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NEW EXPONENTIATED EXTENSION LINDLEY DISTRIBUTION: PROPERTIES WITH SIMULATION AND APPLICATION

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Abstract. In this research, we proposed a novel one-parameter model, where the Inverse Exponential and the Inverse Lindley distributions are sub-models. This study aims to investigate the statistical and mathematical properties of our new model. Also, we estimated its parameter applying maximum likelihood approach and bayesian approach under different loss functions. Furthermore, an algorithm for obtaining a random sample based on the indicated distribution is presented. The approximate confidence interval according to a normal approximation is calculated and the stability of the estimator was conducted by numerical simulation for support the found results. To illustrate the importance of our new uni-modal distribution, we applied it to an actual data set, and it was found that the novel distribution fits considerably more intensely than certain other current distributions.

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1. Introduction

In the literature review, choosing an acceptable model for real-life data has been problematic and thus intensively investigated. However, in several contexts, the traditional models are considered to be insufficient or inaccurate in predicting actual data, and the majority of standard distributions do not adequately fit the real data. Several distributions have really been presented by the authors for analysing actuarial and lifetime data by melanging certain important lifetime distributions.

For a long time, several researchers have been aware of the great importance of combining distributions in actuarial science and survival analysis. For example, in the past, Lindley proposed the Lindley

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distribution as a mixing model for the exponential and gamma distribution [11]. After that, a new discrete distribution has been proposed by S. Denthet and T. Ngamkham [5] which is the mixture of distributions between the negative binomial and Kumaraswamy–Lindley distributions named the negative binomial–Kumaraswamy Lindley distribution. Moreover, Elbatal et al. [6] have introduced a tree parameter model named as the new generalized Lindley distribution by mixing two gamma distributions. In those papers, the authors had studied some of their statistical properties and, in the last one, they estimated the unknown parameters by the maximum likelihood method. On the other hand, A.H Abd Ellah [1] used bayesian and non-bayesian approaches to obtain the estimators of the parameters and reliability function for the inverse Weibull distribution. Next, in 2012, H. Rahman et al. [15] used different symmetric and asymmetric loss functions, such as squared error loss function, quadratic loss function, modified linear exponential loss function and non-linear exponential loss function to study the Bayes estimators of the parameter of Power function distribution. Also, MR. Hasan, and AR. Baizid [8] examined the Bayes estimators of the parameter of exponential distribution under different loss functions and compared among them as well as with the classical estimator.

Motivated by all these studies, we introduce a new model by mixing the Inverse Exponential distribution (studied by AZ. Killer and AR. Kamath [10]) with the Inverse Lindley distribution (created by V. Sharma et al. [19]). The results prove that the mixture distribution enhance fitting of data more than other existing distributions.

The rest of our study contains six essential sections. In section 2, we introduce the new melange via its probability density function (Pdf), cumulative distribution function (CDF), reliability, and hazard functions. Additionally, we indicate the various statistical and mathematical properties of this distribution, such as stochastic orderings, quantile function, entropy, and stress-strength reliability in section 3. The maximum likelihood estimator and the asymptotic confidence interval (IC) of the undetermined parameter for the new proposed model with evaluation of the efficiency and the stability of the suggested estimator are addressed in section 4, whereas bayesian estimator under different loss functions (symmetric and asymmetry loss functions) with a comparison with the first one is given in section 5. In section 6, a set of actual data is fitted. Finally, section 7 draws a conclusion.

2. Creation of the model

2.1. Pdf and CDF. The main idea of our work is to mix the Inverse Exponential distribution with the Inverse Lindley distribution for the scale parameter (α), and melding proportion p. Thus, the Pdf can be formatted as follows:

$$g(x;\alpha) = pg_1(x;\alpha) + pg_2(x;\alpha)$$

where

$p = \frac{\alpha}{1+\alpha}$,

$g_1(x;\alpha) = \alpha \exp(-\alpha/x)/x^2, \alpha > 0, x > 0,$

$g_2(x;\alpha) = \frac{\alpha^2}{1+\alpha}\left(\frac{1+x}{x^3}\right)\exp(-\alpha/x), \alpha > 0, x > 0.$

So, we can present the new model under the name 'New Exponentiated Extention Lindley distribution' with abbreviation NEELD by the following Pdf and CDF:

$$g\left(x;\alpha\right) = \frac{\alpha^2}{\left(1+\alpha\right)^2}\left[\frac{\left(2+\alpha\right)x+1}{x^3}\right]\exp(-\alpha/x);\quad \alpha > 0,\ x > 0, \tag{1}$$

$$G(x;\alpha) = \left[1 + \frac{\alpha}{\left(1+\alpha\right)^2}\frac{1}{x}\right]\exp(-\alpha/x);\quad \alpha > 0,\ x > 0. \tag{2}$$

where, α is a scale parameter.

2.2. **Reliability and hazard rate functions.** The reliability function $R(x)$ and hazard rate function $h(x)$ of the NEEL distribution have been respectively defined by

$$R\left(x;\alpha\right) = 1 - G(x;\alpha) = \frac{\left(1+\alpha\right)^2 x\left[1-\exp(-\alpha/x)\right] - \alpha\exp(-\alpha/x)}{\left(1+\alpha\right)^2 x} \tag{3}$$

$$h\left(x;\alpha\right) = \frac{g(x;\alpha)}{R\left(x;\alpha\right)} = \frac{\alpha^2\left[\left(2+\alpha\right)x+1\right]}{x^2\left[\left(1+\alpha\right)x(\exp(-\alpha/x)-1)-\alpha\right]} \tag{4}$$

The behavior of the NEEL distribution's Pdf (1) and hazard rate function (4) is portrayed in Figure 1 over various values of α, demonstrating that the function (4) is unimodal in x.

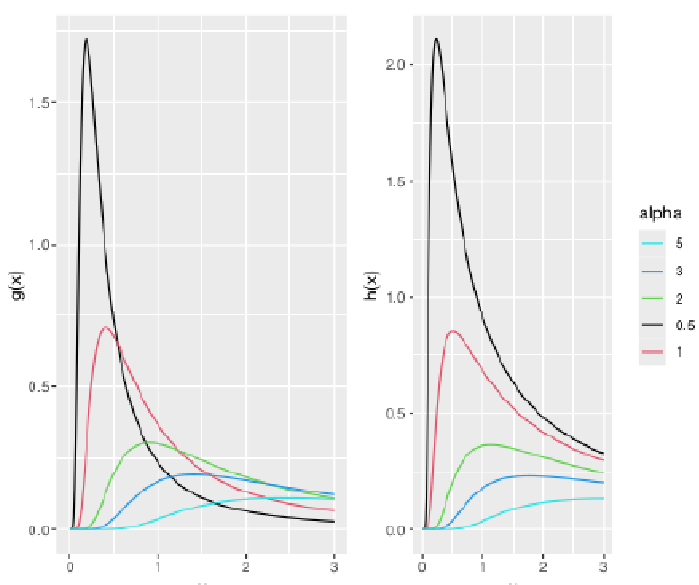

FIGURE 1. NEELD Pdf and hrf plots for chosen values of parameter

2.3. **The mode.** As a way to determine the mode of the Pdf (1) one must find the maximum of the function. Therefore, it is essential to identify the location where the first derivative of this function equals zero. The first derivative of it is provided by:

$$\frac{\partial}{\partial x}g(x;\alpha)=\left(\frac{\alpha}{1+\alpha}\right)^2\exp\left(-\alpha/x\right)/x^5\left[\left(4+2\alpha\right)x^2-\left(2\alpha+\alpha^2-3\right)x-\alpha\right] \tag{5}$$

if we take

$$\phi\left(x\right)=Ax^2+Bx+c,x>0 \tag{6}$$

with:

$A=\left(4+2\alpha\right),$

$B=-\left(2\alpha+\alpha^2-3\right),$

$C=-\alpha.$

It is self-evident that the equation $\phi\left(x\right)$ is a quadratic function, and that, $\phi\left(x\right)=0$ indicates that $\frac{\partial}{\partial x}g(x;\alpha)=0$. The roots of function (6) must then be found in order to establish the mode of the Pdf already presented in (1).

In addition, the function's roots may be found, one of them is negative and the other is positive. However, because X is non-negative, the positive root is only of the relevance, which is defined as:

$$Mo=\frac{(\alpha^2+2\alpha-3)+\sqrt{(\alpha^2+2\alpha-3)^2+(8\alpha^2+16\alpha)}}{(4\alpha+8)} \tag{7}$$

3. Statistical properties of the NEELD

Features of the NEELD are evaluated in this section, specifically the stochastic ordering, quantile function, rényi entropy, stress-strength reliability, and the distribution of the order statistics.

3.1. **Stochastic ordering.** In order to evaluate comparative behavior, it is helpful to use stochastic orderings of non-negative continuous random variables. When the variables Z_1 and Z_2 are independent with CDFs T_{Z_1} and T_{Z_2}, respectively, it is said that Z_1 is larger than Z_2 in the following circumstances:

- The stochastic order $(Z_1\leq_s Z_2)$ if $T_{Z_1}(z)\geq T_{Z_2}(z)\,\forall\, z$,
- The mean residual life order $(Z_1\leq_{mrl} Z_2)$ if $m_{Z_1}(z)\geq m_{Z_2}(z)\,\forall\, z$,
- The hazard rate order $(Z_1\leq_{hr} Z_2)$ if $h_{Z_1}(z)\geq h_{Z_2}(z)\,\forall\, z$,
- The likelihood ratio order $(Z_1\leq_{lr} Z_2)$ if $\frac{t_{z_1}(z)}{t_{z_2}(z)}$ is an decreasing function of z.

Remark 3.1. *The following effects (see [18]) are especially striking:*

$$\underbrace{(\textit{The likelihood ratio order}\Rightarrow\textit{The hasard rate order}\Rightarrow\textit{The mean residual life order})}$$

$$\Downarrow$$

The stochastic order

Theorem 3.2. *Suppose Z_1 and Z_2 two independent random variables with parameters α_1 and α_2 respectively, that follow the NEELD.*

$$\text{If } \alpha_2 > \alpha_1, \text{then } \left(Z_1 \leq_{lr} Z_2\right), \forall z > 0.$$

Proof. We have, for all $x > 0$

$$\begin{aligned}\frac{t_{z_1}(z)}{t_{z_2}(z)} &= \frac{\alpha_1^2 \left(1+\alpha_2\right)^2 \left[\left(2+\alpha_1\right) z+1\right] \exp\left(-\alpha_1/z\right)}{\alpha_2^2 \left(1+\alpha_1\right)^2 \left[\left(2+\alpha_2\right) z+1\right] \exp\left(-\alpha_2/z\right)}, \\ &= \frac{\alpha_1^2 \left(1+\alpha_2\right)^2 \left[\left(2+\alpha_1\right) z+1\right]}{\alpha_2^2 \left(1+\alpha_1\right)^2 \left[\left(2+\alpha_2\right) z+1\right]} \exp\left\{\left(\alpha_2-\alpha_1\right)/z\right\}\end{aligned}$$

It is increasing in z for $\alpha_2 > \alpha_1$. so, in terms of likelihood ratio, Z_1 is stochastically larger than Z_2. □

Corollary 3.1. *If $Z_1 \rightsquigarrow NEELD(\alpha_1)$ and $Z_2 \rightsquigarrow NEELD(\alpha_2)$ with $\alpha_2 > \alpha_1$, then $(Z_1 \leq_{lr} Z_2)$, hence $\left(Z_1 \leq_{hr} Z_2\right), \left(Z_1 \leq_{mrl} Z_2\right)$ and $\left(Z_1 \leq_{s} Z_2\right)$.*

3.2. **Quantile function.** The mean and variance formulae of the NEELD are difficult to be acquired directly since mathematic formulas for related integrals are not obtainable, but the quantiles are straightforward to calculate. Assuming T is an arbitrary random variable with the CDF $G_T(t) = P(T \leq t)$, where $t \in \mathbb{R}$, the quantile function of the NEELD can also be calculated by using the formula $Q(u) = G_T^{-1}(u)$, where $u \in (0,1)$.

Theorem 3.3. *For any $\alpha > 0$. The NEEL distribution's quantile function is supplied by*

$$Q(u) = \left[-\frac{(1+\alpha)^2}{\alpha} - \frac{1}{\alpha} W_{-1}\left(-u(1+\alpha)^2 \exp(-(1+\alpha)^2)\right)\right]^{-1}, u \in (0,1). \tag{8}$$

where W_{-1} represents the negative branch of the Lambert W function.

Proof To any specific $\alpha > 0$, allow $u \in (0,1)$,

$$((1+\alpha)^2 + \frac{\alpha}{Q(u)}) \exp(-\alpha/Q(u)) = u\,(1+\alpha)^2 \tag{9}$$

Both sides of the equation (9) are multiplied by $-\exp(-\left(1+\alpha\right)^2$, we have

$$-\left(1+\alpha\right)^2 - \frac{\alpha}{Q(u)} \exp - \left(\left(1+\alpha\right)^2 + \frac{\alpha}{Q(u)}\right) = -u\left(1+\alpha\right)^2 \exp(-\left(1+\alpha\right)^2) \tag{10}$$

The Lambert W function, as provided by Jodra [9], must be utilized to solve equation (10). It is a multivalued complex function represented as the solution to the equation $W(z)\exp(W(z)) = z$, where z is a complex number.

Equation (10) clearly shows that $-\left((1+\alpha)^2 + \frac{\alpha}{Q(u)}\right)$ is a Lambert W function with real argument $-u\left(1+\alpha\right)^2 \exp(-\left(1+\alpha\right)^2)$.

Next we take

$$W_{-1}\left(-u\,(1+\alpha)^2\exp(-\,(1+\alpha)^2)\right) = -\left((1+\alpha)^2 + \frac{\alpha}{Q(u)}\right) \tag{11}$$

In addition, for any $\alpha > 0$ and $x > 0$, and $u \in (0,1)$, the next inequalities exist:
i) $\left((1+\alpha)^2 + \frac{\alpha}{Q(u)}\right) > 1$,
ii) $-\,u\,(1+\alpha)^2\exp(-\,(1+\alpha)^2 \in (-\frac{1}{e}, 0)$.

By virture of the inequalities (i) and (ii) below, including the Lambert W , the real branch of W associated in (11) is the negative branch W_{-1}, which yields the preferred result, so we have

$$Q(u) = \left[-\frac{(1+\alpha)^2}{\alpha} - \frac{1}{\alpha}W_{-1}\left(-u(1+\alpha)^2\exp(-(1+\alpha)^2)\right)\right]^{-1};\, u \in (0,1)$$

The NEEL distribution's quartiles may be calculated using equation (8).

To that goal, the following assumptions are made: $u = \frac{1}{4}$, $\frac{1}{2}$ and $\frac{3}{4}$. The quartiles of the NEELD are calculated as shown below:

$$\begin{aligned}
Q(\frac{1}{4}) &= \left[-\frac{(1+\alpha)^2}{\alpha} - \frac{1}{\alpha}W_{-1}\left(-\frac{1}{4}(1+\alpha)\right)^2\exp(-(1+\alpha)^2)\right]^{-1} \\
Q(\frac{1}{2}) &= \left[-\frac{(1+\alpha)^2}{\alpha} - \frac{1}{\alpha}W_{-1}\left(-\frac{1}{2}(1+\alpha)\right)^2\exp(-(1+\alpha)^2)\right]^{-1} \\
Q(\frac{3}{4}) &= \left[-\frac{(1+\alpha)^2}{\alpha} - \frac{1}{\alpha}W_{-1}\left(-\frac{3}{4}(1+\alpha)\right)^2\exp(-(1+\alpha)^2)\right]^{-1}
\end{aligned}$$

3.3. **Rényi entropy.** The entropy describes the level of variability in the uncertainty in a random variable's distribution. Plenty of uses of the Rényi entropy can be observed in the areas of statistics, computer science, and econometrics. A high entropy value indicates a high level of uncertainty.

Plenty of uses of the Rényi entropy can be observed in the areas of statistics, computer science, and econometrics.

The expression of the Rényi entropy is given by

$$I_R(\delta) = \frac{1}{1-\delta}\log\left\{I(\delta)\right\},$$

where $I(\delta) = \int g^{\delta}(x)dx$, $\delta > 0$ and $\delta \neq 1$.

Now, consider X as a random variable with the Pdf (1). The entropy of X is thus being described below:

$$I_R(\delta) = \frac{1}{1-\delta}\log\left\{\left(\frac{\alpha}{1+\alpha}\right)^{2\delta}\int\left[\frac{((2+\alpha)\,x+1)^{\delta}}{x^{3\delta}}\right]\exp(-\alpha\delta/x)dx\right\}$$

We know

$$(1+a)^n = \sum_{i=0}^{n}\binom{n}{i}a^i \quad \text{and} \quad \Gamma(\tau)/c^{\tau} = \int_0^{\infty} x^{-\tau-1}\exp(-c/x)dx,$$

As a result, the NEEL distribution's Rényi entropy takes the form

$$I_R(\delta) = \frac{1}{1-\delta}\log\left[\left(\frac{\alpha}{1+\alpha}\right)^{2\delta}\sum_{i=0}^{n}\binom{n}{i}(2+\alpha)^i\frac{\Gamma(3\delta-i-1)}{(\alpha\delta)^{(3\delta-i-1)}}\right] \tag{12}$$

3.4. **Stress-Strength Reliability.** The stress-strength reliability of a system is a measure of its capacity to perform in harsher conditions, and it is theoretically identified as $R = P(Y < X)$, or the probability that the system strength (X) is larger than the environmental stress (Y) when the system is operating. Now, assume that X and Y are independent stress and strength random variables, with parameters α_1 and α_2 that each follow the NEELD. As a result, the stress-strength reliability R is described in this manner:

$$\begin{aligned}
R &= \int_0^\infty\left\{\int_0^x g_Y(y,\alpha_2)dy\right\}g_X(x,\alpha_1)dx \\
&= \int_0^\infty g_X(x,\alpha_1)G_Y(x,\alpha_2)dx \\
&= \int_0^\infty\frac{\alpha_1^2}{(1+\alpha_1)^2}\left[\frac{(2+\alpha_1)x+1}{x^3}\right]\exp(-\alpha_1/x)\left[1+\frac{\alpha_2}{(1+\alpha_2)^2}\frac{1}{x}\right]\exp(-\alpha_2/x)dx \\
&= \frac{\alpha_1^2}{(1+\alpha_1)^2}\int_0^\infty\left[\frac{(2+\alpha_1)x+1}{x^3}\right]\exp\{-(\alpha_1+\alpha_2)/x\}dx \\
&\quad+\frac{\alpha_2}{(1+\alpha_2)^2}\left(\frac{\alpha_1}{1+\alpha_1}\right)^2\int_0^\infty\left[\frac{(2+\alpha_1)x+1}{x^4}\right]\exp\{-(\alpha_1+\alpha_2)/x\}dx
\end{aligned} \tag{13}$$

Finally, using the definition of the inverse gamma density, we arrive to

$$R = \frac{\alpha_1^2\left\{(2+\alpha_1)(1+\alpha_2)^2(\alpha_1+\alpha_2)^2+(\alpha_1+\alpha_2)\left[(1+\alpha_1)^2+\alpha_2(2+\alpha_1)\right]+2\alpha_2\right\}}{(1+\alpha_1)^2(1+\alpha_2)^2(\alpha_1+\alpha_2)^3} \tag{14}$$

3.5. **Order statistics.** Let $X_1, X_2, ...X_n$ represent a random sample from the NEELD and $X_{1:n}, X_{2:n}, ...X_{n:n}$ constitute the associated order statistics. The ith order statistic's Pdf, had said $X_{i:n}$, is offered by

$$\begin{aligned}
g_{j:n}(x) &= \frac{n!}{(j-1)!(n-j)!}F^j(x)\left[1-F^j(x)\right]^{n-j}f(x) \\
&= \frac{n!}{(j-1)!(n-j)!}\sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k}F^{j+k-1}(x)f(x)
\end{aligned}$$

respectively, for $j = 1, 2, ...n$, it follows from (1) and (2) that

$$\begin{aligned}
g_{j:n}(x) &= \frac{n!}{(j-1)!(n-j)!}\sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k}\frac{\alpha^2}{(1+\alpha)^2}\left[\frac{(2+\alpha)x+1}{x^3}\right]\exp(-\alpha/x) \\
&\times\left[\left\{1+\frac{\alpha}{(1+\alpha)^2}\frac{1}{x}\right\}\exp(-\alpha/x)\right]^{j+k-1}
\end{aligned}$$

When $j = 1$ and $j = n$, the Pdf of the minimum and the maximum order statistic of the NEELD are respectively provided by

$$\begin{aligned} g_{1:n}(x) &= n\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\frac{\alpha^2}{(1+\alpha)^2}\left[\frac{(2+\alpha)\,x+1}{x^3}\right]\exp(-\alpha/x) \\ &\quad\times\left[\left\{1+\frac{\alpha}{(1+\alpha)^2}\frac{1}{x}\right\}\exp(-\alpha/x)\right]^k \end{aligned}$$

and

$$g_{n:n}(x) = n\left(\frac{\alpha}{1+\alpha}\right)^2\left[\frac{(2+\alpha)\,x+1}{x^3}\right]\exp(-\alpha/x)\times\left[\left\{1+\frac{\alpha}{(1+\alpha)^2}\frac{1}{x}\right\}\exp(-\alpha/x)\right]^{n-1} \tag{15}$$

4. Maximum likelihood estimation

We present in this section the maximum likelihood estimate of the parameter of the proposed distribution, as well as the unknown parameter's approximate confidence interval.

Let $X_1, X_2, ..., X_n$ be n independent and identical random variables returning to the NEELD with parameter α. For identifying the MLE of α, we now have likelihood function based on observed sample $\underset{\sim}{\mathrm{x}}= (x_1, x_2, ...x_n)$ provided by

$$\ell(\alpha;\underset{\sim}{\mathrm{x}}) = \left(\frac{\alpha}{1+\alpha}\right)^{2n}\prod_{i=1}^{n}\left[\frac{(2+\alpha)\,x_i+1}{x^3}\right]\exp(-\alpha\sum_{i=1}^{n}x_i^{-1}) \tag{16}$$

The log-likelihood function for (16) is

$$\log\ell = 2n\log\alpha - 2n\ln(\alpha+1) + \sum_{i=1}^{n}\ln\left[(2+\alpha)x_i+1\right] - 3\sum_{i=1}^{n}\ln x_i - \alpha\sum_{i=1}^{n}x_i^{-1} \tag{17}$$

The maximum likelihood estimates $\hat{\alpha}_{ML}$ of α could be achieved by solving the following non-linear equation:

$$\frac{\partial\log\ell}{\partial\alpha} = \frac{2n}{\alpha} - \frac{2n}{\alpha+1} + \sum_{i=1}^{n}\left[\frac{x_i}{(2+\alpha)\,x_i+1}\right] - \sum_{i=1}^{n}x_i^{-1} = 0 \tag{18}$$

Numerical solutions to this equation using non-linear optimization techniques, such as the quasi-Newton approach, are frequently more practical . Because the explicit formula for the maximum likelihood estimator of the parameter is not accurate, numerical approaches like the Newton-Raphson method, the Monte Carlo method, the BB method, and others are required.

In the next section, we prove that the MLE, say, $\hat{\alpha}$ is consistent and asymptotically normal under certain regularity condition.

4.1. Asymptotic Confidence Interval.

Because the accurate distribution of the aforementioned estimator cannot be determined directly, we used large sample theory, in this part, to generate confidence interval based on it . Let us recollect that the MLE $\hat{\alpha}$ is consistent and asymptotically normal under some regularity criterion (see section 4.2.2). Hence, the asymptotic distribution of the estimator may be written as

$$(\hat{\alpha}-\alpha) \rightsquigarrow N(0, I^{-1}(\hat{\alpha}))$$

As a result, we can create a two-sided $(1-\tau)\,\%$ asymptotic confidence interval for α with $\tau \in (0,1)$ as $\left[\hat{\alpha}-z_{\tau/2}\sqrt{v(\hat{\alpha})}, \hat{\alpha}+z_{\tau/2}\sqrt{v(\hat{\alpha})}\right]$ where $z_{\tau/2}$ is the upper $(\tau/2)\,th$ percentile of the standard normal distribution. Here, $var\,(\hat{\alpha})$ may be estimated as follows:

$$var\,(\hat{\alpha}) = E\left[-\frac{\partial^2 \log \ell}{\partial^2 \alpha}\right]^{-1}_{\alpha=\hat{\alpha}}$$

Where

$$\frac{\partial^2 \log \ell}{\partial^2 \alpha} = -\frac{2n}{\alpha^2} + \frac{2n}{(\alpha+1)^2} - \sum_{i=1}^{n}\left[\frac{x_i^2}{((2+\alpha)\,x_i+1)^2}\right] \tag{19}$$

4.2. **Monte Carlo simulation study.**

4.2.1. *Generation of the random sample.* The inverse of the cumulative distribution function (CDF) has been the most widely used and easiest way to generate random samples. Hence, in the application of NEEL distribution, the inverse of the CDF can not be derived directly, thus a Lambert W function is necessary for simplification (see [9]). For this objective, we used in our study the equation (8) in conjunction with the R package (LambertW) to generate a random sample of size n.

Also, we can use the algorithm described below.

Algorithm.

Because the NEELD is a combination of Inverse Exponential (α) and Inverse Lindley (α) distributions, simulation from the NEELD may also be done using the description of the mixed distribution. The steps of the algorithm are as follows:

Step 1. Set the value of the parameter α, and desired simple size n.

Step 2. Generate U using the basic uniform density.

Step 3. Generate Y using the *Inverse Exponential*(α).

Step 4. Generate Z an *Inverse Lindley* variate $Z \rightsquigarrow IL(\alpha)$

Step 5. If $U \leq \frac{\alpha}{1+\alpha}$, accept Z as a sample from NEELD(α), so $X = Z$ otherwise $X = Y$.

Step 6. Repeat the steps 2 to 5, n times for obtaining the sample of size n from NEELD with parameter α.

Remark 4.1. *The R-codes,* runif(n,0,1), rilindley(n, alpha, mixture = TRUE) *and* rinvexp(n, $rate$ = 1, scale = 1/$rate$) *may be used to generate samples from the uniform, Inverse Lindley and Inverse Exponentiel distributions, respectively, where the first term* n *defines the number of units to be obtained.*

4.2.2. *Simulation.* This section discusses the simulation research that was carried out in order to evaluate the bias and the mean square error of the suggested estimator with various sample sizes. We chose $\alpha = 0.7, 1, 1.5, 2, 4.$ and $n = 10, 30, 80, 150, 200, 300, 500.$ The average of the simulated value of the maximum likelihood estimator $\hat{\alpha}$ (MLE), the bias ($Bias$), and their mean squared error (MSE) were calculated using the R program (R 4.1.2) and the Barzilai-Borwein (BB) technique [16]. We ended this simulation study by investigating the approximate confidence interval IC and their length LIC of MLE of the parameter α . We executed the process N times (N =5000). Table 1 shows the results of the simulation. The following equations are used to determine the MLE, $Bias$, and MSE of α, based on 5000 simulations.

$$MLE(\alpha) = \frac{1}{N}\sum_{i=1}^{N}\hat{\alpha}_i, \tag{20}$$

$$Bias(\hat{\alpha}) = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\alpha}_i - \alpha\right), \tag{21}$$

$$MSE(\hat{\alpha}) = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\alpha}_i - \alpha\right)^2 \tag{22}$$

Table 1 shows the average estimate value, bias and mean square error of the parameter α. In general, for large values of n, the bias and MSE tend to be zero. That is to say, the estimator $\hat{\alpha}$ is asymptotically unbiased.

It is obvious that α has a strictly positive bias. Furthermore, as α increases, so does this bias. Regarding the mean square error, this table shows that as α increases, the MSE increases.

However, the effect of α in the MSE is very important. In addition, as the sample size increases, the approximate length LIC decreases. Based on the results in this table, it can be concluded that the maximum likelihood technique works well when estimating the NEEL model parameter.

Table 1: Average estimate, Bias, MSE and CI for α by varying sample size when $\alpha = 0.7, 1, 1.5, 2, 4$.

$\alpha = 0.7$					
$Sample\ size$	$MLE(\alpha)$	$Bias(\hat{\alpha})$	$MSE(\hat{\alpha})$	$IC_{95\%}$	$LIC_{95\%}$
10	0.7574	0.0574	0.0451	(0.6755, 0.8393)	0.1638
30	0.7168	0.0168	0.0107	(0.6968, 0.7371)	0.0403
80	0.7053	0.0053	0.0037	(0.7003, 0.7111)	0.0108
150	0.7031	0.0031	0.0020	(0.7012, 0.7050)	0.0038
200	0.7014	0.0014	0.0015	(0.6987, 0.7041)	0.0054
300	0.7020	0.0020	0.0009	(0.7018, 0.7035)	0.0017
500	0.7012	0.0012	0.0005	(0.7004, 0.7019)	0.0015
$\alpha = 1$					

10	1.0712	0.0712	0.0978	(0.8896, 1.2528)	0.3632
30	1.0229	0.0229	0.0242	(0.9765, 1.0693)	0.0928
80	1.0095	0.0095	0.0085	(1.0065, 1.0393)	0.0328
150	1.0032	0.0032	0.0044	(0.9946, 1.0118)	0.0172
200	1.0025	0.0025	0.0031	(0.9965, 1.0085)	0.0120
300	1.0019	0.0019	0.0021	(0.9978, 1.0060)	0.0082
500	1.0014	0.0014	0.0012	(0.9991, 1.0037)	0.0046
$\alpha = 1.5$					
10	1.6276	0.1276	0.2633	(1.1435, 2.1117)	0.9682
30	1.5349	0.0349	0.0612	(1.4174, 1.6524)	0.2350
80	1.5128	0.0128	0.0212	(1.4716, 1.5540)	0.0824
150	1.5100	0.0100	0.0107	(1.4893, 1.5307)	0.0414
200	1.5081	0.0081	0.0083	(1.4920, 1.5242)	0.0322
300	1.5330	0.0033	0.0055	(1.5223, 1.5437)	0.0214
500	1.5030	0.0030	0.0031	(1.4970, 1.5090)	0.0120
$\alpha = 2$					
10	2.1743	0.1743	0.4886	(1.2762, 2.9533)	1.6771
30	2.0552	0.0552	0.1207	(1.8246, 2.2858)	0.4612
80	2.0208	0.0208	0.0397	(1.9438, 2.0978)	0.1540
150	2.0103	0.0103	0.0209	(1.9695, 2.0511)	0.0816
200	2.0069	0.0069	0.0154	(1.9768, 2.0370)	0.0602
300	2.0070	0.0070	0.0103	(1.9869, 2.0271)	0.0402
500	2.0029	0.0029	0.0063	(1.9906, 2.0152)	0.0246
$\alpha = 4$					
10	4.4433	0.4433	2.3874	(0.1493, 8.7374)	8.5880
30	4.1325	0.1325	0.5548	(3.0795, 5.1856)	2.1060
80	4.0429	0.0429	0.1849	(3.6840, 4.4019)	0.7178
150	4.0230	0.0230	0.0967	(3.8344, 4.2117)	0.3773
200	4.0201	0.0201	0.0730	(3.8778, 4.1624)	0.2845
300	4.0124	0.0124	0.0475	(3.9194, 4.1053)	0.1858
500	4.0090	0.0090	0.0288	(3.9526, 4.0654)	0.1127

Figures 2 and 3 round out these findings through plotting the Bias and the MSE of the MLE presented in table 1.

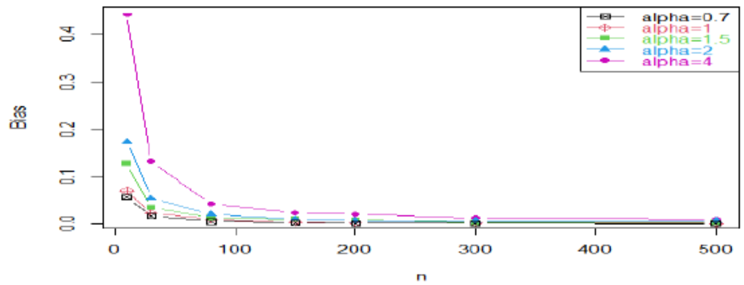

FIGURE 2. Bais of the MLE for different values of the parameter α

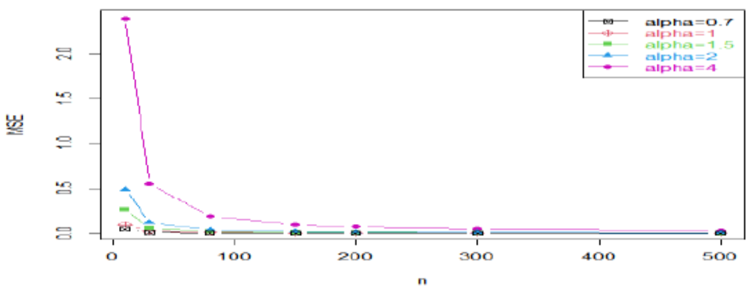

FIGURE 3. MSE of the MLE for different values of the parameter α

The histograms of the MLE values obtained, as well as their forms, were plotted in Figures 4, 5, and 6 for various values of α and sample sizes $n = 20, 100$, and 500, respectively. As a result, we want to get a sense of the MLE distribution. For instance, these graphs reveal that the MLE's overall distribution resembles the bell shape of a normal distribution. Because the sample size is so large, the bell form is more obvious. This is consistent with the MLE's asymptotic normality, which has been widely proven (see [17]). We also see that the modal class of each histogram has the precise value of the parameter α.

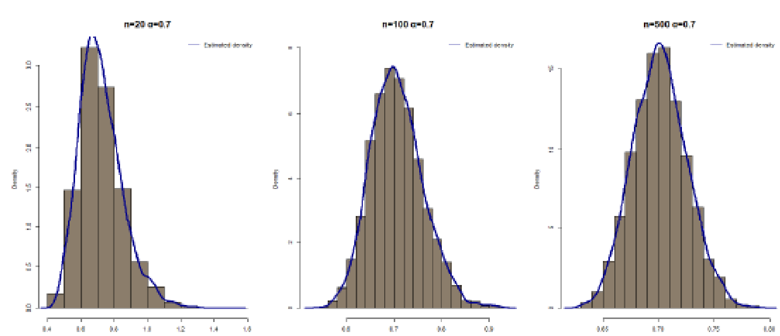

FIGURE 4. Simulated distribution of the MLE for $\alpha = 0.7$, n = 20, 100 and 500

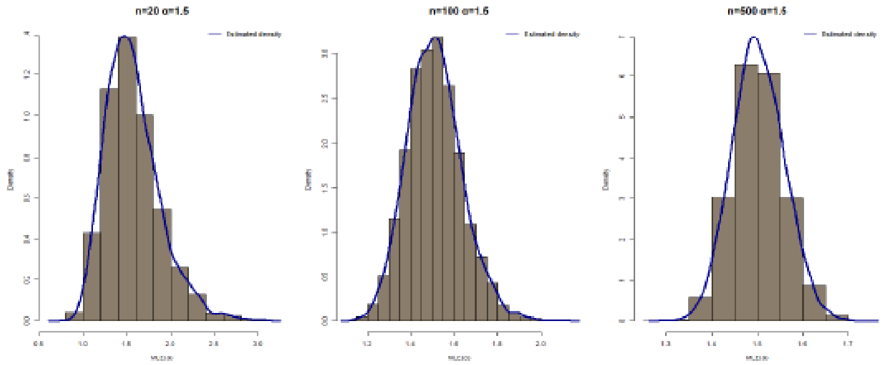

Figure 5. Simulated distribution of the MLE for $\alpha = 1.5$, n = 20, 100 and 500

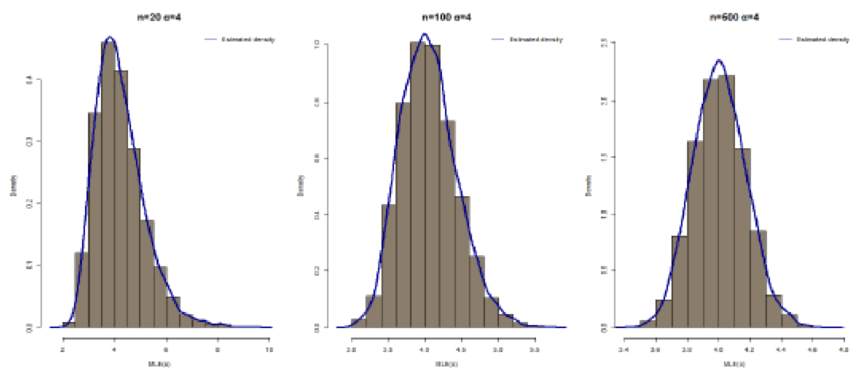

Figure 6. Simulated distribution of the MLE for $\alpha = 4$, n = 20, 100 and 500

4.3. **Stability analysis.** In statistical modelling, an appropriate model should ideally indicate reduced variability and greater precision as additional data is available for estimate. The behaviour of the distribution was tested using bias and standard error and revealed as the sample size increased. The simulation results showed the expected trend: as sample sizes increased, the values of the adequacy parameters (bias and standard error) decreased (Table 1). The pattern shown in Figure 2 and Figure 3 is consistent with predictions for a well-fitting density model.

As a result, falling values of these adequacy indicators with increasing sample numbers might be regarded positively. This finding supports the notion that the NEEL model acts like a well-behaved density function, indicating that it is appropriate for modelling real-world data sets.

In summary, the simulation findings, as summarized in Table 1, show that the NEEL model is a statistically sound and acceptable option for modelling data in a variety of applications.

5. Bayesian estimation

The bayesian estimation is a non-classical technique in statistical inference and is practical in real-world situations. The aim of this section is to study the Bayes estimators of the parameter of the NEEL distribution under different loss functions (symmetric and asymmetric loss functions) and compare them with the MLE.

For the prior distribution, gamma model is a good choice. A comparative situation is studied for different cases.

5.1. **Prior and posterior density functions of the parameter** α**.** Let $X_1, X_2, ...X_n$ be a sample of size n generated from the NEEL model. The likelihood function is then given by (4.1).

In the Bayesian context, we need to specify a prior distribution for the parameter. Consider a conjugate Gamma prior for α having a density function:

$$p(\alpha) = \frac{b^a}{\Gamma(a)}\alpha^{a-1}\exp(-b\alpha); \qquad \alpha, a, b > 0. \tag{23}$$

Then the posterior density function of the parameter α for the given random sample X is given by:

$$\begin{aligned} p(\alpha/x) &= \frac{\ell(\alpha;\underline{x})p(\alpha)}{\int_0^{+\infty}\ell(\alpha;\underline{x})p(\alpha)d\alpha} \\ &= \frac{\frac{\alpha^{2n+a-1}}{(1+\alpha)^{2n}}\prod_{i=1}^{n}\left[(2+\alpha)x_i^{-2}+x_i^{-3}\right]\exp\left[-\alpha\left(b+\sum_{i=1}^{n}x_i^{-1}\right)\right]}{\int_0^{+\infty}\frac{\alpha^{2n+a-1}}{(1+\alpha)^{2n}}\prod_{i=1}^{n}\left[(2+\alpha)x_i^{-2}+x_i^{-3}\right]\exp\left[-\alpha\left(b+\sum_{i=1}^{n}x_i^{-1}\right)\right]d\alpha} \\ &= I^{-1}\times\frac{\alpha^{2n+a-1}}{(1+\alpha)^{2n}}\prod_{i=1}^{n}\left[(2+\alpha)x_i^{-2}+x_i^{-3}\right]\exp\left[-\alpha\left(b+\sum_{i=1}^{n}x_i^{-1}\right)\right] \end{aligned} \tag{24}$$

Where

$$I = \int_0^1 \frac{\alpha^{2n+a-1}}{(1+\alpha)^{2n}}\prod_{i=1}^{n}\left[(2+\alpha)x_i^{-2}+x_i^{-3}\right]\exp\left[-\alpha\left(b+\sum_{i=1}^{n}x_i^{-1}\right)\right]d\alpha$$

5.2. **Different loss functions.** We consider the squared error ,the generalized quadratic, the linex , and the entropy loss functions .

We present the Bayesian estimators with their corresponding posteriors errors in the Table 2

TABLE 2. Loss functions with the corresponding bayesian estimators and the posterior risk for the parameter. ($E_p(.)$ stands for the posterior expectation)

Loss function	expression	Bayes estimator	P.R
Squared error	$L(\alpha,\hat{\alpha}) = (\alpha-\hat{\alpha})^2$	$\hat{\alpha}_{SQE} = E_p(\alpha)$	$E_p((\alpha-\hat{\alpha})^2)$
Linex	$L(\alpha,\hat{\alpha}) = e^{(\hat{\alpha}-\alpha)} - r(\hat{\alpha}-\alpha) - 1$	$\hat{\alpha}_{LE} = \frac{-1}{r}lnE_p(e^{-r\alpha})$	$r(\hat{\alpha}_{SQE}-\hat{\alpha}_{LE})$
Generalized quadratic	$L(\alpha,\hat{\alpha}) = \tau(\alpha)(\alpha-\hat{\alpha})^2$	$\hat{\alpha}_{GQE} = \frac{E_p(\tau(\alpha)\alpha)}{E_p(\tau(\alpha))}$	$E_p(\tau(\alpha)(\alpha-\hat{\alpha}_{GQE})$
Entropy	$L(\alpha,\hat{\alpha}) = (\frac{\hat{\alpha}}{\alpha})^c - cln(\frac{\hat{\alpha}}{\alpha}) - 1$	$\hat{\alpha}_{EE} = [E_p(\alpha)^{-c}]^{\frac{-1}{c}}$	$c[E_p(ln(\alpha)-ln(\hat{\alpha}_{EE}))]$

5.3. **Simulation and comparison study.** In our study, since we cannot calculate the analytical expressions of all these estimators, we generate 5000 samples for each case. We have obtained the estimated values by using MCMC methods as the Merropolis-Hastings algorithm using a self-writhen R code. The results and their graphs are presented in Tables 3, 4, 5.

The performance of different estimation approaches is studied by using simulation, so we generated random samples in such a way that estimators under different estimation techniques can be compared and are in accordance with real-life problems. Here, for simulation, we used the inverse transformation method. The study has been carried out for $n = 50, 100, 150, 200$ and 300, using $\alpha = 1, 2$ and 4. Notice that under the generalized quadratic loss function assuming $\tau(\alpha) = \alpha^{\beta-1}$ for $\beta = 5$, we have taken $c = -3$ for the entropy loss function and $r = 1$ in the case of linex one. The posterior risk (PR) of Bayes estimates has been presented in parenthesis.

Table 3. MLE value (MSE) with different Bayesian estimators (BR)of α for NEEL distribution where $\alpha = 1,\ and\ \beta = 5,\ c = -3, r = 1$ for two senarios (1) with a non informative prior $a = b = 0$ (2) with a conjugate prior $a = 1,\ b = 5$.

n	MLE	senario (1) $a = b = 0$ (BR)				senario (2) $a = 1,\ b = 5$ (BR)			
	(MSE)	$\hat{\alpha}_{SQE}$	$\hat{\alpha}_{LE}$	$\hat{\alpha}_{GQE}$	$\hat{\alpha}_{EE}$	$\hat{\alpha}_{SQE}$	$\hat{\alpha}_{LE}$	$\hat{\alpha}_{GQE}$	$\hat{\alpha}_{EE}$
50	1.0095	1.0631	1.066	1.0748	1.0658	1.0535	1.052	1.0617	1.0554
	(0.01341)	(0.00397)	(0.00138)	(0.00474)	(0.00432)	(0.00285)	()0.00097)	((0.00318)	(0.00306)
100	1.0084	1.0309	1.0305	1.0339	1.0316	1.0238	1.0236	1.026	1.0244
	(0.00647)	(0.00095)	(0.00036)	(0.00095)	(0.00099)	(0.00056)	(0.00026)	(0.00067)	(0.00059)
150	1.0054	1.0451	1.0444	1.0508	1.0464	1.0391	1.0386	1.0432	1.0401
	(0.00458)	(0.00203)	(0.00069)	(0.00199)	(0.00215)	(0.00153)	(0.00049)	(0.00135)	(0.00161)
200	1.0023	1.0222	1.022	1.0237	1.0225	1.0245	1.0242	1.0264	1.0249
	(0.00317)	(0.00049)	(0.00018)	(0.00045)	(0.00050)	(0.00059)	(0.00022)	(0.00058)	(0.00062)
300	1.0027	1.0302	1.0299	1.0327	1.0308	1.0288	1.0285	1.0315	1.0295
	(0.00207)	(0.00091)	(0.00030)	(0.00077)	(0.00094)	(0.000832)	(0.00033)	(0.00085)	(0.00087)

It is clear from the Tables 3, 4 and 5 that the estimated value of the parameter converges to the true value as the sample size increases. While the magnitude of the posterior risk is not stable, the estimates under a conjugate prior are seen to work better than those under a uniform prior for each loss function, and the estimates using a bayesian framework provide better results than the maximum likelihood estimates.

Table 4. MLE value (MSE) with different Bayesian estimators (BR) of α for NEEL distribution where $\alpha = 2,\ and\ \beta = 5,\ c = -3, r = 1$ for two senarios (1) with a non informative prior $a = b = 0$ (2) with a conjugate prior $a = 1,\ b = 5$.

n	MLE	senario (1) $a = b = 0$ (BR)				senario (2) $a = 1,\ b = 5$ (BR)			
	(MSE)	$\hat{\alpha}_{SQE}$	$\hat{\alpha}_{LE}$	$\hat{\alpha}_{GQE}$	$\hat{\alpha}_{EE}$	$\hat{\alpha}_{SQE}$	$\hat{\alpha}_{LE}$	$\hat{\alpha}_{GQE}$	$\hat{\alpha}_{EE}$
50	2.0195	2.0839	2.0805	2.0956	2.0869	2.0612	2.0579	2.0702	2.0638
	(0.06429)	(0.00704)	(0.00338)	(0.11880)	(0.00755)	(0.00374)	(0.00327)	(0.07541)	(0.00406)
100	2.0156	2.0867	2.0832	2.0993	2.0899	2.0652	2.0626	2.0741	2.0675
	(0.03210)	(0.00752)	(0.00358)	(0.12934)	(0.00808)	(0.00425)	(0.00264)	(0.08809)	(0.004557)
150	2.0144	2.0374	2.0359	2.0412	2.0321	2.0444	2.0428	2.049	2.0387
	(0.02025)	(0.00139)	("0.00146)	(0.02787)	(0.00102)	(0.00197)	(0.00165)	(0.03545)	(0.00150)
200	2.0082	2.0322	2.031	2.035	2.0274	2.0365	2.035	2.0404	2.0312
	(0.01467)	(0.00103)	(0.00121)	(0.01756)	(0.00075)	(0.00133)	(0.00147)	0.02905	0.00097
300	2.0063	2.0364	2.0351	2.0398	2.0313	2.0328	2.0315	2.0359	2.0279
	(0.01258)	(0.00132)	(0.00135)	(0.02412)	(0.00098)	(0.00107)	(0.00128)	(0.02057)	(0.00077)

Table 5. MLE value (MSE) with different Bayesian estimators (BR)of α for NEEL distribution where $\alpha = 4,\ and\ \beta = 5,\ c = -3, r = 1$ for two senarios (1) with a non informative prior $a = b = 0$ (2) with a conjugate prior $a = 1,\ b = 5$.

n	MLE	senario (1) $a = b = 0$ (BR)				senario (2) $a = 1,\ b = 5$ (BR)			
	(MSE)	$\hat{\alpha}_{SQE}$	$\hat{\alpha}_{LE}$	$\hat{\alpha}_{GQE}$	$\hat{\alpha}_{EE}$	$\hat{\alpha}_{SQE}$	$\hat{\alpha}_{LE}$	$\hat{\alpha}_{GQE}$	$\hat{\alpha}_{EE}$
50	4.0772	4.053	4.0346	4.0589	3.9696	4.0763	4.0562	4.0851	3.9899
	(0.29906)	(0.00280)	(0.01835)	(0.8581)	(0.00092)	(0.00582)	(0.02008)	(1.72991)	(0.00010)
100	4.0554	4.0775	4.0573	4.0863	3.9909	4.0624	4.0434	4.0695	3.9779
	(0.14688)	(0.00599)	(0.020138)	(1.76168)	$(8.25e-05)$	(0.00389)	(0.01899)	(1.20655)	(0.00048)
150	4.0138	4.0724	4.0524	4.0813	4.0756	4.0527	4.0344	4.0585	4.0551
	(0.09651)	(0.00524)	(0.02001)	(1.78538)	(0.00571)	(0.00277)	(0.01826)	(0.80713)	(0.00303)
200	4.0047	4.0633	4.0442	4.0704	4.066	4.0846	4.0642	4.0939	4.0879
	(0.07363)	(0.00400)	(0.01907)	(1.22492)	(0.00435)	(0.00716)	(0.02044)	(1.94730)	(0.00772)
300	4.0056	4.028	3.9853	4.0863	4.015	4.0076	3.9913	4.0349	4.0066
	(0.06749)	(0.00032)	(0.03261)	(19.34859)	(0.00032)	$(3.181e-05)$	(0.01438)	(7.76343)	$(3.181e-05)$

6. Application of real data

We provide the actual data set that represents the costs of the 31 various kids' wooden toys that were being sold in a Suffolk craft store in April 1991 [20] to demonstrate the validity of the model presented in this study.

The data were: 4.2, 1.12, 1.39, 2, 3.99, 2.15, 1.74, 5.81, 1.7, 2.85, 0.5, 0.99, 11.5, 5.12, 0.9, 1.99, 6.24, 2.6, 3, 12.2, 7.36, 4.75, 11.59, 8.69, 9.8, 1.85, 1.99, 1.35, 10, 0.65, 1.45.

Shafiei et al. [17] believed that this data set to fit the *Inverse Weibull Poisson* (*IWP*), *Inverse Weibull Geometric* (*IWG*) and *Inverse Weibull Logarithmic* (*IWL*) distributions and also compared them with the *Inverse Weibull* (*IW*), *Weibull* (*W*) [13], *Nadarajah-Haghighi* (*NH*) [14], *Generalized Inverse Weibull* (*GIW*) [7], *Exponentiated Exponential Binomial* (*EEB*) [2] and *Exponentiated Weibull–Poisson* (*EWP*) [12].

We compared the fit of the new proposed distribution to this data set with the models mentioned above. For each of those distributions, the subsequent functions provide the corresponding densities (for $x > 0$):

$$\begin{aligned}
IW(z,\zeta,\kappa) &= \zeta\kappa z^{-(\kappa+1)}e^{-\zeta z^{-\kappa}};\\
W(z,\zeta,\kappa) &= \zeta\kappa z^{(\kappa-1)}e^{-\zeta z^{\kappa}};\\
NH(z,\zeta,\kappa) &= \zeta\kappa(1+\kappa z)^{\zeta-1}\exp\left\{1-(1+\kappa z)^{\zeta}\right\};\\
GIW(z,\zeta,\kappa,\nu) &= \nu\kappa\zeta^{\kappa}z^{-(\kappa+1)}e^{-\nu(\zeta/z)^{\kappa}};\\
EEB(z,\zeta,\kappa,\nu) &= \frac{n\zeta\kappa\nu e^{-\nu\lambda z}(1-e^{-\nu z})^{\zeta-1}(1-\kappa(1-e^{-\nu z})^{\zeta})^{n-1}}{1-(1-\beta)^{n}}, \kappa\in[0,1];\\
EWP(z,\zeta,\kappa,\nu,\theta) &= \frac{\zeta\nu\theta\kappa^{\nu}}{e^{\theta-1}}z^{\nu-1}e^{-(\kappa z)^{\nu}}(1-e^{-(\zeta)^{\nu}})^{\zeta-1}e^{\theta(1-e^{-(\kappa z)^{\nu}})^{\zeta}}.
\end{aligned}$$

The previously mentioned pdfs' unidentified parameters have all non-negative real values. The AIC, HQIC, BIC, and CAIC are used for evaluating distributions. These statistics are offered by

$$\begin{aligned}
\textbf{AIC} &= -2\hat{\ell}+2m\\
\textbf{BIC} &= -2\hat{\ell}+m\log(n)\\
\textbf{CAIC} &= -2\hat{\ell}+\frac{2mn}{(n-m-1)}\\
\textbf{HQIC} &= -2\hat{\ell}+2m\log[\log(n)]
\end{aligned}$$

where n is the sample size, m is the number of the previous model's parameters, and $\hat{\ell}$ is the log-likelihood function for the MLE.

The best fit for the data may be determined by choosing the model with fewer values for all of these statistics. To obtain the results, the R program was utilized.

Figure 7 indicates the estimated pdf, estimated *CDF* and the *P-P* plot for the data set, and Figure 8 gives the total time on test plot (TTT plot) and Box plots of them. It can be observed that our data shows a non-monotone failure rate.

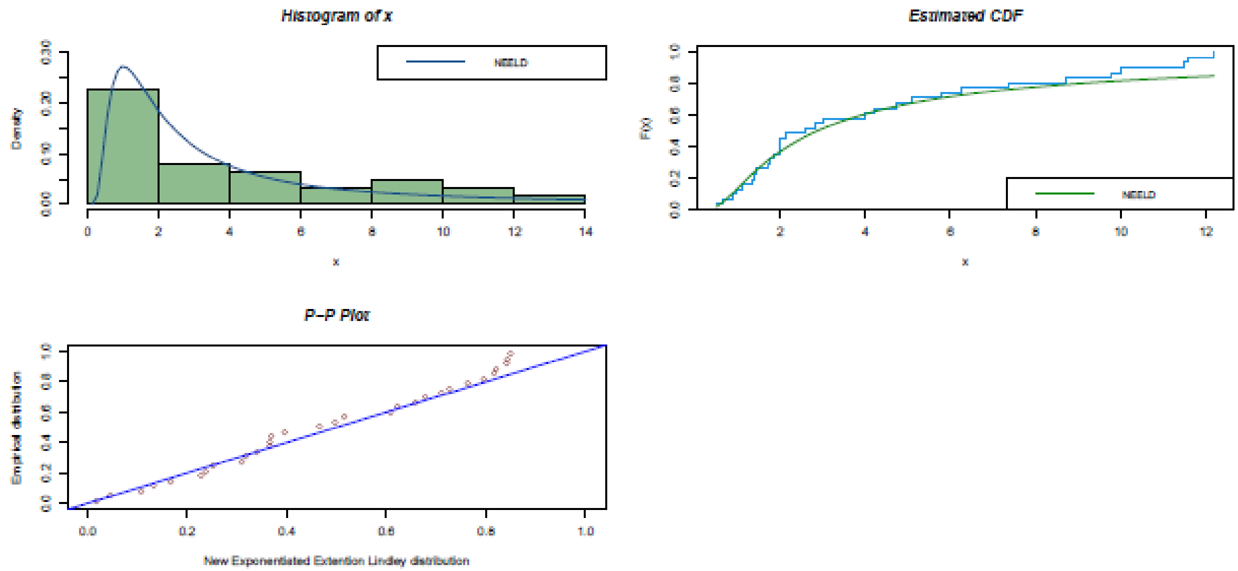

FIGURE 7. Estimated pdf, CDF and P-P plot for the prices of the 31 different children's wooden toys.

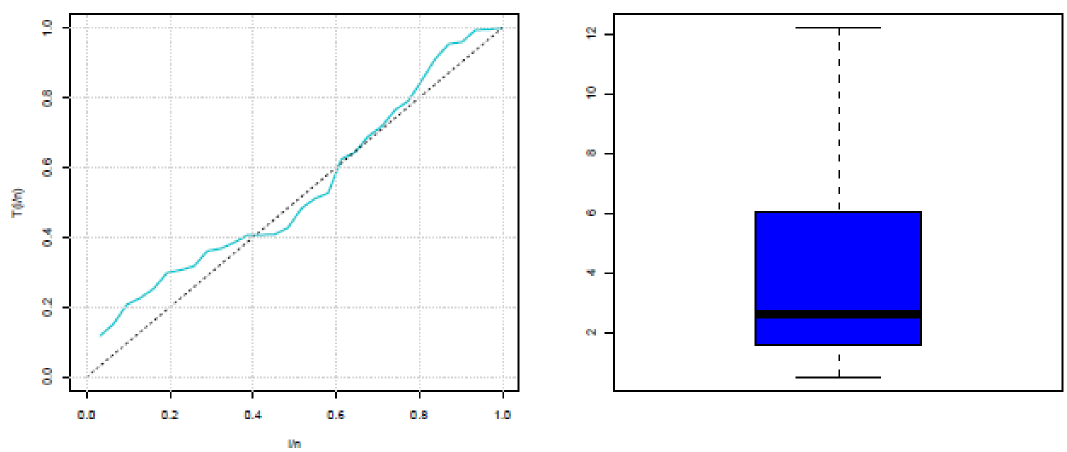

FIGURE 8. TTT and Box plots of the 31 different children's wooden toys.

Table 6 displays the MLEs of the parameters, whereas Table 7 presents a comparison of the NEEL model with the previously mentioned distributions. With all fitted models to these data, the lowest AIC, BIC, CAIC, and HQIC statistics are found in the newly proposed model.

Consequently, it can be chosen as the most effective model out of the bunch. These plots demonstrates that the offered distribution fits the data best than some other models.

TABLE 6. MLEs for the real data.

	Estimates			
Model	$\hat{\zeta}$	$\hat{\kappa}$	$\hat{\nu}$	$\hat{\theta}$
NEELD	**2.209**	–	–	–
IW	2.153	1.214	–	–
W	0.155	1.228	–	–
NH	1.881	0.096	–	–
GIW	1.543	1.214	1.270	–
EEB	1.731	0.036	0.264	–
EWP	20.475	9.112	0.375	0.006

TABLE 7. $AIC, BIC, CAIC$, and $HQIC$ statistics for the real data.

	Statistics			
Model	**AIC**	**BIC**	**CAIC**	**HQIC**
NEELD	**153.478**	**154.912**	**153.615**	**153.940**
IW	153.668	156.563	154.096	154.602
W	153.577	156.445	154.006	154.512
NH	154.590	157.458	155.018	155.524
GIW	155.668	159.970	156.556	157.070
EEB	154.178	158.480	155.066	155.580
EWP	155.198	160.933	156.736	157.067

Table 8 presents a comparison of the parameter estimation within Bayesian and MLE approaches for NEEL model with the previously mentioned loss functions for the two scenarios (informative and non informative priors).

TABLE 8. Real data MLE value (MSE) with different Bayesian estimators (BR)of α for NEELD distribution where $\alpha = 2,\ and\ \beta = 5,\ c = -3, r = 1$ for two senarios (1) with a non informative prior $a = b = 0$ (2) with a conjugate prior $a = 1,\ b = 5$.

n	MLE	senario (1) $a = b = 0$ (BR)				senario (2) $a = 1,\ b = 5$ (BR)			
	(MSE)	$\hat{\alpha}_{SQE}$	$\hat{\alpha}_{LE}$	$\hat{\alpha}_{GQE}$	$\hat{\alpha}_{EE}$	$\hat{\alpha}_{SQE}$	$\hat{\alpha}_{LE}$	$\hat{\alpha}_{GQE}$	$\hat{\alpha}_{EE}$
31	2.2097	2.1544	2.1445	2.1961	2.164	2.1022	2.0964	2.1261	2.1078
	(1.46338)	(0.02384)	(0.00991)	(0.61438)	(0.02690)	(0.010450)	(0.00579)	(0.30438)	(0.01162)

7. Conclusion

In the current work, the NEELD, a one-parameter model, is the focus. The mathematical expression of its probability density is feasible. As a result, we can use this to establish its various statistical characteristics. The technique of maximum likelihood estimation and Bayesian estimation under different loss functions are applied to estimate the parameter. In conjunction with a simulation study, an asymptotic confidence interval for the model parameter is found. Also we used this simulation to evaluate the stability of the Maximum Likelihood Estimate (MLE) for NEELD parameter, exhibiting stable behaviour with larger sample numbers. The evaluation of real data demonstrates the feasibility of our new model (NEELD). This application indicates that it has the opportunity to substantially affect various commonly employed statistical models in the sense of fit. The recently offered distribution might be thought of as a tried-and-true substitute for existing distributions like the IWD, the WD, etc. At last, we believe that the model we have created will be widely used for real data in a variety of fields, including social sciences, engineering, and medicine.

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Authors' Contributions

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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