STABILIZATION OF A SYSTEM OF SCHRÖDINGER EQUATIONS WITH VARIABLE COEFFICIENTS AND DAMPED BY MEMORY BOUNDARY FEEDBACK

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Abstract. We study the energy decay for a coupled system of two complex Schrödinger equations with variable coefficients and damped by memory boundary feedback conditions. The aim of this paper is to prove that we can apply the Riemann geometric approach developed to study the problems of direct stabilization for wave equations (see [10]) and show that the sufficiently smooth solutions decays polynomially at infinity, by adapting the ideas of Alabau in [2] used to obtain indirect stabilization results for a system of two coupled wave equations with constant coefficients.

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1. Introduction

Let Ω be an open bounded domain in $\mathbb{R}^n$ with boundary $\Gamma := \partial\Omega$. It is assumed that Γ consists of two parts Γ_0 and Γ_1 such that $\Gamma_0, \Gamma_1 \neq \emptyset$, $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. Given $T > 0$, let $Q = \Omega\times]0,T[, \Sigma = \Gamma\times]0,T[$ and $\Sigma_l = \Gamma_l\times]0,T[(l = 0,1)$. In Ω, we consider the following coupled complex valued Schr odinger equations with variable coefficients:

$$\begin{aligned}
&iy_t + Ay + az = 0 \;\; in \;\; Q,\\
&iz_t + Az + ay = 0 \;\; in \;\; Q,\\
&y = 0 \;\; on \;\; \Sigma_0, \;\; \frac{\partial y}{\partial \nu_A} = -\int_0^t k'(t-s)y(s)ds - k(0)y(t) - by_t \;\; on \;\; \Sigma_1 \;\; and \;\; z = 0 \;\; on \;\; \Sigma,\\
&y(x,0) = y_0 \;\; and \;\; z(x,0) = z_0 \;\; in \;\; \Omega,
\end{aligned} \tag{1}$$

where

$$Ay = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial y}{\partial x_j}\Big),$$

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is a second order differential operator with real coefficients $a_{ij} = a_{ji}$ of class C^∞ and satisfies the uniform ellipticity condition

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\overline{\xi_j} > a_0 \sum_{i=1}^{n} \xi_i^2 \quad \forall x \in \Omega,\ \zeta \in \mathbb{R}^n,\ \zeta \neq 0, \tag{2}$$

for some positive constant $a_0 > 0.\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ is the outward unit normal to Γ, $\frac{\partial y}{\partial \nu_A} = \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial y}{\partial x_i}\nu_i$, denotes the co-normal derivative with respect to A.
where $k : \Gamma_1 \times \mathbb{R}_+ \to \mathbb{R}_+ \in C^2(\mathbb{R}^+, L^\infty(\Omega))$ and a and b are two functions in $L^\infty(\Omega)$ such that for some constants $a_\star, b^\star > 0$, we have $a_\star \leq a(x)$ for all $x \in \Omega$ and $b(x) \leq b^\star$ for all $x \in \Gamma$.
The stabilization of partial differential equations has been considered by many authors ([6], [2], [5], [10]). Recently, Yao has introduced the Riemann geometric method to study the problem of exact controllability of real valued wave, Euler-Bernoulli and Shalow Shells equations with variable coefficients see ([12]). The authors have used this approach to establish observability estimates for vector valued Maxwell's system with variable coefficients. Using this approach, several papers were devoted to the stability of variable systems (see [4], [6]). More recently, the wave equation with memory and nonlinear feedbacks with constant coefficients has been studied by [9]. This study has been generalised by Chai and Guo [10] for variable coefficients by using a very different method, namely, the Riemannian geometry method. But the same problem was treated by sevral authors using arguments of differential geometry, on the multiplier method and the introduction of appropriate Lyapunov functionals of one wave equation.
On the other hand, the stabilization of one Schrödinger equations with constant coefficients has been studied by Machtyngier and Zuazua [11] in the Neumann boundary conditions, and by auther authors with nonlinear feedbacks. This study has been considered and with constant coefficients acting in the Dirichlet boundary conditions. The asymptotic behaviour of the Schrödinger equation with memory and linear feedbacks with variable coefficients has been studied by Abdesselam and Melkemi [5].
The goal of this work is to prove that we can apply the Riemann geometric on $\mathbf{C^n}$ approach to the coupled complex Schr odinger equations with variables coefficients and show that we can obtain the indirect boundary stabilization of this system, by adapting the method of Alabau developed in the context of coupled real wave equations with constant coefficients. We note here that the coupling coefficient $||a||_{L_\infty(\Omega)}$ is considered as a function with sufficiently small.
The rest of this paper is organized as follows. In Section 2, we present some assumptions and material needed for our work and give the well posedness results of our two systems. Some technical lemmas are presented and proved in Section 3. Finally, we state and prove our main decay results.
To obtain our result, we need some geometric assumptions. The approach adopted uses Riemannian geometry. This method was first introduced to boundary-control problems by Yao [12] for the exactly controllability of wave equations.

2. Preliminary Notes

2.1. **Riemannian metric.** For each $x \in \mathbb{R}^n$, define the inner product and the corresponding norm on the tangent space $T_x\mathbb{R}^n$ by

$$g(X,Y) = \langle X, Y\rangle_g = X \cdot G(x)Y = \sum_{i,j=1}^{n} g_{ij}(x)\alpha_i\beta_j$$

$$|X|_g^2 = \langle X, Y\rangle_g \quad \text{for } X = \sum_{i=1}^{n}\alpha_i\frac{\partial}{\partial x_i}, \quad Y = \sum_{i=1}^{n}\beta_i\frac{\partial}{\partial x_i} \in T_x\mathbb{R}^n.$$

Then $(\mathbb{R}^n, g)$ is a Riemannian manifold with a Riemannian metric g. Denote the Levi-Cevita connection in metric g by D. Let H be a vector field on $(\mathbb{R}^n, g)$. The covariant differential DH of H determines a bilinear form on $T_x\mathbb{R}^n \times T_x\mathbb{R}^n$. For each $x \in \mathbb{R}^n$, by

$$DH(X,Y) = \langle D_X H, Y\rangle_g, \forall X, Y \in T_x\mathbb{R}^n$$

where $D_X H$ is the covariante derivative of H with respect to X. The following lemma provides some useful equalities.

Lemma 1 [12]
Let f, $h \in C^1(\overline{\Omega})$ and let H, X be a vector field on $\mathbb{R}^n$. Then using the above notation, we have

(i)
$$\langle H(x), A(x)X(x)\rangle_g = H(x)X(x), \quad \forall x \in \mathbb{R}^n \tag{3}$$

(ii) The gradient $\nabla_g f$ of f in the Riemannian metric g is given by
$$\nabla_g f(x) = \sum_{i=1}^{n}\Big(\sum_{j=1}^{n} a_{ij}(x)\frac{\partial f}{\partial x_j}\Big)\frac{\partial}{\partial x_i} = A(x)\nabla_0 f. \tag{4}$$

(iii)
$$\frac{\partial y}{\partial \nu_A} = (A(x)\nabla_0 y).\nu = \nabla_g y.\nu. \tag{5}$$

(iv)
$$\langle \nabla_g f, \nabla_g H\rangle_g = \nabla_g f(h) = \nabla_0 f \cdot A(x)\nabla_0 h. \tag{6}$$

(v) An identity. If $f \in C^1(\Omega)$ then
$$\langle \nabla_g f, \nabla_g H(f)\rangle = DH(\nabla_g f, \nabla_g f) + \frac{1}{2}\operatorname{div}_0(|\nabla_g f|_g^2 H)(x) - \frac{1}{2}|\nabla_g f|_g^2 \operatorname{div}_0(H) \quad x \in \mathbb{R}^n. \tag{7}$$

(vi)
$$\begin{aligned} Ay &= -\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial y}{\partial x_j}) \\ &= -\operatorname{div}_0(A(x)\nabla_0 y) = -\operatorname{div}_0(\nabla_g y), \quad y \in C^2(\Omega). \end{aligned} \tag{8}$$

In all this work, we give the counterpart of the Green's formula and the identity (v) for complex valued functions, the following lemma gives further relationships in $\mathbb{C}^n$.

Lemma 2

Let f be a complex valued function and H be a vector field on $(\mathbb{R}^n, g)$. We put

$$H(f) := H(Ref) + iH(Imf) \ \ and \ \ \nabla_g f := \nabla_g Ref + i\nabla_g Imf. \tag{9}$$

Let f_1, f_2 be complex valued functions in $H^2(\Omega)$. Then

$$Re\int_\Omega Af_1 f_2 = Re\int_\Omega \langle \nabla_g f_1, \nabla_g f_2\rangle_g - Re\int_\Gamma \frac{\partial f_1}{\partial \nu_A}\bar{f}_2. \tag{10}$$

Let f be a complex valued function in $C^1(\Omega)$ and H a vector field on $(\mathbb{R}^n, g)$. Then

$$\begin{aligned} &\langle \nabla_g f, \nabla_g H(f)\rangle = DH(\nabla_g Ref, \nabla_g Ref) + DH(\nabla_g Imf, \nabla_g Imf) \\ &+ \frac{1}{2}\operatorname{div}_0(||\nabla_g f||_g^2 H)(x) - \frac{1}{2}||\nabla_g f||_g^2 \operatorname{div}_0(H), \quad x \in \mathbb{R}^n. \end{aligned} \tag{11}$$

We can see that there exist two positive constants α_1 and α_2 such that

$\forall y \in H_1(\Omega)$ we have $\alpha_1, \alpha_2 \in \Omega$

$$\alpha_1\int_\Omega |\nabla_0 y|_g^2 \leq \int_\Omega |\nabla_g y|_g^2 = \int_\Omega Re|\nabla_g y|_g^2 + \int_\Omega Im|\nabla_g y|_g^2 \leq \alpha_2\int_\Omega |\nabla_0 y|_g^2. \tag{12}$$

Let C_1 and C_2 are the positive constants such that

$$\int_\Omega |f|^2 \leq C_1^2\int_\Omega |\nabla_g y|_g^2 \ \ and \ \ \int_{\Gamma_1} |f|^2 \leq C_2^2\int_\Omega |\nabla_g y|_g^2. \tag{13}$$

For all functions f of $H^1_{\Gamma_0}(\Omega)$, C is a generic positive constant which does not depend on the initial data.

2.2. **Statement of main result.** To obtain the boundary stabilization of problem, the following assumptions are made to state our main result. Assume that there exists a real vector field $H \in [C^1(\bar{\Omega})]^n$ on the Riemannian manifold $(\mathbb{R}^n, g)$ a constant $m_0 > 0$ such that

$$DH(X, X) \geq m_0||X||_g^2, \forall X \in \mathbb{R}^n_x, \tag{14}$$

and

$$2m_0 > C_1 C_H, \tag{15}$$

where $C_H = sup_{x\in\Omega}||\nabla_g(div_0 H)||_g$.

$$H(x)\cdot\nu < 0 \quad \text{on } \Gamma_0, \tag{16}$$

$$H(x)\cdot\nu \geq 0 \quad \text{on } \Gamma_1, \tag{17}$$

$$\exists\delta > 0, \ k'' \geq -\delta k' \quad \text{on } \Gamma_1 \times \mathbb{R}_+, \tag{18}$$

$$k' \leq 0 \quad \text{on } \Gamma_1 \times \mathbb{R}_+, \tag{19}$$

$$\varphi = \inf_{(x,t)\in\Gamma_1\times\mathbb{R}^+}(-k') \neq 0. \tag{20}$$

We have the following result of existence and uniqueness of weak solution to problem (1).

Theorem 1

Let

$$\mathbf{A} : D(A) \in H^1_{\Gamma_0}(\Omega) \times H^1_0(\Omega) \to H^1_{\Gamma_0}(\Omega) \times H^1_0(\Omega)$$

be the operator defined by $\mathbf{A}(y,z) = (iAy + iaz, iAz + iay)$ where
$D = D(\mathbf{A}) = \{(y,z) \in H^1_{\Gamma_0}(\Omega) \times H^1_0(\Omega) | (Ay, Az) \in H^1_{\Gamma_0}(\Omega) \times H^1_0(\Omega)$ *and* $\frac{\partial y_0}{\partial \nu_A} = -\int_0^t k'(t-s)y(s)ds - k(0)y(t) - by_t\}$. For all initial data $(y_0, z_0) \in H^1_{\Gamma_0}(\Omega) \times H^1_0(\Omega)$ the system (1) has a unique solution $(y,z) \in C(\mathbb{R}^+; H^1_{\Gamma_0}(\Omega) \times H^1_0(\Omega))$ and that if $(y_0, z_0) \in D$ the system (1) has a unique solution

$$(y,z) \in C(\mathbb{R}^+; D) \cap C^1(\mathbb{R}^+; H^1_{\Gamma_0}(\Omega) \times H^1_0(\Omega)).$$

Proof. we can using the Faedo–Galerkin method. □

3. Main Results and the Proofs

We give our main result and the corresponding proofs in this section. Consider the total energy E of the system defined by, for all $t > 0$

$$E(t) = E(y(t), z(t)) = E_1(y(t) + E_2(z(t)) + Re \int_\Omega ay\bar{z}d\Omega.$$

$$E_1(y(t)) = \frac{1}{2}\int_\Omega |\nabla_g y|_g^2 d\Omega + \frac{1}{2}\int_{\Gamma_1} k|y|^2 \, d\Gamma_1 - \frac{1}{2}\int_0^t \int_{\Gamma_1} k'(t-s)|y(t) - y(s)|^2 \, d\Gamma_1 ds.$$

$$E_2(z(t)) = \frac{1}{2}\int_\Omega |\nabla_g z|_g^2 \, d\Omega.$$

We can see that E is equivalent to $E_1 + E_2$ when we take $||a||_{L_\infty(\Omega)}$ sufficiently small. The dissipative property of the sollution of the system is given by the following lemma.

Lemma 3

For all $t > 0$, we have

$$E'(t) = -\int_{\Gamma_1} b|y_t|^2 \, d\Gamma_1 + \frac{1}{2}\int_{\Gamma_1} k'|y|^2 \, d\Gamma_1 - \frac{1}{2}\int_0^t \int_{\Gamma_1} k''(t-s)|y(t) - y(s)|^2 \, d\Gamma_1 ds \le 0.$$

Proof. We multiply both side the first equation of (1) by $\bar{y}_t$, integrate over Ω, take the real part, use the third Green's formulla, finally we use the boundary condition, we find

$$\begin{aligned} &Re \int_\Omega \langle \nabla_g y_t, \nabla_g y \rangle_g d\Omega + \frac{1}{2}\int_{\Gamma_1} k'|y|^2 \, d\Gamma_1 \\ &+ \mathrm{Re}\int_{\Gamma_1} k\overline{y_t}y \, d\Gamma_1 - \frac{1}{2}\int_0^t \int_{\Gamma_1} k''(t-s)|y(t) - y(s)|^2 \, d\Gamma_1 ds \\ &- \mathrm{Re}\int_0^t \int_{\Gamma_1} k'(t-s)\overline{y_t}(t)(y(t) - y(s)) \, d\Gamma_1 ds + Re\int_\Omega az\bar{y}_t d\Omega = 0 \end{aligned}$$

We observe
$$\text{Re}\int_{\Gamma_1} y\overline{y_t}(k-k(0)-\int_0^t k'(t-s)ds)\,d\Gamma_1=0.$$
We obtain similar identity for z
$$Re\int_\Omega \langle\nabla_g z_t,\nabla_g z\rangle_g d\Omega + Re\int_\Omega az\bar{y}_t d\Omega=0.$$
But
$$E'(t)=Re\int_\Omega \langle\nabla_g y_t,\nabla_g y\rangle_g d\Omega + Re\int_\Omega \langle\nabla_g z_t,\nabla_g z\rangle_g d\Omega + Re\int_\Omega a(y\bar{z})_t d\Omega.$$
Then we find the result. □

Remark

We deduce from the precident Lemma that

$$E(T)\le E(0). \tag{21}$$

and
$$\int_{\Sigma_1}|y_t|^2\le CE(0). \tag{22}$$
Our main result is

Theorem 2

Let $N\ge 1$. For any initial data $(y_0,z_0)\in D(A^N)$, the energy E of the solution of system (1) decays polynomially:
$$E(y(t),z(t))\le \frac{C}{t^N}\sum_{p=0}^{p=N}E(y^{(p)}(0),z^{(p)}(0))\quad \text{for all}\quad t>0.$$

Proof. To prove our result we estimate $\int_0^T E_1(t)$ and $\int_0^T E_2(t)$ then, after summing up these two estimates, we conclude applying the Theorem 3.1 in [2] with $K=1$.

Step I

We prove an estimate useful to estimate the term $\int_0^T E_1(t)$.

For fixed t, we consider w the solution of the problem
$$Aw=0\quad\text{in}\quad \Omega, w=y\quad\text{on}\quad \Gamma.$$
Using elliptic regularity (Lemma 2.1 in [6]), we can see that
$$\int_\Omega |w|^2\le C\int_\Gamma |y|^2\le CE_1(t),$$
and
$$\int_Q |w_t|^2\le C\int_{\Sigma_1}|y_t|^2\le CE(0). \tag{23}$$
On the other hand, we have $Re\int_\Omega Aw\bar{z}=0\Rightarrow Re\int_\Omega\langle\nabla_g w_t,\nabla_g z\rangle_g d\Omega=0.$ Multiplying the conjugate of the second equation of (1) by $(y-w)$, integrating over Q and taking the real part

$$Im\int_Q \bar{z}_t(y-w)+Re\int_Q A\bar{z}(y-w)+Re\int_Q a|y|^2-Re\int_Q a\bar{y}w=0$$

then, by (Lemma 2) and the integration by parts, we obtain

$$\Im\left(\int_\Omega \bar{z}(y-w)\Big|_0^T\right)+\Re\left(\int_\Omega A\bar{z}(y_t-w_t)\right)+\Re\left(\int_Q \langle\nabla_g z_t,\nabla_g y\rangle_g\,d\Omega\right)+\Re\left(\int_Q a|y|^2\right)-\Re\left(\int_Q a\bar{y}w\right)=0,$$

and multiply the first equation of (1) by $(-\bar{z})$, integrate over Q,take the real part, we find

$$|\int_\Omega \bar{z}(y-w)|\leq C(\int_\Omega |y|^2+\int_\Omega |z|^2+\int_\Omega |w|^2)\leq CE(0),$$

then, for all $\epsilon>0$ and $a\ll$, we have

$$\int_Q a|z|^2\leq CE(0)+\frac{\epsilon}{2a_\star}\int_Q a|z|^2+\frac{1}{2\epsilon}\int_Q a|w_t|^2+C||a||_{L_\infty(\Omega)}\int_0^T E_1(t)$$

Using (23) and choosing $\epsilon=a_\star$, we find

$$\int_Q a|z|^2\leq CE(0)+C||a||_{L_\infty(\Omega)}\int_0^T E_1(t). \tag{24}$$

Step II

An estimate of the term $\int_0^T E_1(t)$. Multiplying the first equation by $2H(\bar{y})+div_0H\bar{y}$, integrating over Q and taking the real part, we obtain

$$Im\int_Q \bar{z}_t(2H(\bar{y})+div_0H\bar{y})+Re\int_Q A\bar{z}(2H(\bar{y})+div_0H\bar{y})-Re\int_Q a\bar{y}(2H(\bar{y})+div_0H\bar{y})=0. \tag{25}$$

By integration by parts, we have

$$\int_Q y_tH(\bar{y})=\int_\Omega yH(\bar{y})|_0^T-\int_Q yH(\bar{y}_t).$$

Invoke the standard divergence identity, we obtain

$$\int_Q y_tH(\bar{y})=\int_\Omega yH(\bar{y})|_0^T-\int_\Sigma H.\nu y\bar{y}_t+\int_Q \bar{y}_t div_0 Hy+\int_Q \bar{y}_tH(y).$$

Take the imaginaire, so we have

$$Im\int_Q y_t(2H(\bar{y})+div_0H\bar{y})=Im\int_\Omega yH(\bar{y})|_0^T-Im\int_{\Sigma_1} H.\nu y\bar{y}_t. \tag{26}$$

If we use Lemma 1, we find

$$Re\int_Q Ay(2H(\bar{y})+div_0H\bar{y})=-Re\int_\Sigma \frac{\partial y}{\partial\nu_A}(2H(\bar{y})+div_0H\bar{y})+2\int_Q Re\langle\nabla_g y,\nabla_g(H(y))\rangle_g$$
$$+\int_Q Re\langle\nabla_g y,\nabla_g(div_0Hy)\rangle_g.$$

Indeed, from the identity we obtain

$$Re\int_Q Ay(2H(\bar{y})+div_0H\bar{y})=-Re\int_\Sigma \frac{\partial y}{\partial\nu_A}(2H(\bar{y})+div_0H\bar{y})$$

$$+2\int_Q DH(\nabla_g Rey, \nabla_g Rey) + 2\int_Q DH(\nabla_g Imy, \nabla_g Imy) + \int_Q H.\nabla_0(||\nabla_g y||_g^2)$$

$$+Re\int_Q \langle \nabla_g y, \nabla_g(div_0 Hy)\rangle_g + \int_Q div_0 H||\nabla_g y||_g^2. \tag{27}$$

Recalling the boundary condition on Γ. Since we have $Rey = Imy = 0$ on Γ_0, then we have, (see [12])

$$H(Rey) = \frac{H\cdot\nu}{||\nu_A(x)||_g^2}(\frac{\partial Rey}{\partial \nu_A}) \;\; and \;\; ||\nabla_g Rey||_g^2 = \frac{1}{||\nu_A(x)||_g^2}(\frac{\partial Rey}{\partial \nu_A})^2,$$

and

$$H(Imy) = \frac{H\cdot\nu}{||\nu_A(x)||_g^2}(\frac{\partial Imy}{\partial \nu_A}) \;\; and \;\; ||\nabla_g Imy||_g^2 = \frac{1}{||\nu_A(x)||_g^2}(\frac{\partial Imy}{\partial \nu_A})^2.$$

So

$$H(y) = \frac{H\cdot\nu}{||\nu_A(x)||_g^2}(\frac{\partial y}{\partial \nu_A}) \;\; and \;\; ||\nabla_g y||_g^2 = \frac{1}{||\nu_A(x)||_g^2}|\frac{\partial y}{\partial \nu_A}|^2.$$

Then

$$Re\int_Q Ay(2H(\bar{y}) + div_0 H\bar{y}) = -Re\int_\Sigma \frac{\partial y}{\partial \nu_A}(2H(\bar{y}) + div_0 H\bar{y})$$

Finally, we insert (26) and (27) in (25) to obtain

$$2(\int_Q Dh(\nabla_g Rey, \nabla_g Rey) + \int_Q Dh(\nabla_g Imy, , \nabla_g Imy)) = I_\Omega + I_{\Gamma_0} + I_Q + I_{\Gamma_1}.$$

where

$$I_\Omega = Im\int_\Omega yH(\bar{y})|_0^T,$$

$$I_{\Gamma_0} = \int_{\Gamma_0} \frac{H\cdot\nu}{||\nabla_g y||_g^2}|\frac{\partial y}{\partial \nu_A}|^2,$$

$$I_Q = -Re\int_Q \langle \nabla_g y, \nabla_g(div_0 H\rangle_g \bar{y} - Re\int_Q az(2H(\bar{y}) + div_0 H\bar{y}),$$

$$I_{\Gamma_1} = -Im\int_{\Gamma_1} H.\nu y\bar{y}_t - \int_{\Gamma_1} H.\nu||\nabla_g y||_g^2 + Re\int_{\Gamma_1}(\frac{\partial y}{\partial \nu_A})(2H(\bar{y}) + div_0 H\bar{y}).$$

We can see that by (21)

$$I_\Omega \le CE(0).$$

We have

$$I_{\Gamma_0} \le 0.$$

We also have, for all $\eta > 0$,

$$I_Q \le CE(0) + C\eta\int_0^T + C||a||_{L_\infty(\Omega)} + \int_0^T E_1(t).$$

Step III

Next we estimate $(\frac{\partial y}{\partial \nu_A})^2$ using some idea from [10].

$$\left(\frac{\partial y}{\partial \nu_A}\right)^2 = \left(-\int_0^t k'(t-s)\,y(s)\,ds - k(0)\,y(t) - by_t\right)^2$$

$$= b^2 \left|y_t\right|^2 + \left(-\int_0^t k'(t-s)\, y(s)\, ds\right)^2$$

$$+2k(0)\, y(t) \int_0^t k'(t-s)\, y(s)\, ds + \left(k(0)\, y(t)\right)^2 + 2by_t\left(\int_0^t k'(t-s)\, y(s)\, ds + k(0)\, y(t)\right).$$

Let us exploit the algebraic inequality, we have

$$\left|\int_{\Sigma_1} \left(\frac{\partial y}{\partial \nu_A}\right)^2 d\Sigma_1\right| \leq 2\left[\int_{\Sigma_1} \left(-\int_0^t k'(t-s)\, y(s)\, ds\right)^2 d\Sigma_1 + \int_{\Sigma_1} k(0) \left|y\right|^2 d\Sigma_1 + \int_{\Sigma_1} b^2 \left|y_t\right|^2 d\Sigma_1\right].$$

We have:

$$\int_{\Sigma_1} k(0) \left|y\right|^2 \leq CE(0)$$

and

$$\int_{\Sigma_1} b^2 \left|y_t\right|^2 \leq b^\star E_1(t).$$

we will increase the following term

$$\int_{\Sigma_1} \left|-\int_0^t k'(t-s)\, y(s)\, ds\right|^2$$

Let $e > 0$ verifying $e \inf_{\Gamma_1} k' + 1 > 0$ and posing

$$h(x) = \frac{k(0)}{\delta\left(1 + ek'(0)\right)} \qquad x \in \Gamma_1.$$

Condition (23) implies that $h \geq 0$ et $h \in L^\infty(\Gamma_1)$. Note that

$$I = \left(-\int_0^t k'(t-s)\, y(s)\, ds\right)^2 - h \int_0^t k''(t-s) \left|y(t) - y(s)\right|^2 ds + hk'y^2$$

Applying the inequality of Hölder, we find

$$I \leq \left(\int_0^t -k'(t-s)\, ds\right)\left(\int_0^t -k'(t-s)\, y^2(s)\, ds\right) - h \int_0^t k''(t-s)\, z^2(s)\, ds$$

$$+2hy \int_0^t k''(t-s)\, y(s)\, ds + hk'(0)\, z^2 - hk'y^2 + hk'y^2.$$

It easy to verify $\int_0^t -k'(t-s)\, ds = k(t) - k(0)$.

$$I \leq \left(k(t) - k(0)\right) \int_0^t k'(t-s)\, y^2(s)\, ds - h \int_0^t k''(t-s)\, z^2(s)\, ds + 2hz \int_0^t k''(t-s)\, y(s)\, ds + hk'(0)\, y^2.$$

The Cauchy-Schwarz inequality gives us

$$I \leq k(t) \int_0^t k'(t-s)\, y^2(s)\, ds - k(0) \int_0^t k'(t-s)\, y^2(s)\, ds$$

$$-h \int_0^t k''(t-s)\, y^2(s)\, ds + \frac{h}{e} y^2 + eh\left(\int_0^t k''(t-s)\, y(s)\, ds\right)^2 + hk'(0)\, y^2.$$

On the other hand, the inequality above, it follows

$$I \leq k(t) \int_0^t k'(t-s)\, y^2(s)\, ds - k(0) \int_0^t k'(t-s)\, y^2(s)\, ds$$

$$-h\int_0^t k''(t-s)\,z^2(s)\,ds + h\left(\frac{1}{e}+k'(0)\right)y^2 + eh\left(\int_0^t k''(t-s)\,y(s)\,ds\right)^2.$$

From the inequality of Hölder and (a) of the hypothesis (H_2), we deduce

$$I \le k(t)\int_0^t k'(t-s)\,y^2(s)\,ds + \frac{k(0)}{\delta}\int_0^t k''(t-s)\,y^2(s)\,ds - h\int_0^t k''(t-s)\,y^2(s)\,ds + h\left(\frac{1}{e}+k'(0)\right)y^2$$

$$+eh\left[\left(\int_0^t k'(t-s)\,ds\right)\left(\int_0^t k''(t-s)\,y^2(s)\,ds\right)\right].$$

Also thanks to $\int_0^t -k'(t-s)\,ds = k(t)-k(0)$, the previous estimate gives

$$I \le k(t)\int_0^t k'(t-s)\,y^2(s)\,ds - k(0)\int_0^t k'(t-s)\,y^2(s)\,ds - h\int_0^t k''(t-s)\,y^2(s)ds$$

$$+k'(0)\,y^2 + eh\left(k'(0)-k'(t)\right)\left(\int_0^t k''(t-s)\,y^2(s)\,ds\right).$$

The hypothesis (10) and (11), allow us to undermine equality previous as follows

$$k\int_0^t k'(t-s)\,y^2(s)\,ds < 0$$

and

$$ehk'\int_0^t k''(t-s)\,y^2(s)\,ds < 0$$

and also from the definition of h, we deduce that

$$I \le \frac{1}{e\delta}k(0)\,y^2$$

Consequently

$$\int_{\Sigma_1}\left(-\int_0^t k'(t-s)\,y(s)\,ds\right)^2 d\Sigma_1 \le C_3 E_1(t),$$

where

$$C_3 = 2\left[\frac{||k(0)||_{L^\infty(\Gamma_1)}}{e\delta f} + ||h||_{L^\infty(\Gamma_1)}\right].$$

Step IV

An estimate of the term $\int_0^T E_2(t)$. First we have,

$$\int_Q |z|^2 \le CE(0) = E(y(0), z(0)).$$

If we use this inequality with the derivatives, we obtain

$$\int_Q |z_t|^2 \le CE(0) = E(y_t(0), z_t(0)).$$

On the other hand, if we multiply equation two for system by z, integrate over Q, take the real part and we use (19), we find

$$\int_Q ||\nabla_g z||_g^2 = Im\int_Q z_t\bar{z} - Re\int_Q ay\bar{z}.$$

If we use Cauchy Schwarz, we find

$$E_2(t) \le C(E(y(0), z(0)) + E(y_t(0), z_t(0)).$$

Step V

We can now conclude the result. We have, for all $T > 0$

$$\int_0^T E(y(t), z(t)) = \int_0^T E_1(y(t))\,dt + \int_0^T E_2(z(t))\,dt + \Re\left(\int_Q ay\bar{z}\,dQ\right)$$

$$\leq C(E(y(0), z(0)) + E(y_t(0), z_t(0)).$$

The desired conclusion follows from Theorem 3.1 in [2] with $K = 1$. □

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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