m_2} - \zeta_2 (uv)_x = \Psi_2(v) + f_2(x,t), \quad x \in J, \tag{1.2} $$

where $J = \{x : 0 \le x \le b\}$, $0 < \alpha_1, \alpha_2 \le 1$, $1 < \beta_1, \beta_2 \le 2$, $0 \le t \le 1$, $\Psi_i(w) = \lambda_i w(1-w), i = 1,2,$ are nonlinear functions, with the initial-boundary conditions

$$ u(x,0) = g_1(x), \quad v(x,0) = g_2(x), \tag{1.3} $$

$$ u(0,t) = u(b,t) = 0, \quad v(0,t) = v(b,t) = 0, \tag{1.4} $$

DOI: 10.28924/APJM/11-88

In the above expressions, ϵ_1, ϵ_2, are diffusion coefficient, δ_1, δ_2 be the advection coefficients, λ_1, λ_2 denotes the reaction coefficient, $u(x,t), v(x,t)$ are the velocity components to be determined, $\gamma_i' s, \zeta_i' s, i = 1, 2$ are constants and f_1, f_2, represents the force terms.

Fractional Reaction-Diffusion equations are widely used in demonstrating abnormal slowly diffusion process for non conservative system viz, the FRDEs are the mathematical modeling of the change in time and space of the concentration of one or more modecules of a substance.

In the study of nonlinear physical problems, fractional order reaction-diffusion model plays a significant role, therefore authors are motivated to find the approximate numerical solution of space-time fractional order reaction -diffusion equation and to analyses their physical behavior in porous media associated with Liouville-Caputo fractional order derivative.

To solve fractional differential equations, there exist several methods: Laplace and Fourier transforms, truncated Taylor series, numerical methods, etc. (see [2] and references therein). Recently, a lot of attention has been put on the fractional calculus of variations (see [3], [4] and [5]).

In this paper, we adopt the notations $H^m(\widetilde{J})$ to indicate the usual Sobolve spaces on subinterval $\widetilde{J} \subset J$ equipped with the inner product $(.,.)_{m,\widetilde{J}}$, the norm $\|.\|_{m,\widetilde{J}}$ and the seminorms $|.|_{m,\widetilde{J}}$ for $m \geq 0$. The space $H^1(0,T; H^m(J))$ is defined as:

$$H^1(0,T;H^m(J)) = \{w \in H^m(J); \int_0^T \|\frac{\partial w(t)}{\partial t}\|^2_{m,J} dt < \infty\},$$

where $H^m(J)$ is Hilbert space of order m defined as:

$$H^m(\Omega) = \left\{u : \partial^\nu u \in L^2(\Omega);\ \ \forall \nu\ \ s.t\ |\nu| \leq m\right\}$$

This paper is organized as follows. In section 2, we introduce some of definition of fractional derevative and fractional integral. In Section 3, we establish a weak variational form for problem (1.1)-(1.2). In Section 4 we introduce definitions of weak derivative and discrete weak derivative. In Section 5, we devoted to the continuous and discrete time WG-FEM. In Section 6, we establish the optimal order error estimates in L^2-norm for discrete time WG-FEM. Finally, we present a numerical example to verify theory.

2. Preliminaries

There are several definitions of fractional derivatives and fractional integrals, like Riemann-Liouville, Caputo, Riesz, RieszCaputo, Weyl, Grunwald-Letnikov, Hadamard, Chen, etc. We will present the definitions of the first two of them. Except otherwise stated, proofs of results may be found in [6].

Let $w : [a,b] \to \mathbb{R}$ be a function, α a positive real numbers, n the integer satisfies $n-1 \leq \alpha < n$, and Γ the Gamma function, Then

- The Riemann-Lioville fractional integral of order α defined by

$$I_x^{\alpha} w(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} w(s) ds \tag{2.1}$$

- The Riemann-Lioville fractional derivatives of order α defined by

$${}_a^R D_x^{\alpha} w(x) = \frac{d^n}{dx^n} I_x^{n-\alpha} w(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-s)^{n-\alpha-1} w(s) ds \tag{2.2}$$

- The Caputo fractional derivative of order α defined by

$${}_a^C D_x^{\alpha} w(x) = I_x^{n-\alpha} \frac{d^n}{dx^n} w(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-s)^{n-\alpha-1} w^{(n)}(s) ds \tag{2.3}$$

Based on definitions of fractional derivatives, fractional integrals above, the thierd term in equations (1.1)-(1.2) can be written as:

$$\gamma_1 \, {}_0^C D_x^{\beta_1} u = \gamma_1 \int_0^x k_1(x-s) u_{xx}(s) ds \text{ and } \gamma_2 \, {}_0^C D_x^{\beta_2} v = \gamma_1 \int_0^x k_2(x-s) v_{xx}(s) ds$$

where $k_i(x-s) = \frac{(x-s)^{1-\beta_i}}{\Gamma(2-\beta_i)}, \ i = 1, 2.$

3. Weak Variational Form

In order to defined the WG finite element approximation to problem (1.1)-(1.2), we first need to derive a weak variational form associated with problem (1.1)-(1.2).

Based on definitions of fractional derivatives, fractional integrals, multiplying equation (1.1)-(1.2) by $\chi, \rho \in H_0^1(\Omega)$ and integrating by part, the weak form is find $u, v \in H^1(0,T; H_0^1(\Omega))$ such that

$$({}_0^C D_t^{\alpha_1} u, \chi) \quad + \quad a(u,\chi) = (\Psi_1(u), \chi) + (f_1, \chi), \quad \forall \chi \in H_1^0(J), \tag{3.1}$$

$$({}_0^C D_t^{\alpha_2} v, \rho) \quad + \quad a(v,\rho) = (\Psi_2(v), \rho) + (f_2, \rho), \quad \forall \rho \in H_1^0(J), \tag{3.2}$$

$$u(x,0) \quad = \quad g_1(x), \quad v(x,0) = g_2(x), \ x \in J, \tag{3.3}$$

where

$$\begin{aligned} a(u,\chi) \quad &= \quad (\epsilon_1 u_x, \chi_x) + \gamma_1 \int_0^x k_1(x-s)(u_x(s), \chi_x) ds + \frac{\delta_1}{m_1+2}(u^{m_1} u_x, \chi) - \frac{\delta_1}{m_1+2}(u^{m_1} u, \chi_x) \\ &+ \quad \zeta_1(uv, \chi_x). \end{aligned} \tag{3.4}$$

$$\begin{aligned} a(v,\rho) \quad &= \quad (\epsilon_2 v_x, \rho_x) + \gamma_2 \int_0^x k_2(x-s)(v_x(s), \rho_x) ds + \frac{\delta_2}{m_2+2}(v^{m_2} v, \rho) - \frac{\delta_2}{m_2+2}(v^{m_2} v_x, \rho_x) \\ &+ \quad \zeta_2(uv, \rho_x). \end{aligned} \tag{3.5}$$

4. Weak Galerkin Finite Element Approximation

In this section, we present some of weak derivative concept. Let $\tilde{J} = [x_a, x_b]$ closed interval with interior $J_a = (x_a, x_b)$. A weak function on $\tilde{J}$ refer to a function $w = \{w_0, w_a, w_b\}$. $w_0 = w|_{J_a} \in L^2(J_a)$, values $w_a = w(x_a)$ and $w_b = w(x_b)$ exist.

Note that w_a and w_b may not be necessarily the trace of w_0 at the ends of the interval (x_a, x_b). Denote the weak function space by

$$W(J_a) = \left\{ w = \{w_0, w_a, w_b\} : w_0 \in L^2(J_a), |w_a| + |w_b| < \infty \right\}.$$

Definition 4.1. [7] *Let $w \in W(J_a)$. The weak derivative $\partial_d w$ is defined as a linear functional in the dual space $H^{-1}(J_a)$ whose action on each $q \in H^1(J_a)$ is given by*

$$(\partial_d w, q) = -\int_{I_a} w_0 q' dx + w_b q(x_b) - w_a q(x_a), \quad \forall q \in H^1(J_a), \tag{4.1}$$

It is obvious that, as a bounded linear functional on $H^1(J_a)$, $\partial_d w$ is well defined for all $w \in H^1(J_a)$, considering w as a weak function with components $w_0 = w|_{J_a}$, $w_a = w(x_a)$ and $w_b = w(x_b)$, then by integration by part, we have for any $q \in H^1(J_a)$ that

$$\begin{aligned} \int_{J_a} w' q dx &= \int_{x_a}^{x_b} w' q dx = -\int_{J_a} w_0 q' dx + w_b q_b - w_a q_a \\ &= \int_{J_a} \partial_d w q dx. \end{aligned} \tag{4.2}$$

which means that $\partial_d w = \acute{w}$ is the usual derivative of function w. We then present the discrete weak derivative which is actually used in our analysis. For non-negative integer $k \geq 0$, let $P_k(J_a)$ is the space consisting of all polynomials over J_a with degree less than or equal k. Then, $P_k(J_a)$ is a subspace of $H^1(J_a)$.

Definition 4.2. [8] *For $w \in W(J_a)$, the discrete weak derivative $\partial_{d,k} w \in P_k(J_a)$ is defined as the unique solution of the following equation,*

$$\int_{J_a} \partial_{d,k} w q dx = -\int_{J_a} w_0 q' dx + w_b q_b - w_a q_a, \quad \forall q \in P_k(J_a). \tag{4.3}$$

From (4.1) and (4.3), we have

$$(\partial_d w, q) = (\partial_{d,k} w, q) = \int_{J_a} \partial_{d,k} w q dx \quad \forall q \in P_k(J_a).$$

This shows that $\partial_{d,k} w$ is a discrete approximation of $\partial_d w$ in $P_k(J_a)$. In particular, if $w \in H^1(J_a)$, then from (4.2) and (4.3) that

$$\int_{J_a} (\partial_{d,k} w - w') q dx = 0 \quad \forall q \in P_k(J_a). \tag{4.4}$$

That is, $\partial_{d,k} w$ is the L^2-projection of w' in $P_k(J_a)$ if $w \in H^1(J_a)$.

5. The Discrete Time WG-FEM

Let $J_h : a = x_1 < x_2 < ... < x_{N-1} < x_N = b$ be partition of interval $J = (a, b)$ with elements $J_i = (x_i, x_{i+1})$. Denote the mesh size by $h = \max_i h_i$, where $h_i = x_{i+1} - x_i, i = 1, 2, ..., N-1$. In the weak finite element analysis, for any integer $r \geq 0$, the discrete weak function space defined on the partition J_h defined by [8]

$$W(J_h, r) = \{w : w|_{J_i} \in W(J_i, r), i = 1, 2, ..., N-1\}, \tag{5.1}$$

where

$$W(J_i, r) = \{w = \{w_0, w_i, w_{i+1}\} : w_0 \in P_r(J_i), |w_i| + |w_{i+1}| < \infty\}. \tag{5.2}$$

Note that for a weak function $w \in W(J_i, r)$, the endpoint values $w_i = w(x_i)$ and $w_{i+1} = w(x_{i+1})$ may be independent with the interior value w_0.Recall the discrete weak derivative definition (4.2) for $w \in W(J_i, r)$, its discrete weak derivative $\partial_{d,k} w \in P_k(J_i)$ is given by the following formula

$$\int_{J_i} \partial_{d,k} w q dx = -\int_{J_i} w_0 q' dx + w_{i+1} q_{i+1} - w_i q_i, \quad \forall q \in P_k(J_i), \tag{5.3}$$

where $q_i = q(x_i)$, $q_{i+1} = q(x_{i+1})$.
Therefor the space of discrete weak derivative Y_h defined as :

$$Y_h = \{\vartheta : \vartheta|_{J_i} \in P_k(J_i), i = 1, 2, ..., N-1\} \tag{5.4}$$

In our discussion, with the exception of the weak function $w = \{w_0, w_i, w_{i+1}\} \in W(J_i, r)$, values of end a smooth function w on J_i should be determined by its trace from the interior of J_i. For example, for $z \in H^1(J_i)$,$z_i = z(x_i) = \lim_{x \to x_i} z(x)$, $x \in J_i$. Let $J_L = (x_{i-1}, x_i)$ and $J_R = (x_i, x_{i+1})$ be two adjacent element with the common endpoint x_i, weak function $w|_{\tilde{J}_L} = \{w_0^L, w_{i-1}^L, w_i^L\}$, $w|_{\tilde{J}_R} = \{w_0^R, w_i^R, w_{i+1}^R\}$, we define the jump of weak function w at point x_i by

$$[w]_{x_i} = w_i^R - w_i^L, w \in W(J_h, r).$$

Then, weak function w is unique value at the point x_i iff $[w]_{x_i} = 0$. The weak finite element space X_h is defined by

$$X_h = \{w : w \in W(J_h, r),\ [w]_{x_i} = 0, i = 2, ..., N-1\}. \tag{5.5}$$

We refer to X_h^0 as the subspace of X_h with vanish on endpoints of the interval $[a, b]$; i.e.,

$$X_h^0 = \{w : w \in X_h, w(x_1) = 0, w(x_N) = 0\}, \tag{5.6}$$

Denote the discrete L^2 inner product and norm by

$$(u, w)_h = \sum_{i=1}^{N-1} (u, w)_{J_i} = \sum_{i=1}^{N-1} \int_{J_i} uw dx, \quad \|w\|_h^2 = (w, w)_h.$$

Now, we shall establish the discrete time WG-FEM for time Fractional Reaction-advection- diffusion Equation, let $0 = t_0 < t_1 < .... < t_M = T$ be a partition for time interval $[0, T]$ and the time level $t = t_n = n\tau$ where n is nonnegative integer and denote by $U_n \in X_h^0$ the approximate solution of $u(t_n)$. The Caputo derivative and Caputo integrals are approximated by the following lemmas:

Lemma 5.1. [9] *Let $y(t) \in C^2[0, t_n]$. Then*

$$ {}_0^C D_t^{\alpha} y(t) = D_\tau^{\alpha} y(t) + TR_1^n. \tag{5.7}$$

where

$$D_\tau^{\alpha} y(t) = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}(b_0 y(t_n) - \sum_{\ell=1}^{n-1}(b_{n-\ell-1} - b_{n-\ell})y(t_n) - b_{n-1}y(t_0)),$$

and

$$b_\ell = [(\ell+1)^{1-\alpha} - \ell^{1-\alpha}], \ell \geq 0.$$

The truncation error TR_1^n can be estimated by:

$$|TR_1^n| \leq \frac{1}{1-\alpha}\left[\frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - (1+2^{-\alpha})\right]\tau^{2-\alpha} \max_{0\leq t\leq t_n} |g^{''}(t)|$$

Lemma 5.2. [10] *According to the backward Euler convolution quadrature rule, the integral term discretized as follows:*

$$\begin{aligned}\int_0^{x_n} K(x_n - s)y(s)ds &= \frac{1}{\Gamma(2-\beta)}\int_a^{x_n}(x_n - s)^{1-\beta}y(s)ds \\ &= h^{(2-\beta)}\sum_{q=0}^{n-1} d_q\, y(x_{n-q}) + TR_2^n. \end{aligned}\tag{5.8}$$

where the weight d_q is determined by the generating function $(1-x)^{\beta-2} = \sum_{q=0}^{\infty} d_q\, x^q$

Lemma 5.3. [11] *For $n \geq 0$:*

$$|TR_2^n| \leq Ch\left\{x_n^{1-\beta}\|y(0)\| + \int_0^{x_n}(x_{n-1} - s)^{\beta-1}\|y_x(s)\|ds\right\} \tag{5.9}$$

Now, the fully discrete WG-FEM for problem (1.1)-(1.2):Find $U_n, V_n \in X_h$ such that

$$\begin{aligned}(D_\tau^{\alpha} U_n^0, \chi_0)_h + a_h(U_n, \chi) &= (\Psi_1(U_{n-1}^0), \chi_0) + (f_1^n, \chi_0)_h \quad \forall \chi \in X_h^0, \\ U_n(x, 0) &= g_{1n}(x), \quad x \in J,\end{aligned}\tag{5.10}$$

$$\begin{aligned}(D_\tau^{\alpha} V_n^0, \chi_0)_h + a_h(V_n, \rho) &= (\Psi_2(V_{n-1}^0), \rho_0) + (f_2, \chi_0)_h \quad \forall \chi \in X_h^0, \\ V_n(x, 0) &= g_{2n}(x), \quad x \in J,\end{aligned}\tag{5.11}$$

where, the bilinear form is defined as

$$\begin{aligned} a_h(U_n,\chi)_h &= (\epsilon_1\partial_{d,k}U_n,\partial_{d,k}\chi)_h+\gamma_1 h^{(2-\beta_1)}\sum_{q=0}^{n-1} d_q\,(\partial_{d,k}U_{n-q},\partial_{d,k}\chi)+\frac{\delta_1}{m_1+2}((U_n^0)^{m_1}\partial_{d,k}U_n,\chi_0)\\ &- \frac{\delta_1}{m_1+2}((U^0)^{m_1}U^0,\partial_{d,k}\chi)+\zeta_1(U_n^0V_n^0,\partial_{d,k}\chi). \end{aligned} \tag{5.12}$$

$$\begin{aligned} a_h(V_n,\rho)_h &= (\epsilon_2\partial_{d,k}V_n,\partial_{d,k}\rho)_h+\gamma_2 h^{(2-\beta_2)}\sum_{q=0}^{n-1} d_q\,(\partial_{d,k}V_{n-q},\partial_{d,k}\rho)+\frac{\delta_2}{m_2+2}((V_n^0)^{m_2}\partial_{d,k}V_n,\rho_0)\\ &- \frac{\delta_2}{m_2+2}((V_n^0)^{m_2}V^0,\partial_{d,k}\rho)+\zeta_2(U_n^0V_n^0,\partial_{d,k}\rho). \end{aligned} \tag{5.13}$$

6. Error Analysis

In this section, we will get error estimates for the continuous and discrete time WG-FEM in the L^2-norm and H^1-norm, respectively. To balance the accuracy of approximation between space X_h and $P_k(J_i)$ used to calculate $\partial_{d,k}u_h$, from now on, we fix always index $k=r+1$ in the definition of discrete weak derivative (5.4).

For $\nu>0$, let R_h^ν is the local L^2-projection operator, restricted on each element J_i, $R_h^\nu: w\in L^2(J_i)\to R_h^\nu w\in P_\nu(J_i)$, such that

$$(w-R_h^\nu w,q)=0,\quad \forall q\in P_\nu(J_i), i=1,2,...,N-1.$$

By the Bramble- Hilbert lemma, it is easy to prove that

$$\|w-R_h^\nu w\|_{J_i}+h_i\|w-R_h^\nu w\|_{1,J_i}\le Ch_i^s\|w\|_{s,J_i},\ \ 0\le s\le \nu+1. \tag{6.1}$$

We now define a projection operator $Q_h: w\in H^1(J)\to Q_hw\in W(J_h,r)$ such that

$$Q_hw|_{\tilde J_i}=\left\{Q_h^0w,(Q_hw)_i,(Q_hw)_{i+1}\right\}=\left\{R_h^rw,w(x_i),w(x_{i+1})\right\},\ \ i=1,2,...N-1.$$

Obviously, $Q_hw\in X_h^0$ if $w\in H_0^1$. It follows from (6.1) that

$$\|Q_h^0w-w\|_{J_i}\le Ch_i^s\|w\|_{s,J_i},\ \ 0\le s\le\ r+1. \tag{6.2}$$

Furthermore, using the definition of operator Q_h and the discrete weak derivative $\partial_{d,k}$ in (4.3) we have

$$\begin{aligned} \int_{J_i}\partial_{d,k}Q_hwq\,dx &= -\int_{J_i}Q_h^0wq'dx+(Q_hw)_{i+1}q_{i+1}-(Q_hw)_iq_i\\ &= -\int_{J_i}wq'dx+(w)_{i+1}q_{i+1}-(w)_iq_i\\ &= \int_{J_i}w'q\,dx,\qquad \forall q\in P_k(J_i). \end{aligned} \tag{6.3}$$

Hence $\partial_{d,k}Q_hw=R_h^kw'$ and (noting that $k=r+1$)

$$\|\partial_{d,k}Q_hw-w'\|_{J_i}\le Ch_i^s\|w\|_{s+1,J_i}\quad 0\le s\le r+2. \tag{6.4}$$

Estimates (6.2) and (6.3) show that $Q_h w \in X_h^0$ is a very good approximation for function $w \in H_0^1(J) \cap H^{s+1}(J), s \geq 0$.

In order to do the error analysis, we still need to construct another special projection function.

Lemma 6.1. [7] *For $w \in H^1(J)$, there exists projection function $M_h w \in H^1(J)$, restricted on element J_i, $M_h u \in P_{r+1}(J_i)$ satisfies*

$$((M_h w)', q) = (w', q)_{J_i}, \ \forall q \in P_r(J_i), \ i = 1, 2, ..., N-1, \tag{6.5}$$

$$M_h w(x_i) = w(x_i), \ i = 1, 2, ..., N, \tag{6.6}$$

$$\|w - M_h w\|_{J_i} + h_i \|w' - (M_h w)'\|_{J_i} \leq C h_i^{s+1} \|w\|_{s+1}, \ \ 0 \leq s \leq r+1. \tag{6.7}$$

Lemma 6.2. [12] *Suppose that the nonnegative sequance $\phi^n . g^n \mid n = 0, 1, 2, ...$ satisfy*

$$D_\tau^\alpha \phi^n \leq \lambda_1 \phi^n + \lambda_2 \phi^{n-1} + g^n, n \geq 1$$

where $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ are constants. Then, there exist a positive constant τ^* such that when $\tau \leq \tau^*$.

$$\phi^n \leq 2(\phi^0 + \frac{t_n^\alpha}{\Gamma(1+\alpha)} \max_{0 \leq j \leq n} g^j) E_\alpha(2\lambda t_n^\alpha), 0 \leq n \leq N$$

where $E_\alpha(z) = \Sigma_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k\alpha)}$ is the Mittag-Leffler function and $\lambda = \lambda_1 + \frac{\lambda_2}{(2-2^{1-\alpha})}$.

Lemma 6.3. *Let $u(t), v(t) \in H^1(0, T, H^2(J))$ be the solution of problem (1.1)-(1.4). Then for all $\chi, \rho \in X_h^0$, we have*

$$\begin{aligned}
&({}_0^C D_t^{\alpha_1} u, \chi_0)_h + \epsilon_1 (M_h u_x, \partial_{d,k} \chi)_h + \gamma_1 \int_a^x (k_1(x-s) M_h u_x(s) ds, \partial_{d,k} \chi) \\
&+ \frac{\delta_1}{m_1 + 2} (u^{m_1} u_x, \chi_0)_h - \frac{\delta_1}{m_1 + 2} (M_h u^{m_1+1}, \partial_{d,k} \chi)_h + \zeta_1 (M_h uv, \partial_{d,k} \chi)_h \\
&= (\Psi_1(u), \chi_0) + (f_1, \chi_0)_h,
\end{aligned} \tag{6.8}$$

and

$$\begin{aligned}
&({}_0^C D_t^{\alpha_2} v, \rho_0)_h + \epsilon_2 (M_h v_x, \partial_{d,k} \rho)_h + \gamma_2 \int_a^x (k_2(x-s) M_h v_x(s) ds, \partial_{d,k} \rho) \\
&+ \frac{\delta_2}{m_2 + 2} (v^{m_2} v_x, \rho_0)_h - \frac{\delta_2}{m_2 + 2} (M_h v^{m_2+1}, \partial_{d,k} \rho)_h + \zeta_2 (M_h uv, \partial_{d,k} \rho)_h \\
&= (\Psi_2(v), \rho_0) + (f_2, \rho_0)_h.
\end{aligned} \tag{6.9}$$

Proof. For Eq.(6.8), multiplying equation (1.1) by χ_0 and integrating we obtain

$$\begin{aligned}
({}_0^C D_t^{\alpha_1} u, \chi_0)_h - (\epsilon_1 u_{xx}, \chi_0) \ &- \ \gamma_1 \int_a^x k_1(x-s)(u_{xx}(s), \chi_0) ds + \delta_1 (u_x u^{m_1}, \chi_0) \\
&- \ \zeta_1((uv)_x, \chi_0) = (\Psi_1(u), \chi_0) + (f_1, \chi_0)_h
\end{aligned} \tag{6.10}$$

the nonlinear term of above equation can be written as

$$(u_x u^{m_1}, \chi_0)_h = \frac{1}{m_1 + 2} (u^{m_1} u_x, \chi_0)_h + \frac{1}{m_1 + 2} ((u^{m_1+1})_x, \chi_0)_h,$$

$$= \frac{1}{m_1+2}(u^{m_1}u_x, \chi_0)_h + \frac{1}{m_1+2}((M_h u^{m_1+1})_x, \chi_0)_h,$$

And

$$-((uv)_x, \chi_0)_h = -((M_h uv)_x, \chi_0)_h.$$

By the definition of operator $\partial_{d,k}$, we have

$$((M_h u^{m_1+1})_x, \chi_0)_h = -(M_h u^{m_1+1}, \partial_{d,k}\chi)_h + (M_h u^{m_1+1})_{i+1}\chi_{i+1} - (M_h u^{m_1+1})_i \chi_i,$$

$$-((M_h uv)_x, \chi_0)_h = (M_h uv, \partial_{d,k}\chi)_h - (M_h uv)_{i+1}\chi_{i+1} + (M_h uv)_i \chi_i,$$

since $\chi \in X_h^0$, noting that $\chi_1 = 0$ and $\chi_N = 0$.

Hence,

$$((M_h u^{m_1+1})_x, \chi_0)_h = -(M_h u^{m_1+1}, \partial_{d,k}\chi)_h. \tag{6.11}$$

$$-((M_h uv)_x, \chi_0)_h = (M_h uv, \partial_{d,k}\chi)_h \tag{6.12}$$

Similarly, by using Lemma (6.1) and the definition of operator $\partial_{d,k}$, $\forall \chi \in X_h^0$, we have

$$\begin{aligned} -\epsilon_1(u_{xx}, \chi_0)_h &= -\epsilon_1((M_h u_x)_x, \chi_0)_h \\ &= -\epsilon_1\{-(M_h u_x, \partial_{d,k}\chi)_h + (M_h u_x)_{i+1}\chi_{i+1} - (M_h u_x)_i \chi_i\} \\ &= \epsilon_1(M_h u_x, \partial_{d,k}\chi)_h. \end{aligned} \tag{6.13}$$

and

$$\begin{aligned} &- \gamma_1 \int_a^x k_1(x-s)(u_{xx}(s)ds, \chi_0) = -\gamma_1 \int_a^x k_1(x-s)((M_h u_x(s))_x ds, \chi_0)_h \\ &= -\gamma_1 \int_a^x k_1(x-s)\{-(M_h u_x(s)ds, \partial_{d,k}\chi)_h + (M_h u_x)_{i+1}\chi_{i+1} - (M_h u_x)_i \chi_i\} \\ &= \gamma_1 \int_a^x k_1(x-s)(M_h u_x(s)ds, \partial_{d,k}\chi)_h \end{aligned} \tag{6.14}$$

Substituting (6.11),(6.12), (6.13) and (6.14) into (6.10) we complete the proof.

In the same manner for Eq. (6.9). □

Theorem 6.1. *Let $u(x,t), v(x,t), U_n(x,t)$ and $V_n(x,t)$ are solutions of problem (1.1),(1.2), (5.10) and (5.11) respectively and let $u(t), v(t) \in H^2(0,T,H^{r+1}(J))$ then there exist constant C such that*

$$\begin{aligned} \|(e_h^u)_0\|_h^2 + \|(e_h^v)_0\|_h^2 \;&\le\; C\frac{T^\alpha}{\Gamma 1+\alpha}\{h^{2(r+1)}(\max_{1\le l\le n}\|u(t_l)\|_{r+2}^2 + \max_{1\le l\le n}\|v(t_l)\|_{r+2}^2 \\ &+ \max_{1\le l\le n}\|uv(t_l)\|_{r+1}^2) + \tau^2[\int_0^{t_n}(\|u_{tt}(s)\|_h^2 + \|v_{tt}(s)\|_h^2)ds \\ &+ (\|u(0)\| + \|v(0)\| + \int_0^{x_n}(\|u_x\|^2 + \|v_x\|^2))ds \\ &+ \max_{0\le t\le t_n}(|u_{tt}(t)| + |v_{tt}(t)|)]\}. \end{aligned} \tag{6.15}$$

Proof. Set $t = t_n, x = x_n$ in equation (6.8), we have

$$
\begin{aligned}
& ({}_0^C D_t^{\alpha_1} u_n, \chi_0)_h + \epsilon_1 (M_h u_x(t_n), \partial_{d,k}\chi)_h + \gamma_1 \int_0^{x_n} (k_1(x-s) M_h u_x(s) ds, \partial_{d,k}\chi) \\
+ & \frac{\delta_1}{m_1+2} (u_n^{m_1} u_x, \chi_0)_h - \frac{\delta_1}{m_1+2} (M_h u_n^{m_1+1}, \partial_{d,k}\chi)_h + \zeta_1 (M_h u_n v_n, \partial_{d,k}\chi)_h \\
= & (\Psi_1(u(t_n)), \chi_0) + (f_1(t_n), \chi_0)_h, \qquad (6.16)
\end{aligned}
$$

Subtracting (6.16) from (5.10), we have

$$
\begin{aligned}
& (D_\tau^{\alpha_1} (Q_h^0 u_n - U_n^0), \chi_0)_h + \epsilon_1 (\partial_{d,k}(Q_h u_n - U_n), \partial_{d,k}\chi)_h \\
+ & \gamma_1 h^{(2-\beta_1)} \sum_{q=0}^{n-1} d_q \left(\partial_{d,k}(Q_h u_{n-q} - U_{n-q}), \partial_{d,k}\chi\right) \\
= & (D_\tau^{\alpha_1} Q_h^0 u_n - {}_0^C D_{t_n}^{\alpha_1} u), \chi_0)_h + \epsilon_1 (\partial_{d,k} Q_h u_n, \partial_{d,k}\chi)_h - \epsilon_1 (M_h u_x(t_n), \partial_{d,k}\chi)_h \\
+ & \gamma_1 h^{(2-\beta_1)} \sum_{q=0}^{n-1} d_q \left(\partial_{d,k} Q_h u_{n-q}, \partial_{d,k}\chi\right) - \gamma_1 \int_0^{x_n} (k_1(x-s) M_h u_x(s) ds, \partial_{d,k}\chi) \\
+ & \frac{\delta_1}{m_1+2} \{((U_n^0)^{m_1} \partial_{d,k} U_n, \chi_0)_h - (u^{m_1} u_x, \chi_0)_h\} \\
+ & \frac{\delta_1}{m_1+2} \{(M_h u_n^{m_1+1}, \partial_{d,k}\chi)_h - ((U_n^0)^{m_1} U_n, \partial_{d,k}\chi)_h\} \\
- & \zeta_1 \{(M_h u_n v_n, \partial_{d,k}\chi)_h - (U_n^0 V_n^0, \partial_{d,k}\chi)_h\} \\
+ & (\Psi_1(u_n), \chi_0) - (\Psi_1(U_{n-1}^0), \chi_0) \qquad (6.17)
\end{aligned}
$$

Let $e_n^u = Q_h u_n - U_n$. and $e_n^v = Q_h v_n - V_n$. Taking $\chi = e_n^u$ in equation (6.17) we have

$$
\frac{1}{2} D_\tau^{\alpha_1} \|(e_n^u)_0\|_h^2 + \epsilon_1 \|\partial_{d,k} e_n^u\|_h^2 + \gamma_1 h^{(2-\beta_1)} \sum_{q=0}^{n-1} d_q \left(\partial_{d,k}(e_{n-q}^u), \partial_{d,k} e_n^u\right) = \sum_{i=1}^{7} \Phi^{(i)}, \qquad (6.18)
$$

where

$$
\begin{aligned}
\Phi^{(1)} &= (D_\tau^{\alpha_1} Q_h^0 u_n - {}_0^C D_{t_n}^{\alpha_1} u), (e_n^u)_0)_h \\
\Phi^{(2)} &= \epsilon_1 (\partial_{d,k} Q_h u_n, \partial_{d,k} e_n^u)_h - \epsilon_1 (M_h u_x(t_n), \partial_{d,k} e_n^u)_h. \\
\Phi^{(3)} &= \gamma_1 h^{(2-\beta_1)} \sum_{q=0}^{n-1} d_q \left(\partial_{d,k} Q_h u_{n-q}, \partial_{d,k} e_n^u\right) - \gamma_1 \int_0^{x_n} (k_1(x-s) M_h u_x(s) ds, \partial_{d,k} e_n^u) \\
\Phi^{(4)} &= \frac{\delta_1}{m_1+2} \{((U_n^0)^{m_1} \partial_{d,k} U_n, (e_n^u)_0)_h - (u^{m_1} u_x, (e_n^u)_0)_h\} \\
\Phi^{(5)} &= \frac{\delta_1}{m_1+2} \{(M_h u_n^{m_1+1}, \partial_{d,k} e_n^u)_h - ((U_n^0)^{m_1} U_n, \partial_{d,k} e_n^u)_h\} \\
\Phi^{(6)} &= -\zeta_1 \{(M_h u(t_n) v(t_n), \partial_{d,k} e_n^u)_h - (U_n^0 V_n^0, \partial_{d,k} e_n^u)_h\} \\
\Phi^{(7)} &= (\Psi_1(u_n), (e_n^u)_0) - (\Psi_1(U_{n-1}^0), (e_n^u)_0). \qquad (6.19)
\end{aligned}
$$

For $\Phi^{(1)}$, we can estimate as follows:

$$\begin{aligned}
\Phi^{(1)} &= (D_\tau^{\alpha_1} Q_h^0 u_n -{}_0^C D_{t_n}^{\alpha_1} u, (e_n^u)_0)_h \\
&= (D_\tau^{\alpha_1} Q_h^0 u_n - D_\tau^{\alpha_1} u^n, (e_n^u)_0)_h + (D_\tau^{\alpha_1} u^n -{}_0^C D_{t_n}^{\alpha_1} u, (e_n^u)_0)_h, \\
&\leq \frac{1}{2}\|D_\tau^{\alpha_1} Q_h^0 u_n - D_\tau^{\alpha_1} u^n\| + \frac{1}{2}\|D_\tau^{\alpha_1} u^n -{}_0^C D_{t_n}^{\alpha_1} u\| + \|(e_n^u)_0)\|, \\
&\leq \frac{1}{2}C\|Q_h^0 u_n - u^n\|^2 + \frac{1}{2}\|D_\tau^{\alpha_1} u^n -{}_0^C D_{t_n}^{\alpha_1} u\| + \|(e_n^u)_0)\|, \\
&\leq Ch^{2(r+1)}\|u\|_{r+1}^2 + C\tau^{2(2-\alpha)} \max_{0\leq t\leq t_n} |u''(t)| + \|(e_n^u)_0)\|. \qquad (6.20)
\end{aligned}$$

For $\Phi^{(2)}$

$$\Phi^{(2)} = \epsilon_1(\partial_{d,k} Q_h u(t_n) - u_x(t_n), \partial_{d,k} e_n^u)_h + \epsilon_1(u_x(t_n) - M_h u_x(t_n), \partial_{d,k} e_n^u)_h,$$

Using Youngs' inequality, we have

$$\begin{aligned}
\left|\Phi^{(2)}\right| &\leq \frac{3}{\epsilon_1}\|\partial_{d,k} Q_h u(t_n) - u_x(t_n)\|_h^2 + \frac{\epsilon_1}{12}\|\partial_{d,k} e_n^u\|_h^2 \\
&+ \frac{3}{\epsilon_1}\|u_x(t_n) - M_h u_x(t_n)\|_h^2 + \frac{\epsilon_1}{12}\|\partial_{d,r} e_n^u\|_h^2. \qquad (6.21)
\end{aligned}$$

For $\Phi^{(3)}$

$$\begin{aligned}
\left|\Phi^{(3)}\right| &\leq \gamma_1 h^{(2-\beta_1)} \sum_{q=0}^{n-1} d_q \left(\partial_{d,k} Q_h u_{n-q}, \partial_{d,k} e_n^u\right) - \gamma_1 \int_0^{x_n} (k_1(x-s) M_h u_x(s) ds, \partial_{d,k} e_n^u) \\
&\leq \gamma_1 h^{(2-\beta_1)} \sum_{q=0}^{n-1} d_q \left(\partial_{d,k} Q_h u_{n-q} - u_{x,n-q}, \partial_{d,k} e_n^u\right) \\
&+ \gamma_1 h^{(2-\beta_1)} \sum_{q=0}^{n-1} d_q \left(u_{x,n-q} - M_h u_{x,n-q}, \partial_{d,k} e_n^u\right) \\
&+ (\gamma_1 h^{(2-\beta_1)} \sum_{q=0}^{n-1} d_q\, M_h u_{x,n-q} - \gamma_1 \int_0^{x_n} (k_1(x-s) M_h u_x(s) ds, \partial_{d,k} e_n^u) \\
&\leq \frac{\epsilon_1}{12}\|\partial_{d,r} e_n^u\|_h^2 + \frac{9}{\epsilon_1}\|\gamma_1 h^{(2-\beta_1)} \sum_{q=0}^{n-1} d_q \left(\partial_{d,k} Q_h u_{n-q} - u_{x,n-q}\right)\|_h^2 \\
&+ \frac{9}{\epsilon_1}\|\gamma_1 h^{(2-\beta_1)} \sum_{q=0}^{n-1} d_q \left(u_{x,n-q} - M_h u_{x,n-q}\right)\|_h^2 \\
&+ \frac{9}{\epsilon_1}\|(\gamma_1 h^{(2-\beta_1)} \sum_{q=0}^{n-1} d_q\, M_h u_{x,n-q} - \gamma_1 \int_0^{x_n} (k_1(x-s) M_h u_x(s) ds\|_h^2 \\
&\leq \frac{\epsilon_1}{12}\|\partial_{d,r} e_n^u\|_h^2 + C \max_{0\leq j\leq N} \|\partial_{d,k} Q_h u_j - u_{x,j})\|_h^2 + C \max_{0\leq j\leq N} \|u_{x,j} - M_h u_{x,j})\|_h^2 \\
&+ \frac{9}{\epsilon_1}\|(\gamma_1 h^{(2-\beta_1)} \sum_{q=0}^{n-1} d_q\, M_h u_{x,n-q} - \gamma_1 \int_0^{x_n} (k_1(x-s) M_h u_x(s) ds\|_h^2 \qquad (6.22)
\end{aligned}$$

To estimate $\Phi^{(4)} + \Phi^{(5)}$ we can write $\Phi^{(4)}$ and $\Phi^{(5)}$ as follows:

$$\begin{aligned}
\Phi^{(4)} &= \frac{\delta_1}{m_1+2}\{((U_n^0)^{m_1}\partial_{d,k}U_n, (e_n^u)_0)_h - (u^{m_1}u_x, (e_n^u)_0)_h\} \\
&= \frac{\delta_1}{m_1+2}\{(u^{m_1}(\partial_{d,k}Q_h u - u_x), (e_n^u)_0)_h + (((Q_h^0 u)^{m_1} - u^{m_1})\partial_{d,k}Q_h u, (e_n^u)_0) \\
&+ (((U_n^0)^{m_1} - (Q_h^0 u)^{m_1})\partial_{d,k}Q_h u, (e_n^u)_0) + ((U_n^0)^{m_1}\partial_{d,k}(U_n - Q_h u), (e_n^u)_0)\} \qquad (6.23)
\end{aligned}$$

and

$$\begin{aligned}
\Phi^{(5)} &= \frac{\delta_1}{m_1+2}\{(M_h u_n^{m_1+1}, \partial_{d,k}e_n^u)_h - ((U_n^0)^{m_1}U_n, \partial_{d,k}e_n^u)_h\} \\
&= \frac{\delta_1}{m_1+2}\{((U_n^0)^{m_1}(Q_h^0 u - U_n^0), \partial_{d,k}e_n^u) + (((Q_h^0)^{m_1} - (U_n^0)^{m_1})Q_h^0 u, \partial_{d,k}e_n^u) \\
&+ (u^{m_1+1} - (Q_h^0 u)^{m_1+1}, \partial_{d,k}e_n^u) + (M_h u^{m_1+1} - u^{m_1+1}, \partial_{d,k}e_n^u)
\end{aligned} \qquad (6.24)$$

Hence, we have

$$\begin{aligned}
\Phi^{(4)} + \Phi^{(5)} &= \frac{\delta_1}{m_1+2}\{(u^{m_1}(\partial_{d,k}Q_h u - u_x), (e_n^u)_0)_h \\
&+ (((Q_h^0 u)^{m_1} - u^{m_1})\partial_{d,k}Q_h u, (e_n^u)_0) \\
&+ (((U_n^0)^{m_1} - (Q_h^0 u)^{m_1})\partial_{d,k}Q_h u, (e_n^u)_0) \\
&+ (((Q_h^0)^{m_1} - (U_n^0)^{m_1})Q_h^0 u, \partial_{d,k}e_n^u) \\
&+ (u^{m_1+1} - (Q_h^0 u)^{m_1+1}, \partial_{d,k}e_n^u) \\
&+ (M_h u^{m_1+1} - u^{m_1+1}, \partial_{d,k}e_n^u)\}
\end{aligned} \qquad (6.25)$$

By using Youngs'-inequality and $\partial_{d,k}Q_h u = R_h^\nu u_x$, we have

$$\begin{aligned}
\left|\Phi^{(4)} + \Phi^{(5)}\right| &\le \frac{\delta_1}{m_1+2}\{\frac{1}{2}\|u^{m_1}\|_\infty^2\|\partial_{d,k}Q_h u - u_x\|_h^2 + \frac{1}{2}\|(e_n^u)_0\|^2 \\
&+ C\|u_x\|_\infty^2\|(e_n^u)_0\|^2 + C\frac{6\delta_1}{(m_1+2)\epsilon_1}\|u\|_\infty^2\|(e_n^u)_0\|^2 + \frac{(m_1+2)\epsilon_1}{12\delta_1}\|\partial_{d,k}e_n^u\|_h^2 \\
&+ C\frac{6\delta_1}{(m_1+2)\epsilon_1}\|u^{m_1}\|_\infty^2\|u - Q_h^0 u\|^2 + \frac{(m_1+2)\epsilon_1}{12\delta_1}\|\partial_{d,k}e_n^u\|_h^2 \\
&+ \frac{6\delta_1}{(m_1+2)\epsilon_1}\|M_h u^{m_1+1} - u^{m_1+1}\| + \frac{(m_1+2)\epsilon_1}{12\delta_1}\|\partial_{d,k}e_n^u\|_h^2\} \qquad (6.26)
\end{aligned}$$

For $\Phi^{(6)}$

$$\begin{aligned}
\Phi^{(6)} &= -\zeta_1[(M_h u(t_n)v(t_n) - u(t_n)v(t_n), \partial_{d,k}e_n^u)_h + ((V_n^0 - Q_0^h v(t_n))(Q_h^0 u(t_n) - U_n^0), \partial_{d,k}e_n^u)_h \\
&+ +(Q_h^0 v(t_n)(Q_h^0 u(t_n) - U_n^0), \partial_{d,k}e_n^u)_h + ((Q_h^0 v(t_n) - V_n^0)Q_h^0 u(t_n), \partial_{d,k}e_n^u)_h \\
&+ (v(t_n)(u(t_n) - Q_h^0 u(t_n)), \partial_{d,k}e_n^u)_h + (Q_h^0 u(t_n)(v(t_n) - Q_h^0 v(t_n)), \partial_{d,k}e_n^u)_h],
\end{aligned}$$

By using Youngs'-inequality, we have

$$
\begin{aligned}
\left|\Phi^{(6)}\right| \leq\ & \frac{3|\zeta_1|^2}{\epsilon_1}\|M_h u(t_n)v(t_n) - u(t_n)v(t_n)\|_h^2 + \frac{\epsilon_1}{12}\|\partial_{d,k}e_n^u\|_h^2 \\
+\ & \frac{9|\zeta_1|^2}{\epsilon_1^2}\|(e_n^v)_0\|_h^2 + \frac{1}{4}\|(e_n^u)_0\|_h^2 + \frac{\epsilon_1}{12}\|\partial_{d,k}e_n^u\|_h^2 \\
+\ & \frac{3|\zeta_1|^2}{\epsilon_1}\|v(t_n)\|_\infty^2\|_h^2\|(e_n^u)_0\|_h^2 + \frac{\epsilon_1}{12}\|\partial_{d,k}e_n^u\|_h^2 \\
+\ & \frac{3|\zeta_1|^2}{\epsilon_1}\|u(t_n)\|_\infty^2\|(e_n^v)_0\|_h^2 + \frac{\epsilon_1}{12}\|\partial_{d,k}e_n^u\|_h^2 \\
+\ & \frac{3|\zeta_1|^2}{\epsilon_1}\|v(t_n)\|_\infty^2\|u(t_n) - Q_h^0 u(t_n)\|_h^2 + \frac{\epsilon_1}{12}\|\partial_{d,k}e_n^u\|_h^2). \\
+\ & \frac{3|\zeta_1|^2}{\epsilon_1}\|u(t_n)\|_\infty^2\|v(t_n) - Q_h^0 v(t_n)\|_h^2 + \frac{\epsilon_1}{12}\|\partial_{d,k}e_n^u\|_h^2).
\end{aligned}
\tag{6.27}
$$

For $\Phi^{(7)}$, by using the Lipschitz continuity, we have

$$
\begin{aligned}
\left|\Phi^{(7)}\right| =\ & (\Psi_1(u_n), (e_n^u)_0) - (\Psi_1(U_{n-1}^0), (e_n^u)_0) \\
\leq\ & L\|u_n - U_{n-1}^0\|\|(e_n^u)^0\| \\
\leq\ & \frac{L}{2}\|u_n - u_{n-1}\|^2 + \frac{L}{2}\|u_{n-1} - Q_h^0 u_{n-1}\|^2 + \frac{L}{2}\|Q_h^0 u_{n-1} - U_{n-1}^0\|^2 + \frac{3L}{2}\|(e_n^u)^0\|^2 \\
\leq\ & \frac{L}{2}\tau^2\int_{t_{n-1}}^{t_n}\|u_{tt}(s)\|^2 ds + \frac{L}{2}\|u_{n-1} - Q_h^0 u_{n-1}\|^2 + \frac{L}{2}\|(e_{n-1}^u)^0\|^2 + \frac{3L}{2}\|(e_n^u)^0\|^2
\end{aligned}
\tag{6.28}
$$

Substituting (6.20), (6.21), (6.22), (6.26), (6.27) and (6.28) into (6.18), we get

$$
\begin{aligned}
& \frac{1}{2}D_\tau^{\alpha_1}\|(e_n^u)_0\|_h^2 + \gamma_1 h^{(2-\beta_1)}\sum_{q=0}^{n-1} d_q\,(\partial_{d,k}(e_{n-q}^u), \partial_{d,k}e_n^u) \\
& \leq\ Ch^{2(r+1)}(\|u\|_{r+2}^2 + \|uv\|_{r+2}^2 + \|v\|_{r+2}^2) + C\tau^{2(2-\alpha)}\max_{0\leq t\leq t_n}|u_{tt}(t)| \\
& +\ \frac{L}{2}\tau^2\int_{t_{n-1}}^{t_n}\|u_{tt}(s)\|^2 ds + \frac{L}{2}\|(e_{n-1}^u)_0\|_h^2 + C(\|(e_n^u)_0\|_h^2 + \|(e_n^v)_0\|_h^2) \\
& +\ C\max_{0\leq j\leq N}(\|u_j)\|_{r+3}^2 + \|u_j)\|_{r+2}^2) + C\tau^2(\|u(0)\| + \int_0^{x_n}\|u_x\|^2 ds)
\end{aligned}
\tag{6.29}
$$

In the same manner, we have

$$
\begin{aligned}
& \frac{1}{2}D_\tau^{\alpha_2}\|(e_n^v)_0\|_h^2 + \gamma_2 h^{(2-\beta_2)}\sum_{q=0}^{n-1} d_q\,(\partial_{d,k}(e_{n-q}^v), \partial_{d,k}e_n^v) \\
& \leq\ Ch^{2(r+1)}(\|u\|_{r+2}^2 + \|uv\|_{r+2}^2 + \|v\|_{r+2}^2) + C\tau^{2(2-\alpha)}\max_{0\leq t\leq t_n}|v_{tt}(t)| \\
& +\ \frac{L}{2}\tau^2\int_{t_{n-1}}^{t_n}\|v_{tt}(s)\|^2 ds + \frac{L}{2}\|(e_{n-1}^v)_0\|_h^2 + C(\|(e_n^u)_0\|_h^2 + \|(e_n^v)_0\|_h^2) \\
& +\ C\max_{0\leq j\leq N}(\|v_j)\|_{r+3}^2 + \|v_j)\|_{r+2}^2) + C\tau^2(\|v(0)\| + \int_0^{x_n}\|v_x\|^2 ds)
\end{aligned}
\tag{6.30}
$$

Combin the equation (6.29) with (6.30), and using Lemma 6.2 with $(\alpha_i = \alpha, \beta_i = \beta, \gamma_i = \gamma, i = 1, 2)$, we have

$$\begin{aligned} \|(e_h^u)_0\|_h^2 + \|(e_h^v)_0\|_h^2 &\leq C\frac{T^\alpha}{\Gamma 1+\alpha}\{h^{2(r+1)}(\max_{1\leq l\leq n}\|u(t_l)\|_{r+2}^2 + \max_{1\leq l\leq n}\|v(t_l)\|_{r+2}^2 \\ &+ \max_{1\leq l\leq n}\|uv(t_l)\|_{r+1}^2) + \tau^2[\int_0^{t_n}(\|u_{tt}(s)\|_h^2 + \|v_{tt}(s)\|_h^2)ds \\ &+ (\|u(0)\| + \|v(0)\| + \int_0^{x_n}(\|u_x\|^2 + \|v_x\|^2))ds \\ &+ \max_{0\leq t\leq t_n}(|u_{tt}(t)| + |v_{tt}(t)|)]\}. \end{aligned} \tag{6.31}$$

□

7. Numerical Experiment

In this section, we consider the following one dimensional coupled Burgers' Equation in $J = [0, 1]$ [13]:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - 2u\frac{\partial u}{\partial x} + (u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}) = 0 \tag{7.1}$$

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} - 2v\frac{\partial v}{\partial x} + (v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x}) = 0, \quad x \in J, t > 0, \tag{7.2}$$

with the initial-boundary conditions are taken from the exact solution to above system which can be expressed as

$$u(x,t) = v(x,t) = e^{-t}sin(x), \qquad \forall x \in J, t > 0. \tag{7.3}$$

In this example, the accuracy and effectiveness of the WG-FEM are illustrated where the interval J is divided into N partial intervals with grid size in x-direction $h = 1/N$ and the WG-finite element space X_h consists of constant polynomials on J and ∂J, also the time interval $[0, T]$ was divided into M partial intervals $0 = t_0 < t_1 < ... < t_M = T$ with time step $\tau = T/M$, $M \in Z^+$. Table 1 shows a comparison of numerical and exact solution and absolute error for u and v at $N = 20, M = 50$ and $T = 1$. Table 2 shows the L^2 error and convergance order for u and v at $r = 1$ and $T = 1$. Figure 1 shows a comparison of numerical and exact solution at $N = 80, M = 50$, and $T = 1$ .Figure 2 shows the solution for different values of $(\alpha = 0.1, 0.2, 0.3, ..., 1)$.

x	Exact	Numerical	Error
0	0	0	0
0.05	0.018386	0.014945	3.4409e-03
0.1	0.036727	0.029853	6.8732e-03
0.15	0.054975	0.044687	1.0288e-02
0.2	0.073086	0.059409	1.3677e-02
0.25	0.091015	0.073983	1.7032e-02
0.3	0.10872	0.088374	2.0342e-02
0.35	0.12615	0.10255	2.3598e-02
0.4	0.14326	0.11647	2.6786e-02
0.45	0.16001	0.13013	2.9885e-02
0.5	0.17637	0.14352	3.2854e-02
0.55	0.19229	0.15667	3.5612e-02
0.6	0.20772	0.16974	3.7978e-02
0.65	0.22264	0.18308	3.9560e-02
0.7	0.23699	0.19747	3.9529e-02
0.75	0.25076	0.21454	3.6221e-02
0.8	0.2639	0.2373	2.6604e-02
0.85	0.27638	0.27021	6.1709e-03
0.9	0.28817	0.31624	2.8066e-02
0.95	0.29924	0.36079	6.1556e-02
1	0.30956	0.30956	0

TABLE 1. Comparison of numerical and exact solution at $N = 20$, $M = 50$ and $T = 1$.

h	L^2- error	order
1/20	2.8962e-02	-
1/40	7.2224e-03	2.0036
1/80	1.8100e-03	1.9965
1/160	4.5253e-04	1.9999

TABLE 2. Convergence rate for u and v at r=1 and T=1.

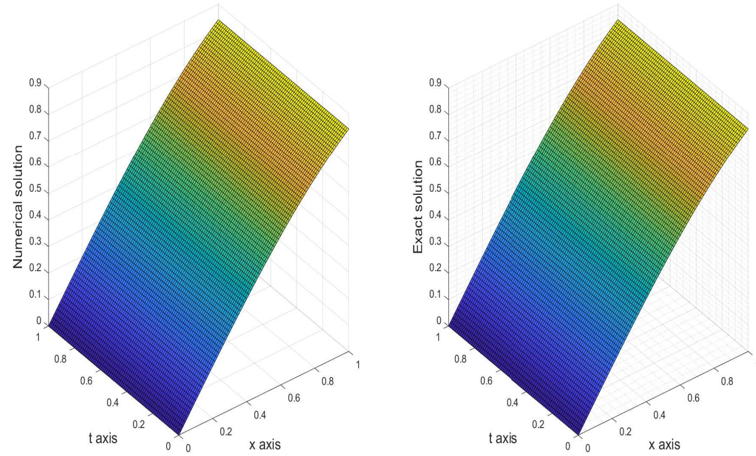

FIGURE 1. Numerical and Exact solution for u = v in case $(N = 80, M = 50)$.

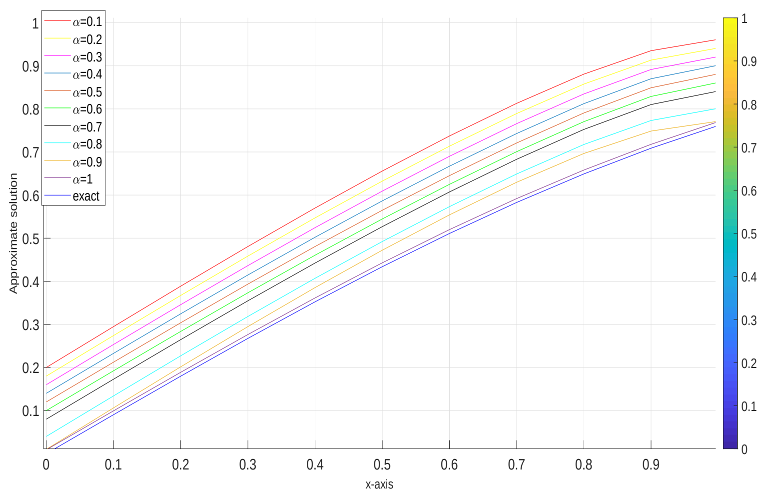

FIGURE 2. The approximate solution at different value of α.

CONFLICTS OF INTEREST

The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

REFERENCES

[1] P. Pandey, S. Kumar, J.F. Gómez-Aguilar, Numerical Solution of the Time Fractional Reaction-advection-diffusion Equation in Porous Media, J. Appl. Comput. Mech. 8 (2022), 84–96. https://doi.org/10.22055/jacm.2019.30946.1796.

[2] O.P. Agrawal, A general formulation and solution scheme for fractional optimal control problems, Nonlinear Dyn. 38 (2004), 323–337. https://doi.org/10.1007/s11071-004-3764-6.

[3] T.M. Atanacković, S. Konjik, S. Pilipović, Variational problems with fractional derivatives: Euler-Lagrange equations, J. Phys. A: Math. Theor. 41 (2008), 095201. https://doi.org/10.1088/1751-8113/41/9/095201.

[4] D.W. Dreisigmeyer, P.M. Young, Nonconservative Lagrangian mechanics: a generalized function approach, J. Phys. A: Math. Gen. 36 (2003), 8297–8310. https://doi.org/10.1088/0305-4470/36/30/307.

[5] M. Klimek, Fractional sequential mechanics-models with symmetric fractional derivative, Czechoslovak J. Phys. 51 (2001), 1348–1354. https://doi.org/10.1023/a:1013378221617.

[6] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and applications of fractional differential equations, vol. 204, North-Holland Mathematics Studies, Elsevier, Amsterdam, 2006.

[7] T. Zhang, L. Tang, A weak finite element method for elliptic problems in one space dimension, Appl. Math. Comp. 280 (2016), 1–10. https://doi.org/10.1016/j.amc.2016.01.018.

[8] Y. Chen, T. Zhang, A weak Galerkin finite element method for Burgers' equation, J. Comp. Appl. Math. 348 (2019), 103–119. https://doi.org/10.1016/j.cam.2018.08.044.

[9] Z. Sun, X. Wu, A fully discrete difference scheme for a diffusion-wave system, Appl. Numer. Math. 56 (2006), 193–209. https://doi.org/10.1016/j.apnum.2005.03.003.

[10] C. Lubich, Discretized fractional calculus, SIAM J. Math. Anal. 17 (1986), 704–719. https://doi.org/10.1137/0517050.

[11] C. Lubich, I.H. Sloan, V. Thomée, Nonsmooth data error estimates for approximations of an evolution equation with a positive-type memory term, Math. Comp. 65 (1996), 1–17.

[12] H. Wang, D. Xu, J. Zhou, J. Guo, Weak Galerkin finite element method for a class of time fractional generalized Burgers' equation, Numer. Meth. Part. 37 (2020), 732–749. https://doi.org/10.1002/num.22549.

[13] A.J. Hussein, H.A. Kashkool, Weak Galerkin finite element method for solving one-dimensional coupled Burgers' equations, J. Appl. Math. Comp. 63 (2020), 265–293. https://doi.org/10.1007/s12190-020-01317-8.