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A NEW TREATMENT OF MULTIPLICATIVE PERTURBATION OF BLACK-SCHOLES OPERATOR

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Abstract. We recall and apply results recently obtained on non autonomous evolution equations concerning mainly non autonomous Cauchy problem. The main contribution is a more general perturbation of Black-Scholes operator considered in the Hilbertian context.

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1. Introduction

The current paper is a natural continuation of recent works on the emergent notion of maximal regularity, particularly [10] and [11] where interesting results related to the form associated to the Black-Scholes operator are detailed. A special interest mainly in [11] has been granted to positivity. By this latter concept, we mean the positive character of the solution of non autonomous evolution equations.
It is worth mentioning that many works in this direction concern the linear case, for instance [20], [1] or [18]. But a big interest begins to be granted to the non-linear case. As a prototype, we cite [12] where the analysis was dedicated to semilinear non-autonomous evolution equations. The linear case developed in the current paper is slightly close to semigroup treatment of evolution equations. The semigroup theory has proved its robustness in solving evolution equations: one may see fruitfully the single parameter theory [15] or bi-parameter one as developed in [2]. However, when the problem is non-autonomous, evolution families give a more efficient tool to deal with evolution problems, see for instance the recent work of Kharou on analytic evolution families [14].
For a class of *regular* operators, the existence and uniqueness are well known and by some classical techniques, one may extend the well-posedness to a large "neighbours" of such operators. This remark leads to the *maximal regularity*. Let us recall the definition of this important property. We start first with

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the autonomous case (i.e., the operator does not depend on time). In the second step, the case of a family of operators that is time-dependent and give a common, different definition in this case.

For a fixed horizon T and two Banach spaces D and X such that $D \hookrightarrow X$, (this notation means that D is continuously and densely embedded into X).

Definition 1. *A single operator $A \in \mathcal{L}(D, X)$ has L^p-maximal regularity or is of category $\mathcal{MR}$ $\quad (p \in (1, \infty))$ if for every $f \in L^p(0, T; X)$ there exists a unique $u \in W^{1,p}(0, T; X) \cap L^p(0, T; D)$ such that*

$$\begin{cases} \dot{u}(t) + Au(t) = f(t) \quad a.e. \ on \quad (0, T), \\ u(0) = 0. \end{cases} \tag{1}$$

In the autonomous case, the well known result of Baillon (Baillon's theorem), as restored in [7], ensures operators with maximal regularity on $\mathcal{C}([0, T], E)$(i.e. space of all continuous mappings on $[0, T]$ with values in a general Banach space E) are exactly bounded operators. This surprising result gives no chance to pursue researches on this kind of spaces. Hence, the interest accorded to L^p-maximal regularity or $\mathcal{C}^\alpha-$ regularity is clarified.

L^p-maximal regularity has many applications in the wide field of PDEs, mainly to prove existence, uniqueness and regularity of solutions of linear and non-linear evolution equations. There are a lot of manners to express this property, for instance, when the horizon T is finite, the maximal regularity of a single operator A is the same thing as saying that $\dot{u}$, Au and f belong to the same space $L^p(0, T, X)$. The space $\mathcal{MR} = W^{1,p}(0, T; X) \cap L^p(0, T; D)$ of all solutions of (1) is a Banach space for the norm:

$$\|u\|_{MR} = \|u\|_{W^{1,p}(0,T;X)} + \|u\|_{L^p(0,T;D)}.$$

and is commonly said of maximal regularity space.

The trace, at zero or at any $t_0 \in [0, T]$ of all elements of $\mathcal{MR}$, namely

$Tr = \{u(0); u \in \mathcal{MR}\} = \{u(t_0); u \in \mathcal{MR}\}$ is known in literature as *the trace space*. When it is endowed with the norm:

$\|x\|_{Tr} = \inf \{\|u\|_{MR} : x = u(0)\}$., or indifferently $\|x\|_{Tr} = \inf \{\|u\|_{MR} : x = u(t_0)\}$, Tr is a Banach space. This space takes its interest in defining all *admissible* start data for well-posedness of the Cauchy problem:

$$\begin{cases} \dot{u} + Au = 0 \quad \text{a.e. on} \quad (0, T), \\ u(0) = x \end{cases} \tag{2}$$

More precisely, the trace space describes the start datum x as being the image of $\{0\}$ by all solutions of (1). More details are given at the end of this introduction.

It is well known that if the operator has the L^p-maximal regularity as given by (1), then for all $x \in Tr$, the problem (2) is well-posed. Naturally, this important bridge between the forcing problem (1) and the homogeneous one leads to the necessity of characterizing operators with this remarkable property.

de Simon characterized completely the maximal regularity in Hilbertian spaces (see [3]) by establishing that this latter property is equivalent, in any Hilbert space, to generation of holomorphic semigroup. In general spaces, holomorphic generator are not $\mathcal{MR}$. A counterexample was given recently by Fackler [9] generalizing by this way some results and counterexamples related to the fail of maximal regularity in general space such as L^p-spaces are discussed by Kalton and Lancien in [17].
Some classical spaces are associated with the treatment of maximal regularity. The most important of these are the maximal regularity spaces, denoted: $\mathcal{MR}_p(0,T) = W^{1,p}(0,T;X) \cap L^p(0,T;D)$ and the trace space $Tr = \{u(0); u \in \mathcal{MR}\}$, as mentioned above, which is a Banach space isometric to an interpolation space between D and X.

2. Preliminaries

When the operator A is not single and forms a family $A(t)_{t\in[0,T]}$, the definition of maximal regularity is not, in general, reduced to the punctual maximal regularity. Other hypotheses on the map $t \mapsto A(t)$ are required to ensure the well-posedness of the evolution problem (1). The time dependence case (i.e., $A(t)$ varies with time) has recently gained great importance, mainly in the hybrid studies (deterministic and stochastic ones). To illustrate this importance, one may see [16].
Let us recall the definition of maximal regularity in this non-autonomous case:

Definition 2. *Fix a real $p \in (1,\infty)$. The non-autonomous operator $A(t)$ is said to have L^p-maximal regularity (for short, we write $A \in \mathcal{MR}p(0,T)$) if for every $f \in L^p(0,T;X)$ and every $[0,\tau]$ where $0 \le \tau < T$ there exists a unique $u \in MR(0,\tau)$ such that*

$$\begin{cases} \dot{u}(t) + A(t)u(t) = f(t) \quad \text{a.e. on } (0,\tau), \\ u(0) = 0. \end{cases} \tag{3}$$

Although all operators $A(t)$ have the $\mathcal{MR}$ property, additional hypotheses on $t \mapsto A(t)$, as mentioned, are required. For example, Prüss and Schnaubelt established the maximal regularity for parabolic equations in [19] under two hypotheses: norm continuity of $t \mapsto A(t)$ and *individual* maximal regularity for each operator $A(t)$. Arendt et al. (see [4]) generalized this result to bounded and measurable functions $t \mapsto A(t)$ using a notion of *relative continuity*, which consists of a local perturbation of a single regular operator.

Among applications of maximal regularity, the most important, to the best of our knowledge, we cite the integration of the homogeneous Cauchy problem with start data in the interpolation space Tr. It is henceforth worth recalling

Theorem 1. *Let $A(t)_{t\ge 0}$ be a family of maximal regular operators such that*
$A : t \in [0,T] \mapsto \mathcal{L}(D,X)$ *is* relatively continuous.

For all $f \in L^p(0,T;X)$ *the problem:*

$$\begin{cases} \dot{u} + A(t)u(t) = f \text{ a.e on } [0,T] \\ u(0) = x \end{cases} \tag{4}$$

has a unique solution for all $x \in Tr$.

The hypothesis "A is relatively continuous" is replaced by two embedding hypotheses, H_1 and H_2, as detailed in [10].

Our focus is on maximal regularity within the Hilbertian framework, specifically examining perturbation results when A coincides with the Black-Scholes operator. The paper is organized as follows: In the first section, we recall essential results on operators associated with sesquilinear forms and reformulate the form associated with the Black-Scholes operator in an appropriate Hilbert space. We then explore the possibility of perturbing the autonomous operator, mentioning some results on multiplicative perturbations of holomorphic operators. In the final section, we apply these results to generalize the perturbation of the Black-Scholes operator within the context of non-autonomous forms.

3. Black-Scholes Operator

Here we introduce and study the maximal regularity of Black-Scholes operator. We restrict ourselves to Hilbetian case and treat the case of operators associated with forms. For a general theoretical study, the best resource is manifestly [5] since it gives, with *good* notations, a simple bridge between forms and associated operators in Hilbert spaces. The ambient space X is a Hilbert one, denoted henceforth H and the form is defined on a dense subspace V densely embedded into H. For more results on operators which arise from elliptic forms, one may consult fruitfully [22] or [21].
In the following, we recall some facts about the Black-Scholes operator as a prototype of an operator associated with a suitable form in a suitable space. To do so, we refer to the results and findings presented in [10] and [11].

Owing to the known Baillon's result on maximal regularity (see again [7]) of operators acting on spaces of continuous functions, the natural framework will be $L^2(0,+\infty)$. To define suitably the BS-operator, consider $H = L^2(0,+\infty)$. For all $(a,b) \in \mathbb{R}^+ \times \mathbb{R}$ the BS-operator is defined as

$$Au = ax^2D^2u + bxDu - bu \tag{5}$$

where $u \in H^2(0,+\infty)$. We recall Einemann method to determine precisely the form associated with the operator A acting on L^2 as detailed in [10].

Consider the form:

$$\mathfrak{a}(u,v) = (xDu, xDv) + (2a-b)(xDu, v) + b(u,v)$$

with domain

$$V = \{u \in W^{1,1}_{loc}(0, +\infty),\ xDu \in H\}.$$

It is known that V associated with the form $\mathfrak{a}$ is a Hilbert space with inner canonical product:

$$(u, v)_V = (u, v) + (xDu, xDv).$$

and in particular $\|.\|_H \leq \|.\|_V$ which we transcribe in functional words by writing that $V \underset{d}{\hookrightarrow} H$ and saying that V is continuously embedded in H.
The form $\mathfrak{a}$ enjoys suitable properties, and we recall in particular these most important ones.

Proposition 1. [10, *Proposition 3.6*] *The form $\mathfrak{a}$ is continuous, densely defined and elliptic (on H). Moreover, if $b \geq \frac{2}{3}a$ then $\mathfrak{a}$ is coercive.*

By continuity and coercivity of the form $\mathfrak{a}$, one should understand that there exists a constant $M > 0$ and $\omega \in \mathbb{R}$ such that

i): $\forall (u, v) \in V \quad |\mathfrak{a}(u, v)| \leq M\|u\|\|v\|$
ii): $\forall u \in V \quad \mathfrak{a}(u) = \mathfrak{a}(u, u) \geq \omega\|u\|^2$.

These properties of the form $\mathfrak{a}$ ensure that the associated operator $-A$ is a generator of an analytic semigroup on H. Given the Hilbertian framework, the Black-Scholes operator possesses the property of maximal regularity (see [10] for technical proofs).

Now, let us address the non-autonomous case. In [11], it was demonstrated that for suitable constants a and b, and for an appropriate time horizon T, the evolution problem associated with the time-dependent form

$$\mathfrak{a}(t; u, v) = (xDu, xDv) + (2a(t) - b)(xDu, v) + b(u, v)$$

is well posed for the special case of $t \mapsto a(t) = a + t$ provided that the coefficients $a(t)$ and b satisfy some estimation. In the next section, we hope to study a more general case, precisely multiplicative perturbation, and give a category of such coefficients susceptible to ensuring well-posedness and preserving maximal regularity of the associated operator.

4. A New Multiplicative Perturbation of $\mathcal{BS}$ Operator

The general framework of this section, which contains the main results, is the multiplicative perturbation of regular operators. One of the most important papers related to this topic is [6], which addresses the right multiplicative perturbation of non-autonomous L^p maximal regularity. Since we restrict ourselves to one dimension in time, all results can be applied to both left and right perturbations. Specifically, the authors treated the case

$$\begin{cases} \dot{u}(t) + A(t)\mathcal{H}(t)u(t) = f(t) \quad a.e \quad \text{on} \quad [0,T], \\ u(0) = 0. \end{cases} \tag{6}$$

Recently, the obtained results were applied and extended to study a class of non-autonomous boundary control and observation linear systems that are governed by non-autonomous multiplicative perturbations (see [13]). Unfortunately, the tools used therein may not conform to our problem, since in the general Hamiltonian equation treated

$$\dot{x}(t,\xi) = \sum_{k=1}^{n} P_k \frac{\partial^k}{\xi^k}[\mathcal{H}(t,\xi)x(t,\xi)] + P_0\mathcal{H}(t,\xi)x(t,\xi). \quad t \geq 0,\ \xi \in [a,b]$$

does not take in account the term of degeneracy x^2 which appears in (4) when A coincides with $\mathcal{BS}$ operator.

To encounter the degeneracy problem, we restrict the study to characterize coefficients $a(.)$ and $b(.)$ in (5) which are "*admissible*" to preserve the maximal regularity of $\mathcal{BS}$ operator in the non-autonomous case. Precisely, we consider the operator family depending on time given by

$$A(t)u = a(t)x^2D^2u + b(t)xDu - b(t)u \quad \forall t \in [0,T]$$

where

- $a : t \longmapsto a(t)$ denotes, financially speaking, the volatility and hence it is a positive quantity and may be assumed piecewise continuous on $[0,T]$ and globally bounded. That means there exist $m_T > 0$ and $M_T > 0$ such that $m_T \leqslant u(t) \leqslant M_T$, $\forall t \in [0,T]$.
- $b : t \longmapsto b(t)$ denotes the instantaneous rate t assumed to be constant by parts (piecewise) on $[0,T]$. In other words, there is a subdivision $0 = t_0 < t_1 < \cdots < t_n = T$ and $b_k \in \mathbb{R}$ for $0 \leq k \leq n-1$ satisfying:

$$b(t) = \sum_{k=0}^{n-1} b_k.\chi(t)_{[t_k,t_{k+1}[} \quad \forall t \in [0,T[\text{ and } b(T) = b_{n-1}.$$

We have

$$\forall t \in [0,T[\quad ,\exists! k \in [|0,n-1|] \quad tq : t \in [t_k,t_{k+1}[$$

and

$$A(t)u = a(t)x^2D^2u + b_kxDu - b_ku$$

$$A(T)u = a(T)x^2D^2u + b_{n-1}xDu - b_{n-1}u$$

According to proposition (1) in *preliminaries* section above, the form $a(t,.,.)$ associated with the operator $A(t)$ is continuous densely defined on H. In addition,

$$\forall k \in [|0,n-1|]\ ;\ \forall t \in [t_k,t_{k+1}[\quad b_k \geqslant \frac{2}{3}a(t) \ and \ b_{n-1} \geqslant \frac{2}{3}a(T)$$

then $a(t,.,.)$ is coercive.

As stated in [10], the operators $(A(t))_{t\in[0,T]}$ share the same domain $D = D(t) = H^2(0,+\infty)$ and the correspondent forms are all defined on the same domain

$$V = \left\{u \in W^{11}_{loc}(0,+\infty); xDu \in H\right\}$$

It is a direct application of the autonomous case for a frozen time in $[0,T]$. Henceforth, the punctual maximal regularity of $(A(t))_{t\in[0,T]}$ is proved. Since $t \mapsto \mathfrak{a}(t,.,.)$ is piecewise continuous, it is enough to prove the global coercivity of the non-autonomous form $\mathfrak{a}$. To do so, we proceed as in the proof of proposition (1)

$$\begin{aligned} a(t,u,v) &\leqslant (3a(t) + 2 \mid b_k \mid) \|u\|\|v\| \\ &\leqslant (3M_T + 2N_T) \|u\|\|v\| \quad \text{where } N_T = max_{0\leq k\leq n-1} \mid b_k \mid \end{aligned}$$

Denoting $M = 3M_T + 2N_T$ then:

$$\forall (u,v) \in V^2; \forall t \in [0,T] \quad a(t,u,v) \leqslant M\|u\|\|v\|.$$

The continuity of $a(t,.,.)$ is then proved.

A similar technical calculus used again in (1) enables us to establish its coercivity

$$\begin{aligned} &\forall u \in V; \forall k \in [|0,n-1|] \text{ and } \forall t \in [t_k, t_{k+1}[\\ &\mathfrak{a}(t,u,u) = (xDu, xDu) + (2a(t) - b_k)(xDu,u) + b_k(u,u) \\ &\qquad = \|xDu\|_H^2 - \frac{(2a(t)-b_k)}{2}\|u\|_H^2 + b_k\|u\|_H^2 \text{ since } (xDu,u) = \frac{-\|u\|_H^2}{2} \\ &\qquad = \|xDu\|_H^2 + \frac{3b_k - 2a(t)}{2}\|u\|_H^2 \\ &\qquad \geqslant \|xD_u\|_H^2 + \frac{3m_T - 2M_T}{2}\|u\|_H^2 \qquad \text{where } m_T = \min_{0\leqslant k\leq n-1} b_k, \end{aligned}$$

which holds for every $0 \leq t \leq T$.

These results allow stating the well-posedness of the problem. They translate otherwise, that the non-autonomous operator $\mathcal{BS}$ of Black-Scholes has the property of maximal regularity provided that $(A(t))_{t\in[0,T]}$ satisfies la condition $\frac{3m_T}{2} - M_T \geqslant 0$ and b is any step function. Moreover, under these hypotheses the form $a(t,\cdot,\cdot)$ remains positive independently on time. This property is financially crucial to avoid important loss in portfolio transactions, as explained at the conclusion of [11].

In more clear words, among other possible multiplicative admissible perturbations, the functions that have the same regularity as a and b are good candidates. The question is:" Are they the only admissible ones?"

Remark. In fact, one might be tempted to justify the preservation of the maximal regularity property using a direct computation. It seems easier to prove that $t \mapsto A(t)$ is *relatively continuous*. However, it is prudent to avoid such seemingly straightforward calculations, as the norm on V is not easily managed. Thus, the method presented is both elementary and effective. A numerical treatment will be provided in our future works.

5. Overcoming Degeneracy by Suitable Perturbation

In this section we come back to the initial model where $a(.)$ and $b(.)$ are considered in a general case, say $\mathcal{C}^1$ functions. To overcome the degeneracy, we consider the sequence of problems $(\mathcal{P}_n)_{n\geq 1}$ given by:

$$(\mathcal{P}_n): \quad Au = a(x^2 + \frac{1}{n})D^2u + b_n xDu - b_n u. \tag{7}$$

It is easy to see that $(\mathcal{P}_n)$ is equivalent to:

$$(\mathcal{Q}_n): \quad Au = D\big(\alpha_n(x)Du(x)\big) + \beta_n(x)u(x). \tag{8}$$

for some suitable functions $\alpha_n(.)$ and $\beta_n(.)$. The form $\mathfrak{a}_n$ that governs the problem $(\mathcal{Q}_n)$ may be written:

$$\mathfrak{a}_n(u,v) = \int_0^T \alpha_n'(x)u'(x)v'(x)dx + \int_0^T \beta_n(x)u(x)v(x)dx$$

which leads (but also results from) naturally, for every $f \in L^2[0,T]$ and every $u, v \in H_0^1[0,T]$ to the variational problem:

$$\int_0^T \alpha_n'(x)u'(x)v'(x)dx + \int_0^T \beta_n(x)u(x)v(x)dx = \int_0^T f(x)v(x)dx.$$

Thanks to compacity of finite horizon $[0,T]$ and to continuity of the coefficient α_n' and β_n', it is so easy to establish that $\mathfrak{a}_n$ is continuous
(i.e. $\forall (u,v) \in H_0^1 : \ \mathfrak{a}_n(u,v) \leq M_n\|u\|_{H_0^1}\|v\|_{H_0^1}$). To apply Lax-Milgram theorem immediately, one must establish the ellipticity property (strict coerciveness). To this end, consider $w \in H_0^1$ and remark that Poincaré inequality allows this

$$\mathfrak{a}_n(w,w) = \mathfrak{a}_n(w) = \int_0^T \alpha_n'(x)w'(x)^2dx + \int_0^T \beta_n(x)w(x)^2dx \geq K_n\|w\|^2_{H_0^1}$$

where $K_n = c\min_{x\in[0,T]}\{\alpha_n(x)\}$ for some strictly positive constant $c > 0$, so also $K_n > 0$. Then $\mathfrak{a}_n$ is an elliptic form for all integer $n \in \mathbb{N}$.

All hypothesis to apply Lax-Milgram theorem are satisfied, then there exists a solution u_n that solves uniquely the approximate problem $(\mathcal{Q}_n)$ or equivalently the problem $(\mathcal{P}_n)$. Since $\|u_n\|_{H_0^1} \leq K\|f\|_{L^2}$, a classical extraction argument completes the claim.

The results obtained will be crowned by a numerical treatment using recent numerical and statistical efficient concepts, mainly *copulas* as initiated by R. Nelsen and explained and improved, for this purpose, in [23] and [8].

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] M. Achache, E.M. Ouhabaz, Lions' maximal regularity problem with $H^{\frac{1}{2}}$-regularity in time, J. Diff. Equ. 266 (2019), 3654–3678. https://doi.org/10.1016/j.jde.2018.09.015.

[2] M. Akkouchi, M. Houimdi, My.H.L. Rhali, A theoretical framework for two-parameter semigroups, Gulf J. Math. 7 (2019), 1–17. https://doi.org/10.56947/gjom.v7i1.14.

[3] H. Amann, Maximal regularity for nonautonomous evolution equations, Adv. Nonlinear Stud. 4 (2004), 417–430. https://doi.org/10.1515/ans-2004-0404.

[4] W. Arendt, R. Chill, S. Fornaro, C. Poupaud, L^p-maximal regularity for non-autonomous evolution equations, J. Diff. Equ. 237 (2007), 1–26. https://doi.org/10.1016/j.jde.2007.02.010.

[5] W. Arendt, A.F.M. ter Elst, From forms to semigroups, in: W. Arendt, J.A. Ball, J. Behrndt, K.H. Förster, V. Mehrmann, C. Trunk (Eds.), Spectral Theory, Mathematical System Theory, Evolution Equations, Differential and Difference Equations, Springer Basel, Basel, 2012: pp. 47–69. https://doi.org/10.1007/978-3-0348-0297-0_4.

[6] B. Augner, B. Jacob, H. Laasri, On the right multiplicative perturbation of non-autonomous L^p-maximal regularity, J. Oper. theory 74 (2015), 391–415. https://www.jstor.org/stable/24718162.

[7] B. Eberhardt, G. Greiner, Baillon's theorem on maximal regularity, in: C.B. Huijsmans, W.A.J. Luxemburg (Eds.), Positive Operators and Semigroups on Banach Lattices, Springer Netherlands, Dordrecht, 1992: pp. 47–54. https://doi.org/10.1007/978-94-017-2721-1_5.

[8] M. El Maazouz, A. Sani, New asymmetric perturbations of FGM bivariate copulas and concordance preserving problems, Moroccan J. Pure Appl. Anal. 9 (2023), 111–126. https://doi.org/10.2478/mjpaa-2023-0008.

[9] S. Fackler, The Kalton–Lancien theorem revisited: Maximal regularity does not extrapolate, J. Funct. Anal. 266 (2014), 121–138. https://doi.org/10.1016/j.jfa.2013.09.006.

[10] C. Hilmi, A. Sani, Maximal regularity for non autonomous evolution equations and Black-Scholes operator, Gulf J. Math. 11 (2021), 73–82. https://doi.org/10.56947/gjom.v11i1.670.

[11] C. Hilmi, A. Sani, S. Elmourchid, Invariance of convex sets: An alternative proof and application to Black-Scholes operator, Gulf J. Math. 13 (2022), 101–109. https://doi.org/10.56947/gjom.v13i1.929.

[12] T. Hossni, A. Mahdi, Maximal regularity for semilinear non-autonomous evolution equations in temporally weighted spaces, Arab. J. Math. 11 (2022), 539–547. https://doi.org/10.1007/s40065-022-00390-0.

[13] B. Jacob, H. Laasri, Well-posedness of infinite-dimensional non-autonomous passive boundary control systems, arXiv:1901.11348 [math.FA] (2019). https://doi.org/10.48550/ARXIV.1901.11348.

[14] Y. Kharou, On the admissibility of observation operators for evolution families, arXiv:2109.10069 [math.OC] (2021). http://arxiv.org/abs/2109.10069.

[15] K.J. Engel, R. Nagel, One-parameter semigroups for linear evolution equations, Springer-Verlag, New York, 2000. https://doi.org/10.1007/b97696.

[16] E. Lakhel, B. Boufoussi, A. Tlidi, Neutral stochastic functional differential evolution with varying-time delays driven by Rosenblatt process in Hilbert spaces, Gulf J. Math. 6 (2018), 89–103. https://doi.org/10.56947/gjom.v6i1.127.

[17] N. Kalton, G. Lancien, A solution to the problem of L^p-maximal regularity, Math. Z. 235 (2000), 559–568. https://doi.org/10.1007/PL00004816.

[18] H. Neidhardt, A. Stephan, V.A. Zagrebnov, Trotter product formula and linear evolution equations on Hilbert spaces, in: T.M. Rassias, V.A. Zagrebnov (Eds.), Analysis and Operator Theory, Springer, Cham, 2019: pp. 271–299. https://doi.org/10.1007/978-3-030-12661-2_13.

[19] J. Prüss, R. Schnaubelt, Solvability and maximal regularity of parabolic evolution equations with coefficients continuous in time, J. Math. Anal. Appl. 256 (2001), 405–430. https://doi.org/10.1006/jmaa.2000.7247.

[20] S.G. Pyatkov, On the maximal regularity property for evolution equations, Azerbaijan J. Math. 9 (2019), 137–155.

[21] A. Sani, Intégrale produit et régularité maximale du problème de Cauchy non autonome, Ph.D. thesis, Université Ibn Zohr, 2014.

[22] A. Sani, H. Laasri, Evolution equations governed by Lipschitz continuous non-autonomous forms, Czech. Math. J. 65 (2015), 475–491. https://doi.org/10.1007/s10587-015-0188-z.

[23] A. Sani, L. Karbil, Functional treatment of asymmetric copulas, arXiv:1907.05915 [math.PR] (2019). http://arxiv.org/abs/1907.05915.