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WIENER-HOSOYA ENERGY OF NON-COMMUTING GRAPH FOR DIHEDRAL GROUPS

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Abstract. Spectral graph theory studies the connection between graph theory and algebra through matrices representation. This research is devoted to the spectrum of the non-commuting graph for the dihedral group. The matrix representation is the Wiener-Hosoya matrix which is a square matrix and the eigenvalues corresponding to the matrix are determined. The result shows that the energy is always similar to twice its spectral radius.

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1. Introduction

A branch of mathematics referred to as chemical graph theory applies graph theory to the mathematical modeling of chemical compounds discussed in [1], which then discusses graph energies as presented in [2], which determines the electron energy of a chemical molecule by considering it as a graph.

As part of spectral graph theory, certain matrices provide information about graphs, such as adjacency matrices, Laplacian matrices, and signless Laplacian matrices. It is possible to describe a graph based on the spectrum of one of these matrices. Spectrum of these various matrices can provide useful information concerning the graph in general. This work aims to describe the behavior of a non-commuting graph, one of the finite groups that a graph can represent.

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Let G be a group and $Z(G)$ be a center of G. The non-commuting graph of G, denoted by Γ_G, has vertex set $G \backslash Z(G)$ and two distinct vertices v_p, v_q in Γ_G are connected by an edge whenever $v_p v_q \neq v_q v_p$ [3]. This work considers the non-abelian dihedral group of order $2n$ as the vertex set for non-commuting graph, where $n \geq 3$, denoted by $D_{2n} = \langle a, b : a^n = b^2 = e, bab = a^{-1} \rangle$ [4]. Throughout this paper, the discussion focuses on Γ_G for D_{2n}, denoted by $\Gamma_{D_{2n}}$.

On the other hand, the Hosoya index is an important molecular descriptor in chemical graph theory. Several results of this tree topic can be found in [5, 6]. The Hosoya matrix was first introduced by Randic in 1994 [7], then Ibrahim et al. [8] defined the formal definition of the Wiener-Hosoya matrix of a graph.

Let d_p be the degree of a vertex v_p, and d_{pq} be the distance between vertex v_p and v_q. The following results are the degree of every vertex and the distance between two vertices in $\Gamma_{D_{2n}}$.

Theorem 1.1. [9] *Let $\Gamma_{D_{2n}}$ be the non-commuting graph on D_{2n}. Then*

(1) $d_{a^i} = n$,

(2) $d_{a^i b} = \begin{cases} 2(n-1), & \text{if } n \text{ is odd} \\ 2(n-2), & \text{if } n \text{ is even.} \end{cases}$

Theorem 1.2. [10] *Let $\Gamma_{D_{2n}}$ be the non-commuting graph on D_{2n}. For two distinct vertices $v_p, v_q \in \Gamma_{D_{2n}}$, then the distance between v_p and v_q*

(1) *for the odd n,* $d_{pq} = \begin{cases} 2, & \text{if } v_p, v_q \in G_1 \\ 1, & \text{otherwise,} \end{cases}$,

(2) *for the even n,* $d_{pq} = \begin{cases} 2, & \text{if } v_p, v_q \in G_1 \\ 2, & v_p \in G_2, v_q \in \left\{ a^{\frac{n}{2}+i} b \right\} \\ 1, & \text{otherwise.} \end{cases}$

Furthermore, the transmission of a vertex v_p, denoted by τ_p, is defined as the sum of distances between v_p and any other vertices in a graph [8]. The Wiener-Hosoya (WH) matrix of order $n \times n$ corresponding to $\Gamma_{D_{2n}}$ is given by $WH(\Gamma_{D_{2n}}) = [wh_{pq}]$ whose (p, q)-th entry is

$$wh_{pq} = \begin{cases} \frac{\tau_p}{2d_p} + \frac{\tau_q}{2d_q}, & \text{if } v_p \neq v_q \text{ and they are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

The energy of $\Gamma_{D_{2n}}$, $E(\Gamma_{D_{2n}})$, is calculated by adding all the absolute values of its eigenvalues [2]. The spectral radius of $\Gamma_{D_{2n}}$ is the maximum of absolute eigenvalues [11]

A detailed discussion of dihedral groups as a vertex set that is spectral problems for commuting and non-commuting graphs involving several degree-based matrices can be found in [12–16]. Meanwhile, Romdhini et al. [17] discussed the signless Laplacian energy. In addition, readers can also see the graph matrix extension in [18]. Therefore, this work discusses the energy of non-commuting graphs corresponding with the Wiener-Hosoya matrix.

Moreover, to formulate the characteristic polynomial of $\Gamma_{D_{2n}}$, we need the determinant properties. If a square matrix has a large order, we will have difficulty finding its eigenvalues. To simplify it, we will use two results from the literature as follows.

Lemma 1.3. [19] *If a, b, c, and d are real numbers, then the determinant of*

$$\begin{vmatrix} (\lambda+a)I_{n_1} - aJ_{n_1} & -cJ_{n_1\times n_2} \\ -dJ_{n_2\times n_1} & (\lambda+b)I_{n_2} - bJ_{n_2} \end{vmatrix}$$

can be simplified as

$$(\lambda+a)^{n_1-1}(\lambda+b)^{n_2-1}\left((\lambda-(n_1-1)a)(\lambda-(n_2-1)b) - n_1n_2cd\right),$$

where $1 \le n_1, n_2 \le n$ and $n_1 + n_2 = n$.

Theorem 1.4. [20] *Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a partitioned matrix with A is a non-singular, then*

$$|M| = \begin{vmatrix} A & B \\ O & D - CA^{-1}B \end{vmatrix} = |A|\left|D - CA^{-1}B\right|.$$

If the form of the matrix is not similar to the above lemma, row and column operations need to be performed to get the characteristic polynomial of $\Gamma_{D_{2n}}$. Let R_i and C_i be the $i-$th row and column of the matrix. Suppose now R_i' and C_i' are the new $i-$th row and column of the matrix obtained from R_i and C_i, respectively.

2. Main Results

To construct the Wiener-Hosoya matrix of $\Gamma_{D_{2n}}$, by the definition, we need the transmission of every vertex in $\Gamma_{D_{2n}}$. The result is given below:

Theorem 2.1. *Let $\Gamma_{D_{2n}}$ be the non-commuting graph on D_{2n}, then the transmission of vertex v_p*

(1) *for the odd n, $\tau_p = \begin{cases} 3n-4, & \text{if } v_p \in G_1 \\ 2(n-1), & \text{if } v_p \in G_2, \end{cases}$, and*

(2) *for the even n, $\tau_p = \begin{cases} 3(n-2), & \text{if } v_p \in G_1 \\ 2(n-1), & \text{if } v_p \in G_2, \end{cases}$*

Proof. (1) For the odd n case, let $G_1 = \{a, a^2, \ldots a^{n-1}\}$ and $G_2 = \{b, ab, a^2b, \ldots a^{n-1}b\}$. There are $n-1$ vertices in G_1 and n vertices in G_2. We know from Theorem 1.2 that $d_{pq} = 2$ if $v_p, v_q \in G_1$ and $d_{pq} = 1$, otherwise. Then the transmission of $v_p \in G_1$ is

$$\tau_p = 2(n-2) + n = 3n - 4.$$

Meanwhile, the transmission of $v_p \in G_2$ is

$$\tau_p = (n-1) + (n-1) = 2(n-1).$$

(2) While for the even n case, let $G_1 = \{a, a^2, \ldots, a^{\frac{n}{2}-1}, a^{\frac{n}{2}+1}, \ldots, a^{n-1}\}$ and $G_2 = \{b, ab, a^2b, \ldots, a^{n-1}b\}$. There are $n-2$ vertices in G_1 and n vertices in G_2. We know from Theorem 1.2 that $d_{pq} = 2$ if $v_p, v_q \in G_1$ or $v_p \in G_2, v_q \in \left\{a^{\frac{n}{2}+i}b\right\}$ and $d_{pq} = 1$, otherwise. Then the transmission of $v_p \in G_1$ is

$$\tau_p = 2(n-3) + n = 3(n-2).$$

Meanwhile, the transmission of $v_p \in G_2$ is

$$\tau_p = (n-2) + (n-2) + 2 = 2(n-1).$$

□

Theorem 2.2. *Let $\Gamma_{D_{2n}}$ be the non-commuting graph on D_{2n}, then the characteristic polynomial of $WH(\Gamma_{D_{2n}})$ is*

(1) *for odd n*

$$P_{WH(\Gamma_{D_{2n}})}(\lambda) = (\lambda)^{n-2}(\lambda+1)^{n-1}\left(\lambda^2 - (n-1)\lambda - \frac{4(n-1)^3}{n}\right),$$

(2) *for even n,*

$$P_{WH(\Gamma_{D_{2n}})}(\lambda) = (\lambda)^{\frac{3(n-2)}{2}}\left(\lambda + 2\left(\frac{n-1}{n-2}\right)\right)^{\frac{n}{2}-1}\left(\lambda^2 - (n-1)\lambda - \frac{(4n^2-13n+12)^2}{4n(n-2)}\right).$$

Proof. (1) Let n is odd with $Z(D_{2n}) = \{e\}$ which implies that there are $2n-1$ vertices for $\Gamma_{D_{2n}}$. It can be seen that $Z(D_{2n}) = \{e\}$ for odd n implies that $\Gamma_{D_{2n}}$ has $2n-1$ vertices. We label the set G_1 as $\{a, a^2, \ldots a^{n-1}\}$ and G_2 as $\{b, ab, a^2b, \ldots a^{n-1}b\}$. Considering the degree of every vertex and in Theorem 1.1 and the transmission of every vertex in Theorem 2.1, also using the definition of WH-matrix, then the $(2n-1)\times(2n-1)$ Weiner-Hosoya matrix for $\Gamma_{D_{2n}}$ is

$$WH(\Gamma_{D_{2n}}) = \begin{array}{c} \\ a \\ a^2 \\ \vdots \\ a^{n-1} \\ b \\ ab \\ \vdots \\ a^{n-1}b \end{array}\begin{array}{c} \begin{array}{cccccccc} a & a^2 & \ldots & a^{n-1} & b & ab & \ldots & a^{n-1}b \end{array} \\ \begin{pmatrix} 0 & 0 & \ldots & 0 & \frac{2(n-1)}{n} & \frac{2(n-1)}{n} & \ldots & \frac{2(n-1)}{n} \\ 0 & 0 & \ldots & 0 & \frac{2(n-1)}{n} & \frac{2(n-1)}{n} & \ldots & \frac{2(n-1)}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & \frac{2(n-1)}{n} & \frac{2(n-1)}{n} & \ldots & \frac{2(n-1)}{n} \\ \frac{2(n-1)}{n} & \frac{2(n-1)}{n} & \ldots & \frac{2(n-1)}{n} & 0 & 1 & \ldots & 1 \\ \frac{2(n-1)}{n} & \frac{2(n-1)}{n} & \ldots & \frac{2(n-1)}{n} & 1 & 0 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{2(n-1)}{n} & \frac{2(n-1)}{n} & \ldots & \frac{2(n-1)}{n} & 1 & 1 & \ldots & 0 \end{pmatrix} \end{array}.$$

Here the WH-matrix of $\Gamma_{D_{2n}}$ can be obtained as the block matrices as follows:

$$WH(\Gamma_{D_{2n}}) = \begin{pmatrix} 0_{n-1} & \frac{2(n-1)}{n}J_{(n-1)\times n} \\ \frac{2(n-1)}{n}J_{n\times(n-1)} & (J-I)_n \end{pmatrix},$$

and the determinant below is the characteristic polynomial for $WH(\Gamma_{D_{2n}})$,

$$P_{WH(\Gamma_{D_{2n}})}(\lambda) = \begin{vmatrix} \lambda I_{n-1} & -\frac{2(n-1)}{n}J_{(n-1)\times n} \\ -\frac{2(n-1)}{n}J_{n\times(n-1)} & (\lambda+1)I_n - J_n) \end{vmatrix}.$$

Repeated application of Lemma 1.3, with $a=0$, $b=1$, $c=d=\frac{2(n-1)}{n}$, $n_1=n-1$ and $n_2=n$, the desired result is obtained.

(2) Suppose now n is even. We write the set G_1 as $\{a, a^2, \ldots, a^{\frac{n}{2}-1}, a^{\frac{n}{2}+1}, \ldots, a^{n-1}\}$ and G_2 as $\{b, ab, a^2b, \ldots, a^{n-1}b\}$. Again considering the degree of every vertex and in Theorem 1.1 and the transmission of every vertex in Theorem 2.1, also using the definition of WH-matrix, which implies $WH(\Gamma_{D_{2n}})$ being the matrix of size $(2n-2)\times(2n-2)$ as follows,

$$\begin{array}{c} \\ a \\ \vdots \\ a^{n-1} \\ b \\ \vdots \\ a^{\frac{n}{2}-1}b \\ a^{\frac{n}{2}}b \\ \vdots \\ a^{n-1}b \end{array}
\begin{array}{c} \begin{array}{ccccccccc} a & \cdots & a^{n-1} & b & \cdots & a^{\frac{n}{2}-1}b & a^{\frac{n}{2}}b & \cdots & a^{n-1}b \end{array} \\
\begin{pmatrix}
0 & \cdots & 0 & \frac{4n^2-13n+12}{2n(n-2)} & \cdots & \frac{4n^2-13n+12}{2n(n-2)} & \frac{4n^2-13n+12}{2n(n-2)} & \cdots & \frac{4n^2-13n+12}{2n(n-2)} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \frac{4n^2-13n+12}{2n(n-2)} & \cdots & \frac{4n^2-13n+12}{2n(n-2)} & \frac{4n^2-13n+12}{2n(n-2)} & \cdots & \frac{4n^2-13n+12}{2n(n-2)} \\
\frac{4n^2-13n+12}{2n(n-2)} & \cdots & \frac{4n^2-13n+12}{2n(n-2)} & 0 & \cdots & \frac{n-1}{n-2} & 0 & \cdots & \frac{n-1}{n-2} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{4n^2-13n+12}{2n(n-2)} & \cdots & \frac{4n^2-13n+12}{2n(n-2)} & \frac{n-1}{n-2} & \cdots & 0 & \frac{n-1}{n-2} & \cdots & 0 \\
\frac{4n^2-13n+12}{2n(n-2)} & \cdots & \frac{4n^2-13n+12}{2n(n-2)} & 0 & \cdots & \frac{n-1}{n-2} & 0 & \cdots & \frac{n-1}{n-2} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{4n^2-13n+12}{2n(n-2)} & \cdots & \frac{4n^2-13n+12}{2n(n-2)} & \frac{n-1}{n-2} & \cdots & 0 & \frac{n-1}{n-2} & \cdots & 0
\end{pmatrix} \end{array}.$$

Now we provide nine block matrices of $WH(\Gamma_{D_{2n}})$ as follows:

$$WH(\Gamma_{D_{2n}}) = \begin{pmatrix} 0_{n-2} & \frac{4n^2-13n+12}{2n(n-2)}J_{(n-2)\times\frac{n}{2}} & \frac{4n^2-13n+12}{2n(n-2)}J_{(n-2)\times\frac{n}{2}} \\ \frac{4n^2-13n+12}{2n(n-2)}J_{\frac{n}{2}\times(n-2)} & \frac{n-1}{n-2}(J_{\frac{n}{2}} - I_{\frac{n}{2}}) & \frac{n-1}{n-2}(J-I)_{\frac{n}{2}} \\ \frac{4n^2-13n+12}{2n(n-2)}J_{\frac{n}{2}\times(n-2)} & \frac{n-1}{n-2}(J-I)_{\frac{n}{2}} & \frac{n-1}{n-2}(J-I)_{\frac{n}{2}} \end{pmatrix}.$$

The characteristic polynomial of $WH(\Gamma_{D_{2n}})$, $P_{WH(\Gamma_{D_{2n}})}(\lambda)$ is as follows

$$\begin{vmatrix} \lambda I_{n-2} & -\left(\frac{4n^2-13n+12}{2n(n-2)}\right)J_{(n-2)\times\frac{n}{2}} & -\left(\frac{4n^2-13n+12}{2n(n-2)}\right)J_{(n-2)\times\frac{n}{2}} \\ -\left(\frac{4n^2-13n+12}{2n(n-2)}\right)J_{\frac{n}{2}\times(n-2)} & (\lambda+\frac{n-1}{n-2})I_{\frac{n}{2}} - \left(\frac{n-1}{n-2}\right)J_{\frac{n}{2}} & -\left(\frac{n-1}{n-2}\right)(J-I)_{\frac{n}{2}} \\ -\left(\frac{4n^2-13n+12}{2n(n-2)}\right)J_{\frac{n}{2}\times(n-2)} & -\left(\frac{n-1}{n-2}\right)(J-I)_{\frac{n}{2}} & (\lambda+\frac{n-1}{n-2})I_{\frac{n}{2}} - \left(\frac{n-1}{n-2}\right)J_{\frac{n}{2}} \end{vmatrix}. \tag{2.1}$$

According to the row operation $R'_{n-2+\frac{n}{2}+i} = R_{n-2+\frac{n}{2}+i} - R_{n-2+i}$, followed by $C'_{n-2+i} = C_{n-2+i} + C_{n-2+\frac{n}{2}+i}$ on Equation 2.1 for $1 \leq i \leq \frac{n}{2}$, then $P_{WH(\Gamma_{D_{2n}})}(\lambda)$ is

$$\left| \begin{array}{ccc} \lambda I_{n-2} & -2\left(\frac{4n^2-13n+12}{2n(n-2)}\right) J_{(n-2)\times\frac{n}{2}} & -\left(\frac{4n^2-13n+12}{2n(n-2)}\right) J_{(n-2)\times\frac{n}{2}} \\ -\left(\frac{4n^2-13n+12}{2n(n-2)}\right) J_{\frac{n}{2}\times(n-2)} & \left(\lambda + 2\left(\frac{n-1}{n-2}\right)\right) I_{\frac{n}{2}} - 2\left(\frac{n-1}{n-2}\right) J_{\frac{n}{2}} & -\left(\frac{n-1}{n-2}\right)(J-I)_{\frac{n}{2}} \\ 0_{\frac{n}{2}\times(n-2)} & 0_{\frac{n}{2}} & \lambda I_{\frac{n}{2}} \end{array} \right|. \tag{2.2}$$

Consequently, Equation 2.2 can be written as

$$P_{WH(\Gamma_{D_{2n}})}(\lambda) = \left| \begin{array}{cc} A_{n-2+\frac{n}{2}} & B_{(n-2+\frac{n}{2})\times\frac{n}{2}} \\ C_{\frac{n}{2}\times(n-2+\frac{n}{2})} & D_{\frac{n}{2}} \end{array} \right|, \tag{2.3}$$

where

$A = \left| \begin{array}{cc} \lambda I_{n-2} & -2\left(\frac{4n^2-13n+12}{2n(n-2)}\right) J_{(n-2)\times\frac{n}{2}} \\ -\left(\frac{4n^2-13n+12}{2n(n-2)}\right) J_{\frac{n}{2}\times(n-2)} & \left(\lambda + 2\left(\frac{n-1}{n-2}\right)\right) I_{\frac{n}{2}} - 2\left(\frac{n-1}{n-2}\right) J_{\frac{n}{2}} \end{array} \right|,$

$B = \left| \begin{array}{c} -\left(\frac{4n^2-13n+12}{2n(n-2)}\right) J_{(n-2)\times\frac{n}{2}} \\ -\left(\frac{n-1}{n-2}\right)(J-I)_{\frac{n}{2}} \end{array} \right|$, $C = \left|0_{\frac{n}{2}\times(n-2+\frac{n}{2})}\right|$, and $D = \left|\lambda I_{\frac{n}{2}}\right|$.

According to Theorem 1.4, since $C = 0$, we then obtain Equation 2.3 as $P_{WH(\Gamma_{D_{2n}})}(\lambda) = |A|\,|D|$. By applying Lemma 1.3 to $|A|$, with $a = 0$, $b = 4\sqrt{2}(n-2)$, $c = 2\left(\frac{4n^2-13n+12}{2n(n-2)}\right)$, $d = \left(\frac{4n^2-13n+12}{2n(n-2)}\right)$, $n_1 = n-2$ and $n_2 = \frac{n}{2}$, and considering D is a diagonal matrix, we then get

$$P_{WH(\Gamma_{D_{2n}})}(\lambda) = (\lambda)^{\frac{3(n-2)}{2}} \left(\lambda + 2\left(\frac{n-1}{n-2}\right)\right)^{\frac{n}{2}-1} \left(\lambda^2 - (n-1)\lambda - \frac{(4n^2-13n+12)^2}{4n(n-2)}\right).$$

□

The following Theorem 2.3 and 2.4 present the Weiner-Hosoya spectral radius, and energy of $\Gamma_{D_{2n}}$.

Theorem 2.3. *Let $\Gamma_{D_{2n}}$ be the non-commuting graph on D_{2n}, then the WH-spectral radius for $\Gamma_{D_{2n}}$ is*

(1) $\rho_{WH}(\Gamma_{D_{2n}}) = \frac{n-1}{2} + \frac{1}{2}\sqrt{\frac{(n-1)^2(17n-16)}{n}}$, *for odd n, and*
(2) $\rho_{WH}(\Gamma_{D_{2n}}) = \frac{n-1}{2} + \frac{1}{2}\sqrt{(n-1)^2 + \frac{(4n^2-13n+2)^2}{n(n-2)}}$, *for even n.*

Proof. (1) The formula of $P_{WH(\Gamma_{D_{2n}})}(\lambda)$ of Theorem 2.2 (1) for odd n result the eigenvalues for $\Gamma_{D_{2n}}$. We have $\lambda_1 = 0$ of multiplicity $n-2$. Then we get $\lambda_2 = -1$ of multiplicity $n-1$, and $\lambda_{3,4} = \frac{n-1}{2} \pm \frac{1}{2}\sqrt{\frac{(n-1)^2(17n-16)}{n}}$ as the roots of the quadratic formula. Hence, the Wiener-Hosoya spectrum for $\Gamma_{D_{2n}}$ is as follows

$$Spec(\Gamma_{D_{2n}}) = \left\{ \left(\frac{n-1}{2} + \frac{1}{2}\sqrt{\frac{(n-1)^2(17n-16)}{n}}\right)^1, (0)^{n-2}, \right.$$
$$\left. \left(\frac{n-1}{2} - \frac{1}{2}\sqrt{\frac{(n-1)^2(17n-16)}{n}}\right)^1, (-1)^{n-1} \right\}.$$

Now for $i = 1, 2, 3, 4$, the maximum of $|\lambda_i|$ is the Wiener-hosoya spectral radius of $\Gamma_{D_{2n}}$,

$$\rho_{WH}(\Gamma_{D_{2n}}) = \frac{n-1}{2} + \frac{1}{2}\sqrt{\frac{(n-1)^2(17n-16)}{n}}.$$

(2) As can be seen in Theorem 2.2 (2), the roots of $P_{WH(\Gamma_{D_{2n}})}(\lambda) = 0$ are the eigenvalues of $\Gamma_{D_{2n}}$. Firstly, we have $\lambda_1 = 0$ of the multiplicity $\frac{3(n-2)}{2}$. Next, the roots are $\lambda_2 = -2\left(\frac{n-1}{n-2}\right)$ of multiplicity $\frac{n}{2} - 1$, and $\lambda_{3,4} = \frac{n-1}{2} \pm \frac{1}{2}\sqrt{(n-1)^2 + \frac{(4n^2-13n+2)^2}{n(n-2)}}$. Therefore,

$$Spec(\Gamma_{D_{2n}}) = \left\{ \left(\frac{n-1}{2} + \frac{1}{2}\sqrt{(n-1)^2 + \frac{(4n^2-13n+2)^2}{n(n-2)}}\right)^1, (0)^{\frac{3n-6}{2}}, \right.$$
$$\left. \left(\frac{n-1}{2} - \frac{1}{2}\sqrt{(n-1)^2 + \frac{(4n^2-13n+2)^2}{n(n-2)}}\right)^1, \left(-2\left(\frac{n-1}{n-2}\right)\right)^{\frac{n}{2}-1} \right\}.$$

Determining the maximum absolute eigenvalues, as a consequence, the Wiener-Hosoya spectral radius of $\Gamma_{D_{2n}}$ is provided as

$$\rho_{WH}(\Gamma_{D_{2n}}) = \frac{n-1}{2} + \frac{1}{2}\sqrt{(n-1)^2 + \frac{(4n^2-13n+2)^2}{n(n-2)}}.$$

□

Theorem 2.4. *Let $\Gamma_{D_{2n}}$ be the non-commuting graph on D_{2n},then the Wiener-Hosoya energy for $\Gamma_{D_{2n}}$ is*

(1) $E_{WH}(\Gamma_{D_{2n}}) = (n-1) + \sqrt{\frac{(n-1)^2(17n-16)}{n}}$, *for odd n, and*
(2) $E_{WH}(\Gamma_{D_{2n}}) = (n-1) + \sqrt{(n-1)^2 + \frac{(4n^2-13n+2)^2}{n(n-2)}}$, *for even n.*

Proof. (1) From the Wiener-Hosoya spectrum in Theorem 2.3 (1) for odd n, we can calculate the Wiener-Hosoya energy for $\Gamma_{D_{2n}}$ is. By the definition of energy, we obtain

$$\begin{aligned} E_{WH}(\Gamma_{D_{2n}}) =& (n-2)|0| + (n-1)\,|-1| + \left|\frac{n-1}{2} \pm \frac{1}{2}\sqrt{\frac{(n-1)^2(17n-16)}{n}}\right| \\ =& (n-1) + \sqrt{\frac{(n-1)^2(17n-16)}{n}}. \end{aligned}$$

(2) According to $Spec(\Gamma_{D_{2n}})$ in Theorem 2.3 (2) for even n, as the same manner of odd n case, we derive the Wiener-Hosoya energy of $\Gamma_{D_{2n}}$ as follows

$$\begin{aligned} E_{WH}(\Gamma_{D_{2n}}) =& \left(\frac{3(n-2)}{2}\right)|0| + \left(\frac{n}{2}-1\right)\left|-2\left(\frac{n-1}{n-2}\right)\right| + \\ & \left|\frac{n-1}{2} \pm \frac{1}{2}\sqrt{(n-1)^2 + \frac{(4n^2-13n+2)^2}{n(n-2)}}\right| \\ =& (n-1) + \sqrt{(n-1)^2 + \frac{(4n^2-13n+2)^2}{n(n-2)}}. \end{aligned}$$

□

Based on Theorem 2.3 and 2.4, we conclude the relationship between WH-energy and WH-spectral radius with the following conclusion:

Corollary 2.5. *For the non-commuting graph* $\Gamma_{D_{2n}}$, $E_{WH}(\Gamma_{D_{2n}}) = 2 \cdot \rho_{WH}(\Gamma_{D_{2n}})$.

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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