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DYNAMICAL ANALYSIS OF A TWO PREY-ONE PREDATOR MODEL

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Abstract. Predator-prey interactions are among the most significant features of ecology. In this paper, we intend to investigate an ecological system with one predator and two prey. The current study aims to deepen our understanding of the combined impacts of fear and toxic substance in ecological system, through consider their impact on a delay predator-prey model. The problem specifically dealt a nonlinear, three-dimensional, ecological system that is affected by toxicity to all species. The situation of prey growth rates affected by predator fear was considered. Also taken into account is the predator's gestation delay. Numerical simulations were used to show how well our theoretical work could support theoretical results and to make clear how the dynamics of the suggested ecological model would change when certain factors, particularly toxic rates, fear levels and gestation time delay, were changed. The findings showed that the systems dynamical behavior displays a range of dynamics without degenerating into chaos and that the presence of toxicity in addition to fear and time delay significantly affects the stability of the dynamics of the system.

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1. Introduction

Prey-predator relationships are among the most fundamental relationships between different species in ecology. Malthus constructed a mathematical model of the dynamics of interactions between a single species early in the nineteenth century. Then the famous logistic growth model was created by adding an intraspecific competition element to that mathematical model to improve it. This improved the Lotka-Volterra model after adding the logistic growth component for the prey. Over the past few decades, numerous scholars have examined many of theoretical works on prey-predator models and their applications to the study of theoretical ecology and evolutionary biology, [1–4].

Globally, there is growing concern about the effects of environmental pollutants, both natural and man-made, on ecosystem health. The main source of pollution worldwide is industrial pollution, which

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can result from solid waste, air pollution, water leaks, and contaminants. The direct impact of toxic substances on the food web system have been a major environmental concern because they have an adverse effect on human life. Numerous studies have examined the effects of toxic substances on the environment. One of the earliest was the population model with effect of environment toxics created by Hallam et al. [5]. In a closed, polluted environment, models based on hazardous chemicals in a single species were published by Friedman and Shukla [6]; however, they also demonstrated how toxic substances affect population through the observation of the toxicant dependent environmental carrying capacity. Chattopadhyay [7] investigated the impact of toxic substances produced by various species. Pal and Samanta [8] investigated the impacts of present of toxins in a Lotka-Volterra competitive system with two species. In their model the impacts of toxic substances on the exposed species (population) has been investigated by considering the rate of environmental toxin consumption into consideration.

According to recent studies, prey species are more impacted by predator species' indirect consequences than by the direct death [9–13]. Consequently, it makes sense to incorporate the fear effect into the predator-focused models that takes rival species' cohabitation into account. In studies using song sparrows, Zanette et al. [14] discovered that the birds' apprehension of the predator resulted in a 40% reduction in the quantity of of spring born. In the light of that, Wang et al. [15] developed a predator-prey model by accounting for the cost of anxiety on prey reproduction. They was found that when predation follows the Holling type I response function, fear costs have no effect on dynamic behavior. When considering the Holling type II response function, however, it can stabilize the system by removing periodic orbits. Since then, several studies employing predator-prey models have come to light, incorporating fear into the process of prey reproduction.

In biological applications, delay differential equations are typically required because of the presence of particular stage structures. In actuality, temporal delays occur in a wide range of biological processes, including food digestion, energy conversion, maturation, inducible defense of prey groups, and more. In mathematical models, time lag is often incorporated to convey the dynamic nature of the models through historical data. This makes it possible to describe the population's development, hunting, and gestation delays inside the ecology of prey and predator mathematically. In the presence of a time delay, the model can become unstable and show more intricate dynamic behaviors, such as Hopf bifurcation and saddle-node behavior. Specifically, the features of periodic solutions resulting from the Hopf bifurcation hold great significance, [16–19] Delayed gestation refers to the interval of time that occurs between consuming the prey and birthing a new predator. So, in ecological models with gestation time delays, the new birth rate of a predator is based on the amount of prey it has previously eaten, [20,21].

The predator-prey relationship is one of the forms of interactions between different species that is of great importance in determining the dynamics of complex ecosystems. Moreover, the dynamics of ecological epidemiological models is one of the main topics in mathematical biology. In last five years, researchers looked at some mathematical models involving two prey and one predator. Kundu and Maitra in [22] studied the effect of prey cooperation and the environmental noises on the dynamic of such type ecosystem. The authors in [23] investigated the global stability of a discrete such model with optimal harvest strategy and Holling Type-III function response. Such ecological model Also studied in [24] by Sahoo and Samanta. They analyzed the effect of switching in predation and fear of predator on the dynamic of the model. In [25], Mondal et al. showed the model has complex dynamic when it include the harvesting and fear impact in preys. Based on the preceding literature analysis and the motivations discussed therein, the purpose of this paper is to investigate the combined effects of fear effect and environmental toxin on an ecosystem containing two prey and one predator with gestation time delay.

2. Mathematical Model

Numerous of novel models of predator-prey with their environment's impact have been examined in recent years (see for example [25–27]). In this section,, to construct the proposed model, we start by considering the key assumptions: the model's populations include three species, two different preys and one predator. The density of first and second prey are given by x_1 and x_2, and the density of predator is given by y .The preys are growth exponentially with growth rates r_1 and r_2 and affected by factors of fear of the predator g_1 and g_2 at respectively . Also, the preys influenced by external toxic substance with rates a_1 and a_2, and have interspecific competition β_1 and β_2. The predator eating the two preys with Holling type I functions response where d_1 and d_2 represent the predations rates, while c_1 and c_2 are represent the conversion rates of preys biomass to predator's biomass. Predator has delayed growth rate due to the time required for the digestive process and gestation. Finally, the predator also effected by external toxic substance with toxicant rate m.
Based on the above assumptions, our model is as follows:

$$\begin{aligned}\frac{dx_1}{dt} &=[\frac{r_1}{1+g_1y} - a_1x_1 - \beta_1x_2 - d_1y]x_1,\\ \frac{dx_2}{dt} &=[\frac{r_2}{1+g_2y} - a_2x_2 - \beta_2x_1 - d_2y]x_2,\\ \frac{dy}{dt} &=c_1d_1x_1(t-T_d)y(t-T_d) + c_2d_2x_2(t-T_d)y(t-T_d) - my^2.\end{aligned} \tag{1}$$

with initial conditions:

$$x_1(0) > 0, x_2(0) > 0, y(0) > 0. \tag{2}$$

In which the constants r_1, r_2, g_1, g_2, a_1, a_2, β_1, β_2, c_1, c_2, d_1 ,d_2, m are positive, while T_d nonnegative.

3. Positivity and Boundedness

A model is guaranteed to function correctly physiologically if it is both positive and bounded. These mathematical aspects of the system (1) without delay are demonstrated by the next two theorems.

Theorem 1. *All solutions of System* (1)*with* $T_d = 0$*, that begin in* R^3_+ *are positive for* $t > 0$*.*

Proof. Using system (1), first equation, we obtain

$$x_1(t) = x_1(0)\exp[\int_0^t \{\frac{r_1}{1+g_1y(\theta)} - a_1x_1(\theta) - \beta_1x_2(\theta) - d_1y(\theta)\}d\theta].$$

Then $x_1(t) > 0$ for $x_1(0) > 0$.
Similarly, based on system (1), second and third equations, can have

$$x_2(t) = x_2(0)\exp[\int_0^t \{\frac{r_2}{1+g_2y(\theta)} - a_2x_2(\theta) - \beta_2x_1(\theta) - d_2y(\theta)\}d\theta].$$

$$y(t) = y(0)\exp[\int_0^t \{c_1d_1x_1(\theta) + c_2d_2x_2(\theta) - my(\theta)\}d\theta].$$

Then $x_2(t) > 0$ and $y(t) > 0$ for $x_2(0) > 0$ and $y(0) > 0$. Hence, the theorem is demonstrated. □

Theorem 2. *Starting in* R^3_+*, the solutions of system* (1) *with* $T_d = 0$*, are uniformly bounded for* $t > 0$*.*

Proof. From system (1), first equation, may gain

$$\begin{aligned}\frac{dx_1}{dt} =& x_1[\frac{r_1}{1+g_1y} - a_1x_1 - \beta_1x_2 - d_1y]\\ \le& x_1[\frac{r_1}{1+g_1y} - a_1x_1]\\ \le& r_1x_1 - a_1x_1^2.\end{aligned}$$

Then for, $t \to \infty$, can have $\lim_{t\to\infty}\sup x_1(t) \le \frac{r_1^2}{4a_1}$.
In similar manner, one can get

$$\lim_{t\to\infty}\sup x_2(t) \le \frac{r_2^2}{4a_2},$$

$$\lim_{t\to\infty}\sup y(t) \le \frac{1}{4m}\Lambda^2,$$

where, $\Lambda = (\frac{c_1d_1r_1^2}{4a_1} + \frac{c_2d_2r_2^2}{4a_2})$.
Now, let W define by $x_1 + x_2 + y$, then one can has

$$\frac{dW}{dt} = \frac{dx_1}{dt} + \frac{dx_2}{dt} + \frac{dy}{dt},$$

and

$$\frac{dW}{dt} \le r_1x_1 + r_2x_2 + c_1d_1x_1y + c_2d_2x_2y.$$

Further, with $= \frac{r_1^2}{4a_1}(r_1+1) + \frac{r_2^2}{4a_2}(r_2+1) + \frac{1}{4m}\Lambda^2[\Lambda+1]$, direct calculation may yields

$$\frac{dw}{dt} + W \leq .$$

Appling Gronwall inequality [29] may obtain

$$0 < W(t) \leq (1-e^{-t}) + W(x_1(0), x_2(0), y(0))e^{-t}.$$

Hence, $0 < W(t) \leq$, fot $t \to \infty$.

Consequently, every solutions for system (1) enters the area:

$\Re = \{(x_1, x_2, y) : 0 < x_1(t) \leq \frac{r_1^2}{4a_1}; 0 < x_2(t) \leq \frac{r_2^2}{4a_2}; 0 < y(t) \leq \frac{1}{4m}\Lambda^2; 0 < W(t) \leq \}$. □

4. Existence of the System Equilibrium Points

We define the prerequisites for the system's equilibrium points existence in this section. The system has seven potential nonnegative equilibria, which we can find by equating system's (1) right side to zero, namely $\Xi_0(0,0,0)$, $\Xi_1(x_1^*,0,0)$, $\Xi_2(0,x_2^*,0)$, $\Xi_3(x_1^*,x_2^*,0)$, $\Xi_4(x_1^*,0,y^*)$, $\Xi_5(0,x_2^*,y^*)$and $\Xi_6(x_1^*,x_2^*,y^*)$. One can note Ξ_0 exist trivially. We demonstrate that additional equilibria exist in the following ways:

(i): Ξ_1's existence.

System (1) with $x_2 = y = 0$, yields: $x_1(r_1 - a_1x_1) = 0$, from which we have $x_1^* = \frac{r_1}{a_1}$, thus $\Xi_1(x_1^*,0,0)$ exists and take the form $x_1^* = (\frac{r_1}{a_1}, 0, 0_)$.

(ii): Ξ_2's existence.

System (1) with $x_1 = y = 0$, offers: $x_2(r_2 - a_2x_2) = 0$, and hence $x_2^* = \frac{r_2}{a_2}$. Therefore the equilibrium Ξ_2 exists and given by $\Xi_2(0,x_2^*,0) = \Xi_2(0,\frac{r_2}{a_2},0)$.

(iii): Ξ_3's existence.

System (1) with $y = 0$, reduced to algebraic equations:

$$x_1(r_1 - a_1x_1 - \beta_1x_2) = 0,$$

$$x_2(r_2 - a_2x_2 - \beta_2x_1) = 0.$$

Their solution directly given by

$$(x_1^*, x_2^*) = (\frac{r_2\beta_1 - r_1a_2}{\beta_1\beta_2 - a_1a_2}, \frac{r_1\beta_2 - r_2a_1}{\beta_1\beta_2 - a_1a_2}).$$

Thus, $\Xi_3(x_1^*,x_2^*,0) = \Xi_3(\frac{r_2\beta_1 - r_1a_2}{\beta_1\beta_2 - a_1a_2}, \frac{r_1\beta_2 - r_2a_1}{\beta_1\beta_2 - a_1a_2}, 0)$ exists if one of the following conditions set met

$$r_2\beta_1 > r_1a_2, \tag{3}$$

$$r_1\beta_2 > r_2a_1, \tag{4}$$

or

$$r_2\beta_1 < r_1 a_2, \tag{5}$$

$$r_1\beta_2 < r_2 a_1, \tag{6}$$

(iv): Ξ_4's existence.

System (1) with $x_2 = 0$, provides:

$$x_1[\frac{r_1}{1+g_1 y} - a_1 x_1 - d_1 y] = 0, \tag{7}$$

$$y[c_1 d_1 x_1 - my] = 0. \tag{8}$$

For $y \neq 0$, last equation gives, $x_1 = \frac{my}{c_1 d_1}$. Use this in equation (7) gives the polynomial

$$A_1 y^2 + A_2 y + A_3 = 0.$$

where,

$$A_1 = g_1 c_1 d_1^2 + g_1 a_1 m > 0,$$

$$A_2 = a_1 m + c_1 d_1^2 > 0,$$

$$A_3 = -r_1 c_1 d_1 < 0.$$

So, according to the descartes' rule of sign, there is a single positive root that is supplied by

$$y = \frac{-A_2 + \sqrt{A_2^2 - 4A_1 A_3}}{2A_1}.$$

Therefore, $\Xi_4(x_1^*, 0, y^*)$ exists and has the value $\Xi_4(\frac{my^*}{c_1 d_1}, 0, \frac{-A_2+\sqrt{A_2^2-4A_1A_3}}{2A_1})$.

(v): Ξ_5's existence.

Applied $x_1 = 0$, in System (1) yields:

$$x_2[\frac{r_2}{1+g_2 y} - a_2 x_2 - d_2 y] = 0, \tag{9}$$

$$y[c_2 d_2 x_2 - my] = 0. \tag{10}$$

For $y \neq 0$, one can get, $x_2 = \frac{my}{c_2 d_2}$, and is the positive root of the next polynomial

$$B_1 y^2 + B_2 y + B_3 = 0.$$

where,

$$B_1 = g_2 c_2 d_2^2 + g_2 a_2 m > 0,$$

$$B_2 = a_2 m + c_2 d_2^2 > 0,$$

$$B_3 = -r_2 c_2 d_2 < 0.$$

So, there is a unique positive root for this polynomial provided by

$$y = \frac{-B_2 + \sqrt{B_2^2 - 4B_1B_3}}{2B_1}.$$

Therefore, $\Xi_5(0, x_2^*, y^*)$ exists and has the value $\Xi_5(0, \frac{my^*}{c_2d_2}, \frac{-B_2+\sqrt{B_2^2-4B_1B_3}}{2B_1})$.

(vi): Ξ_6's existence.

Equating (1) to zero gives:

$$x_1[\frac{r_1}{1+g_1y} - a_1x_1 - \beta_1x_2 - d_1y] = 0, \tag{11}$$

$$x_2[\frac{r_2}{1+g_2y} - a_2x_2 - \beta_2x_1 - d_2y] = 0, \tag{12}$$

$$y[c_1d_1x_1 + c_2d_2x_2 - my] = 0. \tag{13}$$

From equation (13) we get,

$$x_1 = \frac{my - c_2d_2x_2}{c_1d_1}. \tag{14}$$

Let $K = c_2d_2\beta_2 - c_1d_1a_2$, and let $J = 1 + g_2y$, and using (14) and (12) one may find

$$x_2 = \frac{m\beta_2yJ - c_1d_1r_2 + c_1d_1d_2yJ)}{JK}. \tag{15}$$

Also, by using (15) and (14) one can have,

$$x_1 = \frac{myJK - c_2d_2m\beta_2yJ + c_1c_2d_1d_2r_2 - c_1c_2d_1d_2^2yJ}{c_1d_1JK}. \tag{16}$$

Now, setting $h = 1 + g_1y$ and applying (15) and (16) into (11) we obtain,

$$C_1y^3 + C_2y^2 + C_3y + C_4 = 0. \tag{17}$$

Where,

$$\begin{aligned}
C_1 =&(g_1g_2)\big[(a_1m + c_1d_1^2)(c_2d_2\beta_2 - c_1d_1a_2) + (m\beta_2 + c_1d_1d_2)(c_1d_1\beta_1 - c_2d_2a_1)\big],\\
C_2 =&(g_1 + g_2)\big[(a_1m + c_1d_1^2)(c_2d_2\beta_2 - c_1d_1a_2) + (m\beta_2 + c_1d_1d_2)(c_1d_1\beta_1 - c_2d_2a_1)\big],\\
C_3 =&\big[(a_1m + c_1d_1^2)(c_2d_2\beta_2 - c_1d_1a_2) + (m\beta_2 + c_1d_1d_2)(c_1d_1\beta_1 - c_2d_2a_1)\\
&+g_1c_1d_1r_2(c_2d_2a_1 - c_1d_1\beta_1) + g_2c_1d_1r_2(c_1d_1a_2 - c_2d_2\beta_2)\big],\\
C_4 =&c_1d_1r_2(c_2d_2a_1 - c_1d_1\beta_1) + c_1d_1r_1(c_1d_1a_2 - c_2d_2\beta_2).
\end{aligned}$$

Now for applying Descarte's sign rule , we have the next two cases for gain the positive value y^*:

Case (i): When the following conditions are met

$$c_2d_2\beta_2 < c_1d_1a_2, \tag{18}$$

$$c_1d_1\beta_1 < c_2d_2a_1. \tag{19}$$

We can have $C_1 < 0$, $C_2 < 0$,$C_3 < 0$,$C_4 > 0$. Hence one can say y has only one positive value. Moreover, the values of x_1^* and x_2^* also positive under the next condition.

$$\frac{c_2d_2r_2}{a_2m + c_2d_2^2} < J^*y^* < \frac{c_1d_1r_2}{m\beta_2 + c_1d_1d_2}. \tag{20}$$

where, $J^* = 1 + g_2y^*$

Case (ii): If the next conditions are met

$$c_2d_2\beta_2 > c_1d_1a_2, \tag{21}$$

$$c_1d_1\beta_1 > c_2d_2a_1. \tag{22}$$

One can have,$C_1 > 0$, $C_2 > 0$,$C_3 > 0$,$C_4 < 0$.Then y has a unique positive value. Hence the values of x_1^* and x_2^* are positive when

$$\frac{c_1d_1r_2}{m\beta_2 + c_1d_1d_2} < J^*y^* < \frac{c_2d_2r_2}{a_2m + c_2d_2^2}. \tag{23}$$

So, we have the following theorem.

Theorem 3. *The equilibrium point Ξ_6 exists when one of the set of conditions (18)- (20) or (21)- (23) are fulfilled.*

5. Local Stability Analysis for $T_d = 0$

The eigenvalues of the Jacobian matrices can be used to find the local stability conditions of the equilibrium locations. At this points, for $T_d = 0$, system's (1) Jacobian matrix is provided by:

$$\mathbf{J} = \begin{bmatrix} A_* & B_* & C_* \\ D_* & E_* & F_* \\ N_* & M_* & H_* \end{bmatrix}$$

where,

$A_* = \frac{r_1}{h} - 2a_1x_1 - \beta_1x_2 - d_1y$, $B_* = -\beta_1x_1$, $C_* = -(\frac{g_1r_1x_1}{h^2} + d_1x_1)$,
$D_* = -\beta_2x_2$, $E_* = \frac{r_2}{J} - 2a_2x_2 - \beta_2x_1 - d_2y$, $F_* = -(\frac{g_2r_2x_2}{J^2} + d_2x_2)$,
$N_* = c_1d_1y$, $M_* = c_2d_2y$, $H_* = c_1d_1x_1 + c_2d_2x_2 - 2my$.

(i): For $\Xi_0 = (0, 0, 0)$:

$$\mathbf{J}|_{\Xi_0} = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So, $\lambda_{01} = r_1 > 0$, $\lambda_{02} = r_2 > 0$, $\lambda_{03} = 0$, Since there are two positive eigenvalues, point is unstable.

(ii): For $\Xi_1 = (\frac{r_1}{a_1}, 0, 0)$:

$$\mathbf{J}|_{\Xi_1} = \begin{bmatrix} -r_1 & \frac{-\beta_1 r_1}{a_1} & \frac{-g_1 r_1^2 - d_1 r_1}{a_1} \\ 0 & \frac{r_2 a_1 - \beta_2 r_1}{a_1} & 0 \\ 0 & 0 & \frac{c_1 d_1 r_1}{a_1} \end{bmatrix}$$

So, $\lambda_{11} = -r_1 < 0$, $\lambda_{12} = \frac{r_2 a_1 - \beta_2 r_1}{a_1}$, and $\lambda_{13} = \frac{c_1 d_1 r_1}{a_1} > 0$, We note that there is negative and positive eigenvalues, so the point Ξ_1 is saddle point (unstable).

(iii): For $\Xi_2 = (0, \frac{r_2}{a_2}, 0)$:

$$\mathbf{J}|_{\Xi_2} = \begin{bmatrix} \frac{a_2 r_1 - \beta_1 r_2}{a_2} & 0 & 0 \\ \frac{-\beta_2 r_2}{a_2} & -r_2 & \frac{-g_2 r_2^2 - d_2 r_2}{a_2} \\ 0 & 0 & \frac{c_2 d_2 r_2}{a_2} \end{bmatrix}$$

So, $\lambda_{21} = \frac{a_2 r_1 - \beta_1 r_2}{a_2}$, $\lambda_{22} = -r_2 < 0$, and $\lambda_{23} = \frac{c_2 d_2 r_2}{a_2} > 0$. Also can observe that the point Ξ_2 is saddle point (unstable).

(iv): For $\Xi_3 = (\frac{r_2\beta_1 - r_1 a_2}{\beta_1\beta_2 - a_1 a_2}, \frac{r_1\beta_2 - r_2 a_1}{\beta_1\beta_2 - a_1 a_2}, 0)$:

$$\mathbf{J}|_{\Xi_3} = \begin{bmatrix} A_3 & B_3 & C_3 \\ D_3 & E_3 & F_3 \\ 0 & 0 & H_3 \end{bmatrix}$$

$A_3 = \frac{a_1[r_1 a_2 - r_2\beta_1]}{\beta_1\beta_2 - a_1 a_2}$, $B_3 = \frac{\beta_1[r_1 a_2 - r_2\beta_1]}{\beta_1\beta_2 - a_1 a_2}$, $C_3 = \frac{(g_1 r_1 + d_1)(r_1 a_2 - r_2\beta_1)}{\beta_1\beta_2 - a_1 a_2}$,

$D_3 = \frac{\beta_2[r_2 a_1 - r_1\beta_2]}{\beta_1\beta_2 - a_1 a_2}$,$E_3 = \frac{a_2[r_2 a_1 - r_1\beta_2]}{\beta_1\beta_2 - a_1 a_2}$, $F_3 = \frac{(g_2 r_2 + d_2)(r_2 a_1 - r_1\beta_2)}{\beta_1\beta_2 - a_1 a_2}$,

$H_3 = \frac{c_1 d_1[r_2\beta_1 - r_1 a_2] + c_2 d_2[r_1\beta_2 - r_2 a_1]}{\beta_1\beta_2 - a_1 a_2}$.

Where,

$$[\lambda^2 + (-A_3 - E_3)\lambda + (A_3 E_3 - B_3 D_3)](H_3 - \lambda) = 0. \tag{24}$$

Due to the conditions (3)-(4) or the conditions (5)-(6), the above characteristic equation has two eigenvalues with negative real part, while the third eigenvalue given by $\lambda_{33} = H_3$ is positive. So the point Ξ_3 is saddle point (unstable).

(v): For $\Xi_4 = (x_1^*, 0, y^*)$:

$$\mathbf{J}|_{\Xi_4} = \begin{bmatrix} A_4 & B_4 & C_4 \\ 0 & E_4 & 0 \\ N_4 & M_4 & H_4 \end{bmatrix}$$

where,

$$A_4 = \frac{r_1}{h^*} - 2a_1x_1^* - d_1y^*, \qquad B_4 = -\beta_1 x_1^*, \qquad C_4 = -(\frac{g_1 r_1 x_1^*}{h^{*2}} + d_1 x_1^*),$$
$$E_4 = \frac{r_2}{J^*} - \beta_2 x_1^* - d_2 y^*, \qquad N_4 = c_1 d_1 y^*, \qquad M_4 = c_2 d_2 y^*,$$
$$H_4 = c_1 d_1 x_1^* - 2my^*.$$

where $h^* = 1 + g_1 y^*$.

The first eigenvalue of the above Jacobin matrix is $\lambda_{41} = E_4 < 0$, provided

$$J^* y^* > \frac{c_1 d_1 r_2}{\beta_2 m + c_1 d_1 d_2}. \tag{25}$$

The residue eigenvalues are the roots of the equation

$$\lambda^2 + b_{41}\lambda + b_{42} = 0, \tag{26}$$

where, $b_{41} = -A_4 - H_4 > 0$, and $b_{42} = A_4H_4 - C_4N_4 > 0$. Hence, Routh Hurwitz criterion [28] gain the roots (eigenvalues) of (26) have negative real parts. Thus, we can obtain the following theorem.

Theorem 4. *The point Ξ_4 is locally asymptotically stable, If condition (25) are met.*

(vi): For $\Xi_5 = (0, x_2^*, y^*)$:

$$\mathbf{J}|_{\Xi_5} = \begin{bmatrix} A_5 & 0 & 0 \\ D_5 & E_5 & F_5 \\ N_5 & M_5 & H_5 \end{bmatrix}$$

where,

$$A_5 = \frac{r_1}{h^*} - \beta_1 x_2^* - d_1 y^*, \qquad D_5 = -\beta_2 x_2^*, \qquad E_5 = \frac{r_2}{J^*} - 2a_2 x_2^* - d_2 y^*,$$
$$F_5 = -(\frac{g_2 r_2 x_2^*}{J^{*2}} + d_2 x_2^*), \qquad N_5 = c_1 d_1 y^*, \qquad M_5 = c_2 d_2 y^*,$$
$$H_5 = c_2 d_2 x_2^* - 2my^*.$$

The first eigenvalue of the above Jacobin matrix is $\lambda_{51} = A_5 < 0$, provided

$$h^* y^* > \frac{c_2 d_2 r_1}{\beta_1 m + c_2 d_1 d_2}. \tag{27}$$

The residue eigenvalues are the roots of the equation

$$\lambda^2 + b_{51}\lambda + b_{52} = 0, \tag{28}$$

where, $b_{51} = -E_5 - H_5 > 0$, and $b_{52} = E_5H_5 - F_5M_5 > 0$. Than, due to Routh Hurwitz criterion the last equation (28) has two roots (eigenvalues) with negative real parts. Consequently, the next theorem can be obtained.

Theorem 5. *The point Ξ_5 is locally asymptotically stable, If condition (27) are met.*

(vii): Now for $\Xi_6 = (x_1^*, x_2^*, y^*)$:

$$\mathbf{J}|_{\Xi_6} = \begin{bmatrix} A_6 & B_6 & C_6 \\ D_6 & E_6 & F_6 \\ N_6 & M_6 & H_6 \end{bmatrix}$$

where,

$$\begin{aligned} A_6 &= -a_1 x_1^*, & B_6 &= -\beta_1 x_1^*, & C_6 &= -(\frac{g_1 r_1 x_1^*}{h^{*2}} + d_1 x_1^*), \\ D_6 &= -\beta_2 x_2^*, & E_6 &= -a_2 x_2^*, & F_6 &= -(\frac{g_2 r_2 x_2^*}{J^{*2}} + d_2 x_2^*), \\ N_6 &= c_1 d_1 y^*, & M_6 &= c_2 d_2 y^*, & H_6 &= -my^*. \end{aligned}$$

The characteristic equation of $\mathbf{J}|_{\Xi_6}$ is given by

$$\lambda^3 + b_{61}\lambda^2 + b_{62}\lambda + b_{63} = 0, \tag{29}$$

where,

$$b_{61} = -H_6 - A_6 - E_6 > 0,$$

$$b_{62} = A_6 H_6 + E_6 H_6 + A_6 E_6 - B_6 D_6 - F_6 M_6 - C_6 N_6 > 0,$$

$$b_{63} = -A_6 E_6 H_6 - B_6 F_6 N_6 - C_6 D_6 M_6 + B_6 D_6 H_6 + F_6 M_6 A_6 + C_6 N_6 E_6 > 0.$$

The two conditions (18) and (19) yield that $b_{63} > 0$ and $b_{61} b_{62} - b_{63} > 0$, thereby satisfying all the requirements of the Routh-Hurwitz criterion. Hence, we have the following theorem.

Theorem 6. *The point Ξ_6 is locally asymptotically stable, If condition (18) and (19) are met.*

6. Region of Attraction for $T_d = 0$

In the stability analysis of population dynamics, global stability is a crucial and necessary concept. In complex ecological models, we typically study global stability in sub region called a region of attraction. In this talk, we'll look at the sub-region of stability of a system that has one predator species and two prey species. We can use the Lyapunov function to demonstrate that.

Theorem 7. *Point $\Xi_4 = (x_1^*, 0, y^*)$ is globally asymptotically stable in sub-regions where the following requirements are met:*

(i): $ma_1 > \frac{1}{4}[d_1 - c_1 d_1 + \frac{g_1 r_1}{hh^*}]^2$,
(ii): $y^* > \frac{c_1 d_1 r_2}{c_1 c_2 d_1 d_2 - \beta_1 m}$,
(iii): $c_2 < 1$.

Proof. Consider the Lyapunov function:

$$z_1(x_1, x_2, y) = (x_1 - x_1^* - x_1^* \log \frac{x_1}{x_1^*}) + x_2 + (y - y^* - y^* \log \frac{y}{y^*}).$$

After differentiation we get

$$\frac{dz_1}{dt} = (\frac{x_1 - x_1^*}{x_1})x_1' + x_2' + (\frac{y - y^*}{y})y'.$$

$$\begin{aligned}\frac{dz_1}{dt} =&(x_1 - x_1^*)[\frac{r_1}{h} - a_1x_1 - \beta_1x_2 - d_1y - \frac{r_1}{h^*} + a_1x_1^* + d_1y^*] + \frac{r_2x_2}{J} - a_2x_2^2 - \beta_2x_1x_2\\ &-d_2x_2y + (y - y^*)[c_1d_1x_1 + c_2d_2x_2 - my - c_1d_1x_1^* + my^*],\end{aligned}$$

$$\begin{aligned}\frac{dz_1}{dt} =&(x_1 - x_1^*)[-(x_1 - x_1^*)a_1 - \beta_1x_2 - (y - y^*)d_1 - (y - y^*)\frac{g_1r_1}{hh^*}] + \frac{r_2x_2}{J} - a_2x_2^2 - \beta_2x_1x_2\\ &-d_2x_2y + (y - y^*)[(x_1 - x_1^*)c_1d_1 + c_2d_2x_2 - (y - y^*)m],\end{aligned}$$

$$\begin{aligned}\frac{dz_1}{dt} \leq& -(x_1 - x_1^*)^2a_1 - (x_1 - x_1^*)\beta_1x_2 - (x_1 - x_1^*)(y - y^*)[d_1 - c_1d_1 + \frac{g_1r_1}{hh^*}] + r_2x_2\\ &- a_2x_2^2 - \beta_2x_1x_2 - d_2x_2y + (y - y^*)c_2d_2x_2 - (y - y^*)^2m.\end{aligned}$$

We will get a perfect square

$$\begin{aligned}\frac{dz_1}{dt} \leq& -[(x_1 - x_1^*)\sqrt{a_1} + (y - y^*)\sqrt{m}]^2 + d_2x_2y(c_2 - 1) + x_2(\beta_1x_1^* + r_2 - c_2d_2y^*)\\ &- \beta_1x_1x_2 - a_2x_2^2 - \beta_2x_1x_2.\end{aligned}$$

From the conditions of Theorem 7 we will get $\frac{dz_1}{dt} < 0$. Accordingly, Ξ_4 is an asymptotically stable point for every trajectory that begins at a location within the region that meets the above conditions. □

Similarly, can be proved the next two theorems about the sub-regions of globally asymptotically of Ξ_5 and Ξ_6 of system (1) for $T_d = 0$.

Theorem 8. $\Xi_5 = (0, x_2^*, y^*)$ *is globally asymptotically stable in sub- regions where the following requirements are met:*

(i): $ma_2 > \frac{1}{4}[d_2 - c_2d_2 + \frac{g_2r_2}{JJ^*}]^2$,
(ii): $y^* > \frac{c_2d_2r_1}{c_1c_2d_1d_2 - \beta_2m}$,
(iii): $c_1 < 1$.

Theorem 9. $\Xi_6 = (x_1^*, x_2^*, y^*)$ *is globally asymptotically stable in sub- regions where the following requirements are met:*

(i): $a_1a_2 \geq (\beta_1 + \beta_2)^2$,
(ii): $a_1m \geq [d_1 - c_1d_1 + \frac{g_1r_1}{hh^*}]^2$,
(iii): $a_2m \geq [d_2 - c_2d_2 + \frac{g_2r_2}{JJ^*}]^2$.

7. Local Stability and Hopf Bifurcation for Delay Model (1)

System (1) has the generalized variational matrix:

$$\mathbf{J}_d = \begin{bmatrix} A_* & B_* & C_* \\ D_* & E_* & F_* \\ N_* e^{-\lambda T_d} & M_* e^{-\lambda T_d} & Q_* \end{bmatrix} \tag{30}$$

where, $Q_* = [c_1 d_1 x_1 + c_2 d_2 x_2] e^{-\lambda T_d} - 2my$. In what follows, we examine the local stability and the Hopf bifurcation near Ξ_4, Ξ_5, Ξ_6, by utilizing time gestation delay T_d as the bifurcation parameter.

(i): for $\Xi_6 = (x_1^*, x_2^*, y^*)$:

The characteristic equation of (1) at Ξ_6

$$det \begin{bmatrix} A_6 - \lambda & B_6 & C_6 \\ D_6 & E_6 - \lambda & F_6 \\ N_6 e^{-\lambda T_d} & M_6 e^{-\lambda T_d} & Q_6 - \lambda \end{bmatrix} = 0$$

where, $Q_6 = R_\diamond e^{-\lambda T_d} - 2my^*$ and $R_\diamond = c_1 d_1 x_1^* + c_2 d_2 x_2^*$, it is equivalent to:

$$\lambda^3 + \mathtt{p}_1 \lambda^2 + \mathtt{p}_2 \lambda + \mathtt{p}_3 + (\mathtt{q}_1 \lambda^2 + \mathtt{q}_2 \lambda + \mathtt{q}_3) e^{-\lambda T_d} = 0, \tag{31}$$

where,

$$\mathtt{p}_1 = 2my^* - (A_6 + E_6) > 0, \mathtt{p}_2 = A_6 E_6 - B_6 D_6 - 2my^*(A_6 + E_6),$$

$$\mathtt{p}_3 = 2my^*(A_6 E_6 - B_6 D_6), \mathtt{q}_1 = -(c_1 d_1 x_1^* + c_2 d_2 x_2^*) = -R_\diamond < 0,$$

$$\mathtt{q}_2 = A_6 R_\diamond + E_6 R_\diamond - (F_6 M_6 + C_6 N_6),$$

$$\mathtt{q}_3 = -A_6 E_6 R_\diamond - B_6 F_6 N_6 - C_6 D_6 M_6 + B_6 D_6 R_\diamond + F_6 M_6 A_6 + C_6 N_6 E_6.$$

At first, when $T_d = 0$, equation (31) reduces to:

$$\lambda^3 + (\mathtt{p}_1 + \mathtt{q}_1)\lambda^2 + (\mathtt{p}_2 + \mathtt{q}_2)\lambda + (\mathtt{p}_3 + \mathtt{q}_3) = 0. \tag{32}$$

So, with the help of Routh-Hurwitz criterion, theorem (6) acquire that Ξ_6 is locally asymptotically stable. Now, whenever T_d is greater than zero, may (31) has a pair of purely imaginary roots, let this pair represented by $\lambda = \mp i\omega$,$(\omega > 0)$. Through replacing $\lambda = i\omega$ (or $\lambda = -i\omega$) in equation (31), we derive that

$$-iw^3 - \mathtt{p}_1 w^2 + i\mathtt{p}_2 w + \mathtt{p}_3 + (-\mathtt{q}_1 w^2 + i\mathtt{q}_2 w + \mathtt{q}_3)(\cos wT_d - i \sin wT_d) = 0.$$

From separating the previous equation's real and imaginary parts, it produces

$$(\mathtt{q}_3 - \mathtt{q}_1 w^2) \cos wT_d + \mathtt{q}_2 w \sin wT_d = \mathtt{p}_1 w^2 - \mathtt{p}_3, \tag{33}$$

$$\mathtt{q}_2 w \cos wT_d - (\mathtt{q}_3 - \mathtt{q}_1 w^2) \sin wT_d = w^3 - \mathtt{p}_2 w^2.$$

After squaring $(33)_1$ and $(33)_2$, we may sum the resulting equations to obtain

$$w^6 + \hbar_1 w^4 + \hbar_2 w^2 + \hbar_3 = 0, \tag{34}$$

where,

$$\hbar_1 = \mathrm{p}_1^2 - \mathrm{q}_1^2 - 2\mathrm{p}_2 > 0,$$

$$\hbar_2 = \mathrm{p}_2^2 - 2\mathrm{p}_1\mathrm{p}_3 - \mathrm{q}_2^2 + 2\mathrm{q}_1\mathrm{q}_3,$$

$$\hbar_3 = \mathrm{p}_3^2 - \mathrm{q}_3^2 = (\mathrm{p}_3 + \mathrm{q}_3)(\mathrm{p}_3 - \mathrm{q}_3).$$

Further, assume that the next condition holds.

$$\mathrm{q}_3 > \mathrm{p}_3. \tag{35}$$

In the event that conditions (18)- (20) and (35) are fulfilled, $h_3 < 0$. So, there is a unique positive root, let's say ω_0, satisfying equation (34), in accordance with Descartes' rule of signs. Consequently, $\pm i\omega_0$ represents two imaginary roots of equation (31). Moreover, if substituting ω_0 in $(33)_1$ and $(33)_2$, and solving the resulting equations for T_d, we might have

$$T_{03} = \frac{1}{w_0}\cos^{-1}\frac{(\mathrm{q}_2 - \mathrm{q}_1\mathrm{p}_1)w_0^4 + (\mathrm{q}_3\mathrm{p}_1 + \mathrm{q}_1\mathrm{p}_3 - \mathrm{q}_2\mathrm{p}_2)w_0^2 - \mathrm{q}_3\mathrm{p}_3}{\mathrm{q}_1^2 w_0^4 + (\mathrm{q}_2^2 - 2\mathrm{q}_1\mathrm{q}_3)w_0^2 + \mathrm{q}_2^3}. \tag{36}$$

Next, we need to prove the next transversality requirement in order to establish Hopf bifurcation at $T_d = T_{03}$.

$$sing[\frac{d(Re\lambda(T_d))}{dT_d}]_{T_d = T_{03}} > 0. \tag{37}$$

To accomplish that, assume that a root of equation (31) fulfilling $\alpha(T_{03}) = 0$ is $\lambda(T_d) = \alpha(T_d) + i\omega(T_d)$, where $\omega(T_{03}) = \omega_0$.

Now, through the use of $\lambda(T_d)$ in equation (31) and differentiation it with respect to T_d, one may obtain that

$$[3\lambda^2 + 2\mathrm{p}_1\lambda + \mathrm{p}_2 + (2\mathrm{q}_1\lambda + \mathrm{q}_2)e^{-\lambda T_d} - T_d(\mathrm{q}_1\lambda^2 + \mathrm{q}_2\lambda + \mathrm{q}_3)e^{-\lambda T_d}]\frac{d\lambda}{dT_d}$$

$$= \lambda(\mathrm{q}_1\lambda^2 + \mathrm{q}_2\lambda + \mathrm{q}_3)e^{-\lambda T_d}. \tag{38}$$

Constantly,

$$(\frac{d\lambda}{dT_d})^{-1} = \frac{(3\lambda^2 + 2\mathrm{p}_1\lambda + \mathrm{p}_2)e^{\lambda T_d}}{(\mathrm{q}_1\lambda^2 + \mathrm{q}_2\lambda + \mathrm{q}_3)\lambda} + \frac{2\mathrm{q}_1\lambda + \mathrm{q}_2}{(\mathrm{q}_1\lambda^2 + \mathrm{q}_2\lambda + \mathrm{q}_3)\lambda} - \frac{T_d}{\lambda}. \tag{39}$$

For $T_d = T_{03}$, and $\lambda = i\omega_0$, can have that

$$(\mathrm{q}_1\lambda^2 + \mathrm{q}_2\lambda + \mathrm{q}_3)\lambda = -\mathrm{q}_2\omega_0^2 + i\omega_0(\mathrm{q}_3 - \mathrm{q}_1\omega_0^2),$$

$$2\mathrm{q}_1\lambda + \mathrm{q}_2 = \mathrm{q}_2 + 2i\mathrm{q}_1\omega_0,$$

$$\frac{T_d}{\lambda} = -i\frac{T_{03}}{\omega_0}.$$

further,

$$(3\lambda^2+2\mathrm{p}_1\lambda+\mathrm{p}_2)e^{\lambda T_d}=(\mathrm{p}_2-3\omega_0^2+i2\mathrm{p}_1\omega_0)(\cos\omega_0 T_{03}+i\sin\omega_0 T_{03})$$

$$=[(\mathrm{p}_2-3\omega_0^2)\cos\omega_0 T_{03}-2\mathrm{p}_1\omega_0\sin\omega_0 T_{03}]+i[2\mathrm{p}_1\omega_0\cos\omega_0 T_{03}+(\mathrm{p}_2-3\omega_0^2)\sin\omega_0 T_{03}].$$

Than,

$$\begin{aligned}Re[\frac{d(\lambda(T_d))}{dT_d}]^{-1}_{T_d=T_{03}} =&Re[\frac{(2\mathrm{q}_1\lambda+\mathrm{q}_2)+(3\lambda^2+2\mathrm{p}_1\lambda+\mathrm{p}_2)e^{\lambda T_d}}{(\mathrm{q}_1\lambda^2+\mathrm{q}_2\lambda+\mathrm{q}_3)\lambda}-\frac{T_d}{\lambda}]_{\lambda=i\omega_0}\\ =&\frac{1}{Q_0}[3\omega_0^6+2(\mathrm{p}_1^2-\mathrm{q}_1^2-2\mathrm{p}_2)\omega_0^4+(\mathrm{p}_2^2-2\mathrm{p}_1\mathrm{p}_3+2\mathrm{q}_1\mathrm{q}_3-\mathrm{q}_2^2)\omega_0^2]\\ =&\frac{\omega_0^2}{Q_0}h(\omega_0^2),\end{aligned} \tag{40}$$

where,
$Q_0=\mathrm{q}_2^2\omega_0^4+\omega_0^2(\mathrm{q}_3-\mathrm{q}_1\omega_0^2)^2>0$,
$h(\omega_0^2)=3\omega_0^4+2h_1\omega_0^2+h_2$.
Let $\chi=\omega_0^2>0$ and $\Psi=[\frac{d(Re\lambda(T_d))}{dT_d}]$, then from complex analysis one can show that

$$sing(\Psi)_{T_d=T_{03}}=singRe[\frac{d(\lambda(T_d))}{dT_d}]^{-1}_{T_d=T_{03}}=sign[h(\chi)].$$

Since $h'(\chi)=6\chi+2h_1>0$. So, gain that $h(\chi)$ monotonously increases in $[0,+\infty)$. Furthermore,under the next condition

$$\mathrm{p}_2^2-2\mathrm{p}_1\mathrm{p}_3>\mathrm{q}_2^2-2\mathrm{q}_1\mathrm{q}_3. \tag{41}$$

we obtain $h(0)>0$, and $h(\chi)>0$ for $\omega>0$.
Considering the aforementioned, the transversal condition (37) is satisfied. Keeping the above condition in view, we can obtain the following theorem:

Theorem 10. *Assume that the conditions* (18)- (20) *and* (35) *are hold, then:*

- Ξ_6 *is locally asymptotically stable for* $T_d<T_{03}$
- Ξ_6 *is unstable for* $T_d>T_{03}$
- *System* (1) *undergoes Hopf bifurcations at* Ξ_6 *for* $T_d=T_{03}$*, where* T_{03} *is defined in equation* (31)*, if the condition* (41) *is met.*

Likewise, the local stability and Hopf bifurcation may also be investigated close to Ξ_4,Ξ_5 by using time gestation delay T_d as the bifurcation parameter. Consequently, this allows us to ascertain the threshold values $T_{02}(T_{01})$ such that $\Xi_5(\Xi_4)$ is locally asymptotically stable for $T_d<T_{02}(T_d<T_{01})$, and unstable for $T_d>T_{02}$ $(T_d>T_{01})$. Also, we can show the system (1) undergoes Hopf bifurcations at Ξ_5 for $T_d=T_{02}$ (or at Ξ_4 for $T_d=T_{01}$).

8. Numerical Simulations and Discussion

This section covered the numerical stability behaviors of the delay model (1) with the assistance of ($MATLAB$ 2018) $software$. Because actual data is not readily available, we have used here some hypothetical values of the model parameters.

$$r_1 = 0.9, \quad r_2 = 0.95, \quad g_1 = 1.5, \quad g_2 = 1.5, \quad c_1 = 0.85, \quad c_2 = 0.85, \quad \beta_1 = 0.25,$$
$$\beta_2 = 0.25, \quad d_1 = 0.52, \quad d_2 = 0.5, \quad a_1 = 0.205, \quad a_2 = 0.2, \quad m = 0.55. \tag{42}$$

Firstly, for model (1) with $T_d = 0$, according to analytically analysis the equilibria Ξ_0 ,Ξ_1,Ξ_2, and Ξ_3 are always exist, but they are not stable. Consequently, we were unable to obtain numerical stability for these equilibria. On other hand, also the equilibria Ξ_4, and Ξ_5 are always exist, and they may be asymptotically stable under some analytically conditions. Furthermore, if the steady state solution Ξ_6 exist under specific conditions then it's asymptotically stable. From Fig.1 it can be seen that the equilibrium points Ξ_4,Ξ_5 and Ξ_6 exist for the model (1) and they are locally asymptotically stable.

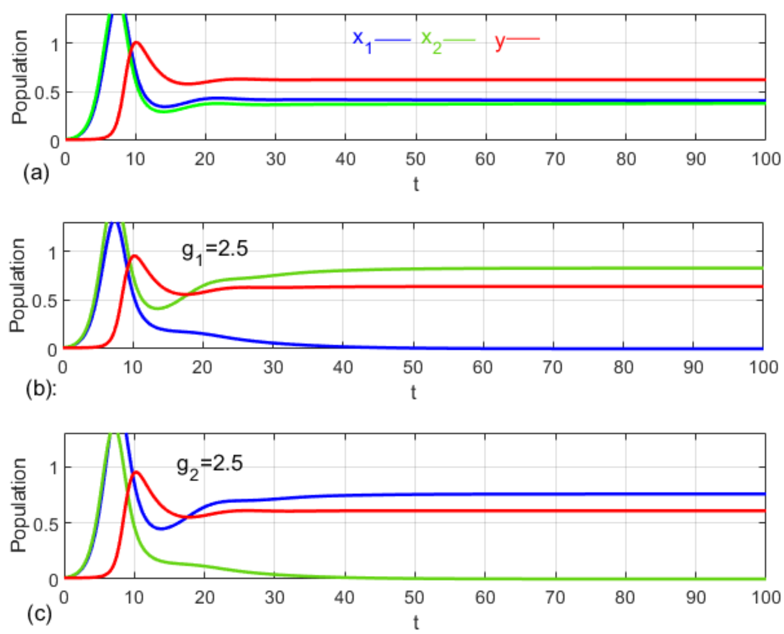

Figure 1. Stability of different equilibrium points through time series solutions. (a) Coexistence equilibrium Ξ_6 stable; (b) First prey-free equilibrium Ξ_5 stable; (c) Second prey-free equilibrium Ξ_4 stable.

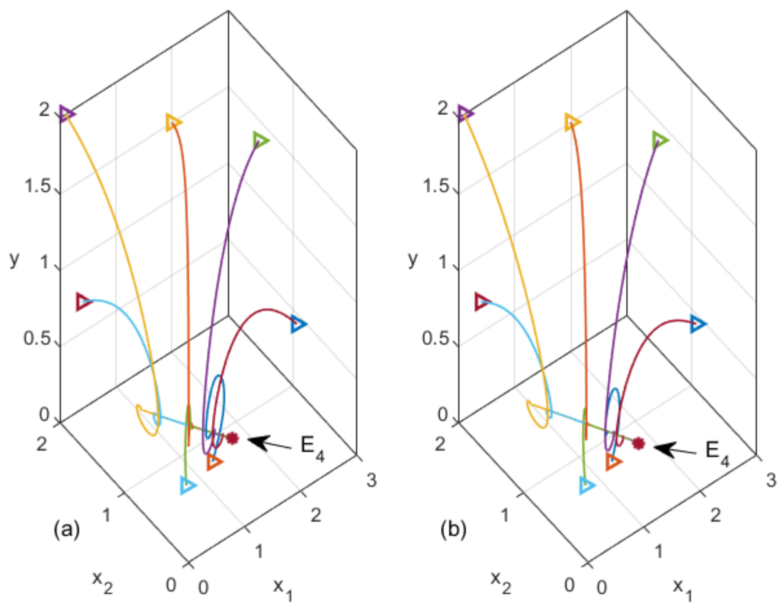

FIGURE 2. Global stability of $\Xi_4 = (0.9075, 0, 0.5730)$ of system (1) with the parameters values as given in Fig. 1 except: (a) $a_1 = 0.17$ (b) $m = 0.7$.

The data set (42) used in Fig. 1 satisfied the existence and stability conditions of Ξ_6 that given in theorem (3) and (6) as shown in subfigure (a), This data set with $g_1 = 2.5$ satisfied the stability conditions of Ξ_5 that given in theorem (5) as plotted in subfigure (b), while with $g_2 = 2.5$ this data set satisfied the stability conditions of Ξ_4 that given in theorems (4) as depicted in subfigure (c). It is apparent from that the ecosystem may become unstable due to the impact of fear of predato on prey's growth. But it may support warmer environments to stabilize.
Figure 2, used the data set (42) with $a_1 = 0.17$ in subfigure 2(a), and $m = 0.7$ in subfigure 2(b), and represents the global stability of Ξ_4 for the proposed model. From this figure one can show the prey x_2 may go to extinction with decreasing toxicity first prey rate a_1 or increasing toxicity predator rate m.

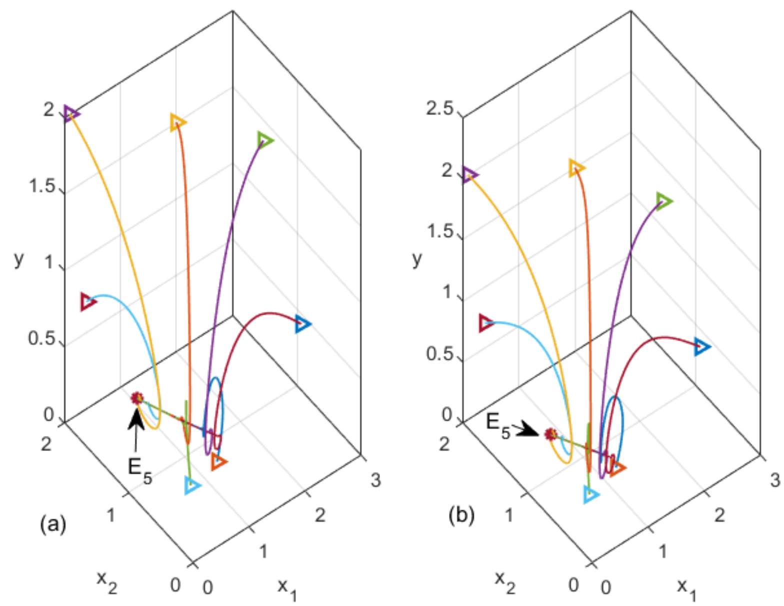

FIGURE 3. Global stability of $\Xi_5 = (0, 0.6421, 0.6822)$ of system (1) with the parameters values as given in Fig. 1 except: (a) $a_2 = 0.15$, (b) $m = 0.4$.

Figure 3, represents the global stability of for Ξ_5 the proposed model for $T_d = 0$. This figure plotted with data set (42) and used $a_2 = 0.15$ in subfigure 3(a) and $m = 0.4$ in subfigure 3(b). It is seen that the first prey x_1 may go to extinction with decreasing toxicity rates a_2 or m. These two figures confirmed the analytic results given in theorems (7) and (8).
Moreover, phase diagram of non-delay model of (1) shows bistability between Ξ_4 and Ξ_5 due to the change of toxicity rates of preys and predator as depicted in Fig. 4. Due the model's complexity, the results indicate that, for lower values of toxicity rates the system solution moves between boundary equilibrium points depends on the initial population sizes.

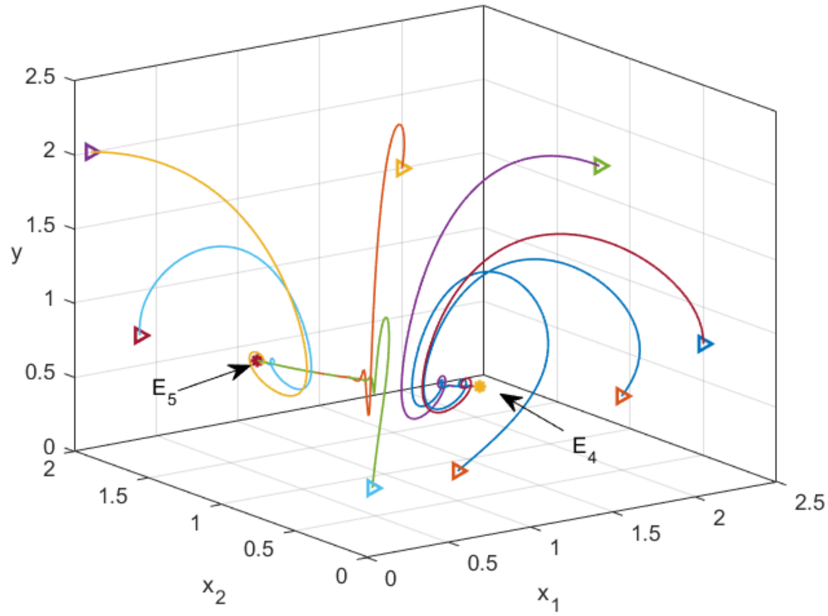

FIGURE 4. Bistability between Ξ_4 and Ξ_5 of system (1) with the parameters values as given in Fig. 1 except: $a_1 = 0.1$, $a_2 = 0.12$, and $m = 0.25$.

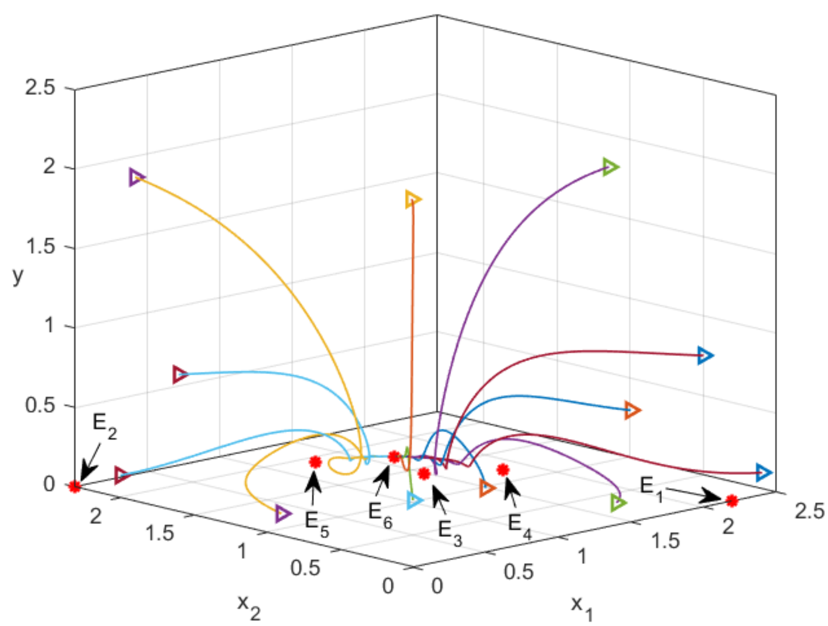

FIGURE 5. Phase portrait of system (1) at the parameters values $r_1 = 0.9, r_2 = 0.95, g_1 = 1.5, g_2 = 1.5, c_1 = 0.85, c_2 = 0.85, \beta_1 = 0.25, \beta_2 = 0.25, d_1 = 0.52, d_2 = 0.5, a_1 = 0.41, a_2 = 0.41$, and $m = 0.55$.

Furthermore, in Fig. 5, the phase portrait of non-delay model of (1) has been presented. It is observed that the data set used in this figure fulfilled the conditions of theorem (8) in addition to the conditions for the existence of all equilibria $\Xi_i, i = 1, 2, ..., 6$ for this model. So, in this figure we show only Ξ_6 is globally stabile while other equilibria are exist but unstable. Therefore, depending on it values, toxicity rate can either promote stability or cause instability.

The above numerical simulations validate the analytical findings of system (1)at $T_d = 0$. In other hand, to investigate the effect of time delay on the dynamic of the model, we showcase numerical tests conducted on the system (1)with $T_d \neq 0$. To confirm the theorem (10)'s analytical result, we used the data set in figure 5 with $m = 0.3$. Due to this data, equation (34) has $\hbar_1 = 0.1358, \hbar_2 = -0.0029, \hbar_3 = -4.6314E - 06$ and $w_0 = 0.1412$. This indicates that the equation (31) has the pure complex roots $\pm i0.1412$. Further, the transversality criterion (37) is also met because, using $T_d = 11.9682$ from equation (36), we obtain $sing(\Psi(T_d = 11.9682)) = sing(h(0.3758)) = sign(0.0031) > 0$. The next two figures 6 and 7 validate the result of theorem (10). Where the time series and the phase portrait of model (1) with $T_d = 10.9682 < T_{03}$ are plotted in Fig. 6 and show that The solution begins with a periodic oscillation and then goes asymptotically to a steady state $\Xi_6 = (0.1464, 0.3005, 0.6444)$. While Fig. 7 plotted with $T_d = 12.9682 > T_{03}$ and show the model has a Hopf bifurcation around Ξ_6.

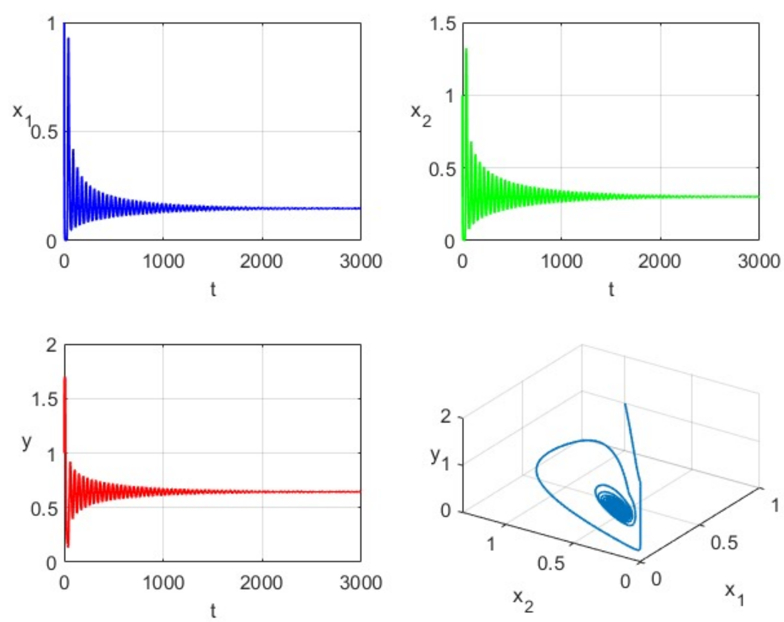

Figure 6. Solutions and Phase portrait of system (1) with $T_d = 10.9682 < T_{03}$, $m = 0.3$ and other parameters values as used in Fig. 5.

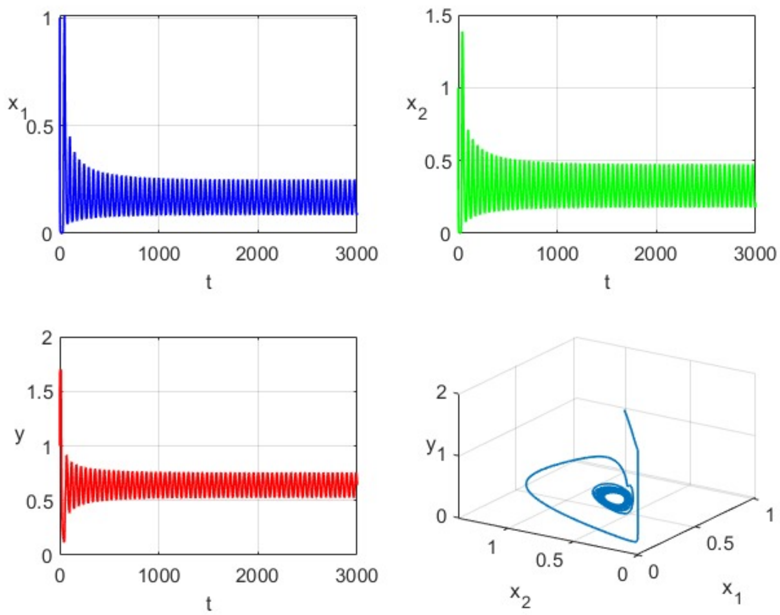

FIGURE 7. Solutions and Phase portrait of system (1) with $T_d = 12.9682 > T_{03}$, $m = 0.3$ and other parameters values as used in Fig. 5.

Furthermore, by altering the values of g_1, g_2, a_1, a_2, and m, it is also possible to get the dynamic behavior suggested by theorem (10). In these cases, as the values of these parameters vary, so does the delay threshold value T_{03}.

In order to verify our claim about the effect of time delay T_d on the stability and instability of Ξ_5, the data set (42) with $a_2 = 0.15$ is now applied in the next two figures, 8 and 9. Figure 8, plotted with $T_d = 6.6986 < T_{02} = 7.6986$, illustrates how the solutions arrive at a free-prey state Ξ_5 following an oscillation phase. While Fig. 9, plotted with $T_d = 8.6986 > T_{02} = 7.6986$, demonstrates that the delay model features a Hopf bifurcation near Ξ_5.

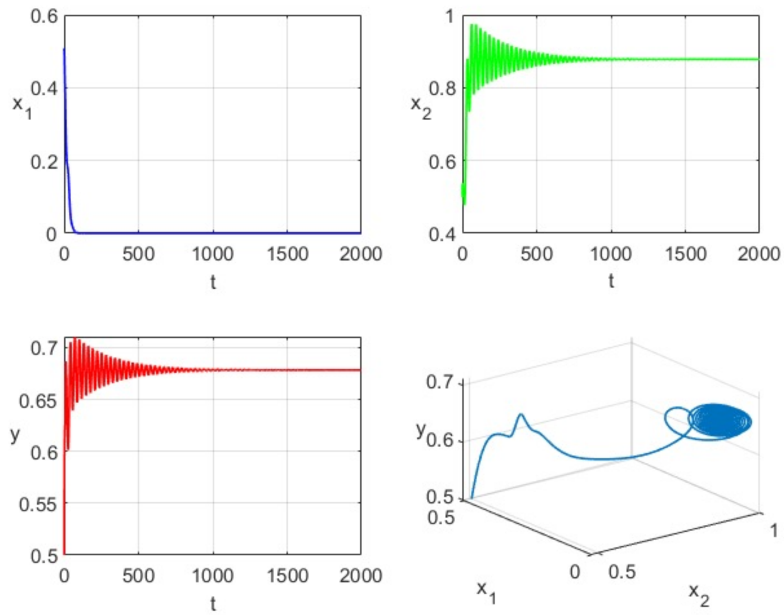

FIGURE 8. Solutions and Phase portrait of system (1) with $T_d = 6.6986 < T_{02}$, $a_2 = 0.15$ and other parameters values as given in (42).

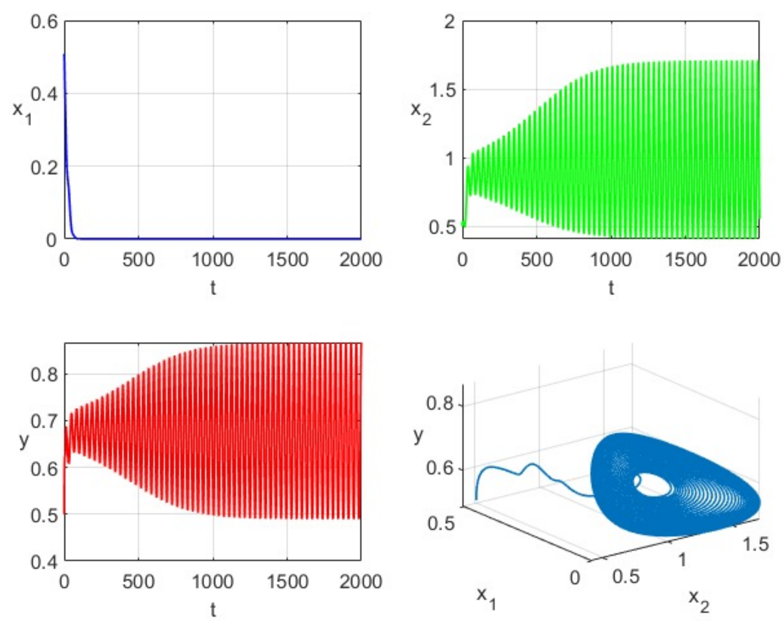

FIGURE 9. Solutions and Phase portrait of system (1) with $T_d = 8.6986 > T_{02}$, $a_2 = 0.15$ and other parameters values as given in (42).

Finally, figures 10 and 11 are plotted with data set in (42) with $a_1 = 0.175$, to confirm our assertion on the impact of time delay T_d on the stability and instability of Ξ_4. Figure 10 plotted with $T_d =$

$8.5 < T_{01} = 9.05$, and demonstrating that the solution starts with a periodic oscillation and progresses asymptotically to a stable state Ξ_4. While Fig. 11 plotted with $T_d = 9.5 > T_{01} = 9.05$, and reveals that there is a Hopf bifurcation in the delay model close to Ξ_4.

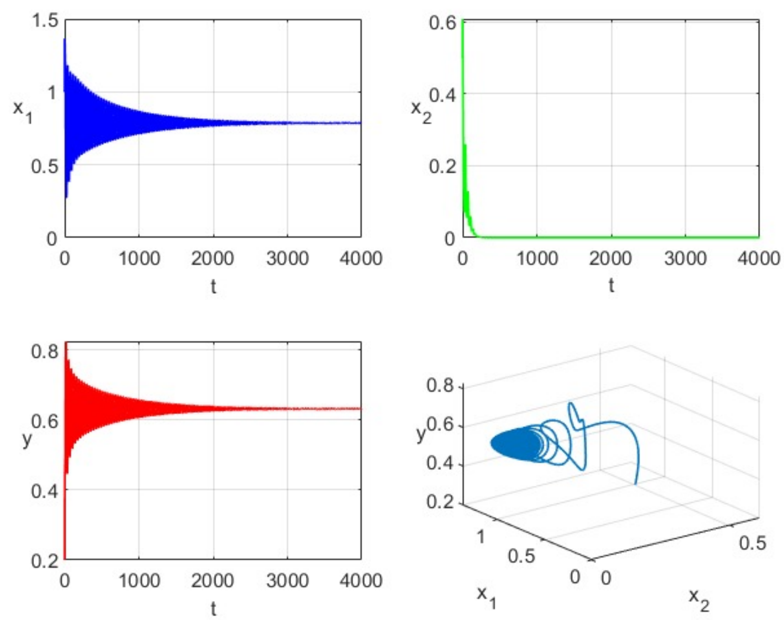

FIGURE 10. Solutions and Phase portrait of system (1) with $T_d = 8.5 < T_{01}$, $a_1 = 0.17$, and other parameters values as given in (42).

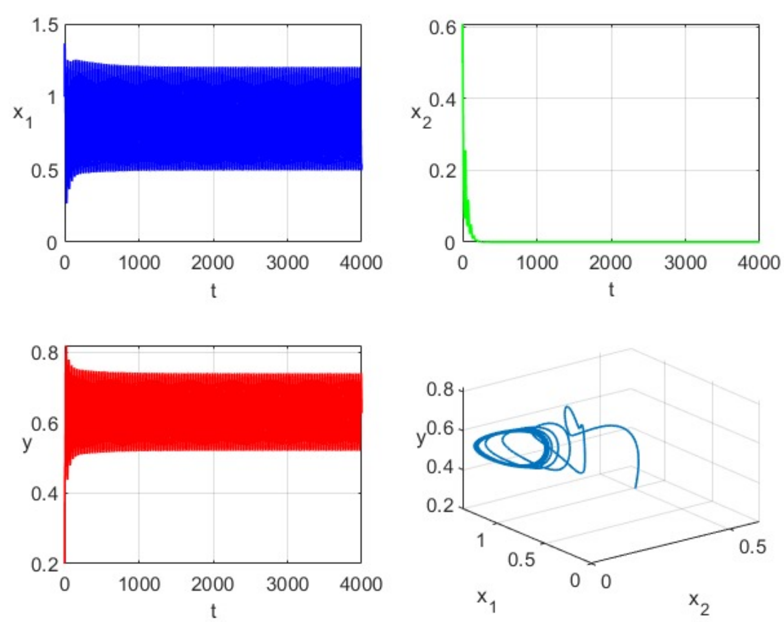

FIGURE 11. Solutions and Phase portrait of system (1) with $T_d = 9.5 > T_{01}$, $a_1 = 0.17$, and other parameters values as given in (42).

9. Conclusions

This work develops a dynamical model of three species, two different preys and one-predator ,in the setting of the effects of fear and toxic substances. The solution's characteristics were examined. It was determined that one internal equilibrium point and six boundary equilibrium points are present. The system (1)'s stability study (locally and globally) was looked into. It is discovered that the internal, x_1-free, and x_2-free equilibria are the only ones that may be stable (conditionally stable) for system (1) at $0 \leq T_d < T_{0i}, i = 1, 2, 3$. Further, the bistability between x_1-free and x_2-free equilibria indicates that the system possesses global stability within sub-regions. Also, we see that the system without delay always $(T_d = 0)$ reaches equilibrium states; in other words, the oscillatory behavior does not arise from the system. This indicates that the predator is constantly dependent on both prey species. In this regard, this model does not predict oscillatory predation between the two prey species. For $T_d < T_{di}$, this oscillation disappears over time; for $T_d > T_{di}$, it moves to the hopf bifurcation. In this case, predator species exhibit oscillatory predation between the two prey species. Further, we observe that the effect of toxicity on predator has no effect on the instability of the predator extinction equilibria Ξ_1,Ξ_2 and Ξ_3. This is due to the fact that predators might depend on the other prey to save themselves from going extinct. Theoretical conditions and numerical results show that incrossing the fear levels of predations $g_1$andg_2, may move the model to extinct the preys. Additionally, the toxicity levels a_1, a_2, and m have a significant impact on stability and instability of the model solutions', and on the value of the gestation delay threshold values at which the hopf bifurcation arises. Overall analysis shows that, for whatever value of the parameters, the system cannot collapse due to the intrinsic instability of Ξ_0.

Authors' Contributions

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] A. Das, G.P. Samanta, Modelling the fear effect in a two-species predator?prey system under the influence of toxic substances, Rend. Circ. Mat. Palermo, II. Ser. 70 (2020), 1501–1526. https://doi.org/10.1007/s12215-020-00570-x.

[2] M. Das, G.P. Samanta, A prey-predator fractional order model with fear effect and group defense, Int. J. Dyn. Control 9 (2020), 334–349. https://doi.org/10.1007/s40435-020-00626-x.

[3] Y. Xu, A.L. Krause, R.A. Van Gorder, Generalist predator dynamics under Kolmogorov versus non-Kolmogorov models, J. Theor. Biol. 486 (2020), 110060. https://doi.org/10.1016/j.jtbi.2019.110060.

[4] S.J. Majeed, R.M. Adbulkareem, Stability analysis of a prey-predator model with additional food, refuge, and variable carrying capacity, AIP Conf. Proc. 2834 (2023), 080109. https://doi.org/10.1063/5.0162386.

[5] T.G. Hallam, J.T. de Luna, Effects of toxicants on populations: A qualitative, J. Theor. Biol. 109 (1984), 411–429. https://doi.org/10.1016/s0022-5193(84)80090-9.

[6] H.I. Freedman, J.B. Shukla, Models for the effect of toxicant in single-species and predator-prey systems, J. Math. Biol. 30 (1991), 15–30. https://doi.org/10.1007/bf00168004.

[7] J. Chattopadhyay, Effect of toxic substances on a two-species competitive system, Ecol. Model. 84 (1996), 287–289. https://doi.org/10.1016/0304-3800(94)00134-0.

[8] A.K. Pal, G.P. Samanta, A single species population in a polluted environment, Int. J. Biomath. 03 (2010), 187–204. https://doi.org/10.1142/s1793524510000933.

[9] S.L. Lima, Nonlethal effects in the ecology of predator-prey interactions, BioScience 48 (1998), 25–34. https://doi.org/10.2307/1313225.

[10] S. Creel, D. Christianson, S. Liley, J.A. Winnie Jr., Predation risk affects reproductive physiology and demography of elk, Science 315 (2007), 960–960. https://doi.org/10.1126/science.1135918.

[11] S. Creel, D. Christianson, Relationships between direct predation and risk effects, Trends Ecol. Evol. 23 (2008), 194–201. https://doi.org/10.1016/j.tree.2007.12.004.

[12] W. Cresswell, Non?lethal effects of predation in birds, Ibis 150 (2008), 3–17. https://doi.org/10.1111/j.1474-919x.2007.00793.x.

[13] S.J. Majeed, S.F. Ghafel, Stability analysis of a prey-predator model with prey refuge and fear of adult predator, Iraqi J. Sci. (2022), 4374–4387. https://doi.org/10.24996/ijs.2022.63.10.24.

[14] L.Y. Zanette, A.F. White, M.C. Allen, M. Clinchy, Perceived predation risk reduces the number of offspring songbirds produce per year, Science 334 (2011), 1398–1401. https://doi.org/10.1126/science.1210908.

[15] X. Wang, L. Zanette, X. Zou, Modelling the fear effect in predator?prey interactions, J. Math. Biol. 73 (2016), 1179–1204. https://doi.org/10.1007/s00285-016-0989-1.

[16] Z. Jing, J. Yang, Bifurcation and chaos in discrete-time predator?prey system, Chaos Solitons Fractals 27 (2006), 259–277. https://doi.org/10.1016/j.chaos.2005.03.040.

[17] Y. Kuang, Delay differential equations: With applications in population dynamics, Academic Press, 1993.

[18] S. Li, X. Liao, C. Li, Hopf bifurcation in a Volterra prey?predator model with strong kernel, Chaos Solitons Fractals 22 (2004), 713–722. https://doi.org/10.1016/j.chaos.2004.02.048.

[19] J. Wei, M.Y. Li, Hopf bifurcation analysis in a delayed Nicholson blowflies equation, Nonlinear Anal.: Theory Meth. Appl. 60 (2005), 1351–1367. https://doi.org/10.1016/j.na.2003.04.002.

[20] H. Smith, An introduction to delay differential equations with applications to the life sciences, Springer, New York, 2011. https://doi.org/10.1007/978-1-4419-7646-8.

[21] R. Naji, S. Majeed, The dynamical analysis of a delayed prey-predator model with a refuge-stage structure prey population, Iran. J. Math. Sci. Inf. 15 (2020), 135–159. https://doi.org/10.29252/ijmsi.15.1.135.

[22] S. Kundu, S. Maitra, Asymptotic behaviors of a two prey one predator model with cooperation among the prey species in a stochastic environment, J. Appl. Math. Comp. 61 (2019), 505–531. https://doi.org/10.1007/s12190-019-01251-4.

[23] R. Banerjee, P. Das, D. Mukherjee, Global dynamics of a Holling type-III two prey?one predator discrete model with optimal harvest strategy, Nonlinear Dyn. 99 (2020), 3285–3300. https://doi.org/10.1007/s11071-020-05490-0.

[24] D. Sahoo, G.P. Samanta, Impact of fear effect in a two prey-one predator system with switching behaviour in predation, Diff. Equ. Dyn. Syst. 32 (2021), 377–399. https://doi.org/10.1007/s12591-021-00575-7.

[25] A. Mondal, A.K. Pal, G.P. Samanta, Complex dynamics of two prey?one predator model together with fear effect and harvesting efforts in preys, J. Comp. Math. Data Sc. 6 (2023), 100071. https://doi.org/10.1016/j.jcmds.2022.100071.

[26] G.S. Kumar, C. Gunasundari, Dynamical analysis of two-preys and one predator interaction model with an Allee effect on predator, Malays. J. Math. Sci. 17 (2023), 263–281. https://doi.org/10.47836/mjms.17.3.03.

[27] A.A. Thirthar, S. Jawad, S.J. Majeed, K.S. Nisar, Impact of wind flow and global warming in the dynamics of prey?predator model, Results Control Optim. 15 (2024), 100424. https://doi.org/10.1016/j.rico.2024.100424.

[28] E.X. DeJesus, C. Kaufman, Routh-Hurwitz criterion in the examination of eigenvalues of a system of nonlinear ordinary differential equations, Phys. Rev. A 35 (1987), 5288–5290. https://doi.org/10.1103/physreva.35.5288.

[29] F. Verhulst, Nonlinear differential equations and dynamical systems, Springer, 2006. https://doi.org/10.1007/978-3-642-61453-8.