A NEW TYPE OF MONOTONOUS FUNCTION USED KKM-MAPPING INVOLVING HEMIVARIATIONAL INEQUALITY

AYED E. HASHOOSH[1,*], EHSAN M. HAMEED[2], SALIM DAWOOD MOHSEN[3]

[1] University of Thi-Qar, College of Education of Pure Sciences, Iraq

[2] University of Thi-Qar, College of Computer Science and Mathematics, Iraq

[3]Mustansiriyah University, College of Education, Iraq

[*]Corresponding author: ayed.hashoosh@utq.edu.iq

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Abstract. The purpose of this work is to establish the concept to be of a $(\eta\sigma\psi - p)$-monotone operator in the context of a hybrid-type mapping $\beta : K \times \Gamma^* \to \Gamma^*$. It explores the presence of $\mathrm{HVI}(\beta\eta\sigma\psi)$ using KKM-mapping in within reflexive Banach spaces and establishes that the solution to $\mathrm{HVI}(\beta\eta\sigma\psi)$ is equivalent to $\mathrm{HVI}(\beta\eta)$. Our findings constitute substantial and noteworthy improvements over the previously published findings.

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1. Introduction and Preliminaries

The studying of nonlinear issues is one of the important topics in functional analysis, difficulties with differential inclusion, optimization, economics, engineering, hemi-equilibrium, complementarity and Nash equilibrium are all examples of specific types of equilibrium issues (see, for instance [1]- [5]). Researchers in this area have considered the existence of equilibrium problems and the uniqueness of their solutions by using a wide range of methodologies for example KKM theories, fixed point theories, critical point theory and forced operators (see, for example, [5]- [9]).

It's now common to study quasi-monotonicity, relaxed monotonicity, semi-monotonicity, α-monotonicity, and Nmonotonicity, as well as hemi-equilibrium issues and equilibrium issues. You can find more information about these topics in [10]- [14]. The study of variational inequalities, initially introduced in 1964, holds significant relevance in the exploration of economics, optimization and engineering sciences. Moreover, variational inequalities find application and generalization in nonlinear analysis.

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One of the applications in variational inequalities is "emergence," which is a major tool for analyzing various issues in industrial, physical, and dynamical systems (see [15]- [19]).

The following paragraphs will discuss a few definitions and ideas that are relevant to the primary results. In the beginning, we make the assumption that Γ is a Banach space and that Γ^* is a dual space of Γ.

Definition 1.1. [20] *For each u in Γ, we say that a function $A : \Gamma \to \mathbb{R}$ is Locally Lipschitz if, for any v and w in U, there exists a neighborhood U of u and a constant $L_u > 0$ such that*

$$|A(w) - A(v)| \leq L_u \|w - v\|_\Gamma$$

Definition 1.2. [21] *Considering $J : \Gamma \to \mathbb{R}$ is a Locally Lipschitz functional(LLF), we can describe $J^0(u; v)$, which is the generalized derivative of J at $u \in S$ in the direction of $v \in \Gamma$, as:*

$$J^0(u; v) = \lim_{w \to u, \lambda \perp 0} \frac{J(w + \lambda v) - J(w)}{\lambda}.$$

Lemma 1.1. [21] *Considering $J : \Gamma \to \mathbb{R}$ is (LLF) of rank L_u near the point $u \in \Gamma$. So*

i. *There is a sub-additive function $v \mapsto J^0(u; v)$ that is finite, positively, homogeneous, and meets the situation that $\left|J^0(u; v)\right| \leq L_u \|v\|_\Gamma$;*
ii. *$J^0(u; v)$ is upper semi-continuous.*
iii. *$J^0(u; -v) = (-J)^0(u; v)$.*

Definition 1.3. [22] *Assume that K is a convex subset of Γ that is not empty. Let there be two mappings, denoted as $T : K \to 2^{\Gamma^*}$ and $\eta : K \times K \to \Gamma$. Consequently, T is referred to as $\eta-$ Hemicontinuous if, for any $\theta, \vartheta \in K$, there exists a mapping $g : [0, 1] \to 2^{\mathbb{R}}$, which is denoted by $\eta - HC$ and defined by*

$$g(t) = \mathrm{U}_{u_t \in T(\theta_l)} \langle u_t, \eta(\vartheta, \theta) \rangle, \textit{ where } \theta_t = \theta + t(\vartheta - \theta) \textit{ is usc at } 0^+.$$

Definition 1.4. [22] *Assume that $\mathrm{N} : \mathrm{K} \times \Gamma^* \to \mathrm{E}^*,\ \mathrm{T} : \mathrm{K} \to 2^{\Gamma^*}$ and $\varpi : \mathrm{K} \times \mathrm{K} \to \Gamma$ are three mappings and a function $\mathcal{L} : \mathrm{K} \times \mathrm{K} \to \mathbb{R} \cup \{+\infty\}$ is proper. T is titled to be w-coercive for $\mathcal{L}$ if, there is an $\theta_0 \in \mathrm{K}$ in which for any $u \in T(\theta)$ and $u_0 \in \mathrm{T}(\theta_0)$*

$$\frac{\langle N(\theta, u) - N(\theta_0, u_0), \varpi(\theta, \theta_0) \rangle + \mathcal{L}(\theta, \theta_0)}{\|\varpi(\theta_0, \theta)\|} \to \infty$$

whenever $\|\theta\| \to \infty$.

2. Main Results

In this section, we employ a novel monotone function known as the ($\eta\sigma\psi - p$) -monotone operator with respect to the 2 -arg of a hybrid-type mapping $\beta : K \times \Gamma^* \to \Gamma^*$. In this context, K is a nonempty set of real reflexive B.S. Γ, and Γ^* is a dual of Γ. The outcomes we have obtained include the establishment of some existence results for $HVI(\beta\eta\sigma\psi)$ through the use of KKM-mapping in reflexive B.Ss. Furthermore,

There is a direct correspondence between the $HVI(\beta\eta\sigma\psi)$ solution and the $HVI(\beta\eta)$ solution.
We present here a new definition we constructed for a quasi-monotonary operator. The idea has made an important contribution to our primary findings.

Definition 2.1. *A mapping $T : K \to 2^{\Gamma^*}$ is called a $(\eta\sigma\psi - p)$-monotone operator for the 2-arg of a hybrid-type mapping $\beta : K \times \Gamma^* \to \Gamma^*$ if it has a set value. $\eta : K \times K \to \Gamma$ and $\sigma : \Gamma \to \mathbb{R}$ such that $\sigma(tz) = \Lambda(t)\sigma(z)$ for $z \in \Gamma$, where $\Lambda : (0, \infty) \to (0, \infty)$ is a function such that $\lim_{\mathrm{t}\to 0} \frac{\Lambda(\mathrm{t})}{\mathrm{t}} = 0$, and $\psi : K \times K \to \mathbb{R}$ by*

$$\mathrm{Lim}_{\mathrm{t}\to 0}\left[\frac{\Lambda(\mathrm{t})\alpha(\vartheta-\theta)}{\mathrm{t}} + \frac{\beta(\theta, \mathrm{t}\vartheta + (1-\mathrm{t})\theta)}{\mathrm{t}}\right] = 0,$$

where for any $\theta, \vartheta \in K$

$$\langle \beta(\theta, u), \eta(\vartheta, \theta)\rangle + J^{\circ}(\theta, \vartheta - \theta) \geq 0 \text{ for all } u \in T(\theta). \tag{2.1}$$

Implies

$$\langle \beta(\theta, v), \eta(\vartheta, \theta)\rangle + J^{\circ}(\theta, \vartheta - \theta) \geq \sigma(\vartheta - \theta) + \psi(\theta, \vartheta) \text{ for all } v \in T(\vartheta). \tag{2.2}$$

Theorem 2.1. *Consider that a non-empty closed convex K is subset of real reflexive Banach space , $T : K \to 2^{\Gamma^*}$ is an η-H.C and $(\eta\kappa\sigma\psi - p)$-monotone operator for the 2-arg of a hybrid-type mapping $\beta : K \times \Gamma^* \to \Gamma^*$. Consider that*

i. $\eta(\theta, \theta) = 0 \forall \theta \in K$;
ii. $\theta \mapsto \eta(., \theta)$ *is convex.*
iii. $J : S \to R$ *is a locally Lipschitz functional.*

Then, $\theta \in K$ is a solution of $HVI(\beta\eta)$ if and only if it is a solution of $\mathrm{HVI}(\beta\eta\sigma\psi)$.
Before proving Theorem 2.1, we make Remark 2.1.

Remark 2.1. i) *If $\beta(\theta, u) = u, \beta(\theta, v) = v$ and $\psi(\theta, \vartheta) = 0$ in Definition (2.1) reduces to that for all $\theta, \vartheta \in K$*

$$\langle u, \eta(\vartheta, \theta)\rangle + J^{\circ}(\theta, \vartheta - \theta) \geq 0 \text{ for all } u \in T(\theta)$$

implies

$$\langle v, \eta(\vartheta, \theta)\rangle + J^{\circ}(\theta, \vartheta - \theta) \geq \sigma(\vartheta - \theta) \text{ for all } v \in T(\vartheta)$$

ii) *If $\beta(\theta, u) = u, \beta(\theta, v) = v, \eta(\vartheta, \theta) = \vartheta - \theta$ and $\psi(\theta, \vartheta) = 0$ in Definition (2.1) reduces to that for all $\theta, \vartheta \in K$*

$$\langle u, \vartheta - \theta\rangle + J^{\circ}(\theta, \vartheta - \theta) \geq 0 \text{ for all } u \in T(\theta)$$

Implies

$$\langle v, \vartheta - \theta\rangle + J^{\circ}(\theta, \vartheta - \theta) \geq \sigma(\vartheta - \theta) \text{ for all } v \in T(\vartheta)$$

iii) *If* $\beta(\theta, u) = u, \beta(\theta, v) = v, \eta(\vartheta, \theta) = \vartheta - \theta$ *and* $\psi(\theta, \vartheta) = 0 = J^\circ$ *in Definition* (2.1) *reduces to that for all* $\theta, \vartheta \in K$

$$\langle u, \vartheta - \theta\rangle \geq 0 \text{ for all } u \in T(\theta).$$

Implies

$$\langle v, \vartheta - \theta\rangle \geq \sigma(\vartheta - \theta) \text{ for all } v \in T(\vartheta)$$

iv) *If* $\psi \equiv 0$ *and* $\lim_{t\to 0} \frac{\Lambda(t)}{t} = 0$, *then in Definition* 2.1, *reduces to that*

$$\langle \beta(\theta, u), \eta(\vartheta, \theta)\rangle + J^\circ(\theta, \vartheta - \theta) \geq 0 \text{ for all } u \in T(\theta)$$

implies

$$\langle \beta(\theta, v), \eta(\vartheta, \theta)\rangle + J^\circ(\theta, \vartheta - \theta) \geq \sigma(\vartheta - \theta) \text{ for all } v \in T(\vartheta)$$

See the topics of generalization for pseudomonotone ([24]- [26]).

Proof. As per Definition 2.1, if θ is solve of $HVI(\beta\eta)$, it is also a solution of $HVI(\beta\eta\sigma\psi)$. Let θ be a solution of the $HVI(\beta\eta\sigma\psi)$ equation and $\vartheta \in K$. The set K contains $\theta_t = t\vartheta + (1-t)\theta$, where $t \in [0, 1]$, If $HVI(\beta\eta\sigma\psi)$ holds, then for $u_t \in T(\theta_t)$

$$\begin{aligned}\langle \beta(\theta, u_t), \eta(\theta_t, \theta)\rangle + J^\circ(\theta, \theta_t - \theta) &\geq \sigma(\theta_t - \theta) + \psi(\theta, \theta_t) \\ &= \sigma(t(\vartheta - \theta)) + \psi(\theta, \theta_t) \\ &= \Lambda(t)\sigma(\vartheta - \theta) + \psi(\theta, \theta_t). \end{aligned} \tag{2.3}$$

As a result of circumstances (i) and (ii), we obtain

$$\begin{aligned}&\langle \beta(\theta, u_t), \eta(\theta_t, \theta)\rangle + J^\circ(\theta, \theta_t - \theta) \\ &= \langle \beta(\theta, u_t), \eta(t\vartheta + (1-t)\theta, \theta)\rangle + J^\circ(\theta, (t\vartheta + (1-t)\theta) - \theta) \\ &\leq t\langle \beta(\theta, u_t), \eta(\vartheta, \theta)\rangle + (1-t)\langle \beta(\theta, u_t), \eta(\theta, \theta)\rangle + tJ^\circ(\theta, \vartheta - \theta) + (1-t)J^\circ(\theta, \theta - \theta) \\ &= t[\langle \beta(\theta, u_t), \eta(\vartheta, \theta)\rangle + J^\circ(\theta, \vartheta - \theta)]. \end{aligned} \tag{2.4}$$

It follows from (2.3) and (2.4)

$$\langle \beta(\theta, u_t), \eta(\vartheta, \theta)\rangle + J^\circ(\theta, \vartheta - \theta) \geq \frac{k(t)}{t}\sigma(\vartheta - \theta) + \frac{1}{t}\psi(\theta, \theta_t) \tag{2.5}$$

for all $\vartheta \in K, u_t \in T(\theta_t)$.

Given T is $\eta - H.C$ and $t \to 0$ in (2.5), one can obtain

$$\langle \beta(\theta, u), \eta(\vartheta, \theta)\rangle + J^\circ(\theta, \vartheta - \theta) \geq 0, \text{ for all } \vartheta \in K \text{ and } u \in T(\theta).$$

Then, $\theta \in K$ is a solution of $HVI(\beta\eta)$. □

Theorem 2.2. *Consider that the set K is a convex set of a reflexive B.S.E. that is bounded. Considering $T: K \rightarrow 2^{\Gamma^*}$ be an η-H.C and ($\eta\kappa\sigma\psi - p$)-monotone operator for the 2 -argument of a map $\beta: K \times \Gamma^* \rightarrow \Gamma^*$. If*

i. $\eta(\theta, \theta) = 0$ *for all* $\theta \in K$;
ii. $\theta \mapsto \eta(\theta, \cdot)$ *is usc and convex in 1-arg ;*
iii. $\sigma: \Gamma \rightarrow \mathbb{R}$ *and* $\psi(., \theta)$ *weakly lsc.*

Consequently, there is at least one solve of $\mathrm{HVI}(\omega\eta)$.

Proof. Consider that $F, G: K \rightarrow 2^{\Gamma}$ are set-valued mappings that defined as follows for each $\vartheta \in K$

$$F(\vartheta) = \{\theta \in K : \langle \beta(\theta, u), \eta(\vartheta, \theta)\rangle + J^{\circ}(\theta, \vartheta - \theta) \geq 0, u \in T(\theta)\}$$

$$G(\vartheta) = \{\theta \in K : \langle \beta(\theta, v), \eta(\vartheta, \theta)\rangle + J^{\circ}(\theta, \vartheta - \theta) \geq \sigma(\vartheta - \theta) + \psi(\theta, \vartheta), v \in T(\vartheta)\}.$$

As far as we are concerned, F is a KKM mapping. Should F not be a KKM mapping, then there is $\{\vartheta_1, \vartheta_2, \ldots, \vartheta_n\} \subset K$ such that $\operatorname{co}\{\vartheta_1, \vartheta_2, \ldots, \vartheta_n\} \not\subset \bigcup_{i=1}^{n} F(\vartheta_i)$,i.e., there is a $\vartheta_0 \in \operatorname{co}\{\vartheta_1, \vartheta_2, \ldots, \vartheta_n\}$, so, $\vartheta_0 = \sum_{i=1}^{n} t_i \vartheta_i$, where $t_i \geq 0 i = 1, \ldots, n, \sum_{i=1}^{n} t_i = 1$, but $\vartheta_0 \notin \mathrm{U}_{i=1}^{n} F(\vartheta_i)$. By the definition of F, we get for $\mathrm{i} = 1, 2, \ldots, \mathrm{n}$

$$\langle \beta(\vartheta_0, u), \eta(\vartheta_i, \vartheta_0)\rangle + J^{\circ}(\vartheta_0, \vartheta_i - \vartheta_0) < 0, \text{ for some } u \in T(\vartheta_0).$$

It is evident from both (i) and (ii) that

$$\begin{aligned}
0 &= \langle \beta(\vartheta_0, u), \eta(\vartheta_0, \vartheta_0)\rangle + J^{\circ}(\vartheta_0, \vartheta_0 - \vartheta_0) \\
&= \left\langle \beta(\vartheta_0, u), \eta\left(\sum_{i=1}^{n} t_i \vartheta_i, \vartheta_0\right)\right\rangle + J^{\circ}\left(\vartheta_0, \sum_{i=1}^{n} t_i \vartheta_i - \vartheta_0\right) \\
&\leq \sum_{i=1}^{n} t_i \langle \beta(\vartheta_0, u), \eta(\vartheta_i, \vartheta_0)\rangle + \sum_{i=1}^{n} t_i J^{\circ}(\vartheta_0, \vartheta_i - \vartheta_0) \\
&= \sum_{i=1}^{n} t_i [\langle \beta(\vartheta_0, u), \eta(\vartheta_i, \vartheta_0)\rangle + J^{\circ}(\vartheta_0, \vartheta_i - \vartheta_0)] \\
&< 0
\end{aligned}$$

for some $u \in T(\vartheta_0)$, This is a contradiction in terms. It may be deduced from this that F is a KKM mapping. We can now demonstrate that

$$F(\vartheta) \subset G(\vartheta), \text{ for all } \vartheta \in K.$$

For any $\vartheta \in K$, letting $\theta \in F(\vartheta)$, we get

$$\langle \beta(\theta, u), \eta(\vartheta, \theta)\rangle + J^{\circ}(\theta, \vartheta - \theta) \geq 0 \text{ for } u \in T(\theta).$$

Since T is $(\eta\kappa\sigma\psi - p)$-monotone operator

$$\langle \beta(\theta, v), \eta(\vartheta, \theta)\rangle + J^{\circ}(\theta, \vartheta - \theta) \geq \sigma(\vartheta - \theta) + \psi(\theta, \vartheta) \text{ for } v \in T(\vartheta).$$

It follows that $\theta \in G(\vartheta)$, so

$$F(\vartheta) \subset G(\vartheta), \text{ for all } \vartheta \in K$$

which implies that G is a KKM mapping.

Conversely, if $\langle \theta_n \rangle$ is a net in $G(\vartheta)$ that weakly approaches θ_0, then for every v in T(y),

$$\langle \beta\left(\theta_n, v\right), \eta\left(\vartheta, \theta_n\right)\rangle + J^{\circ}\left(\theta_n, \vartheta - \theta_n\right) \geq \sigma\left(\vartheta - \theta_n\right) + \psi\left(\theta_n, \vartheta\right).$$

Given that σ and ψ are weakly lsc and $\theta \mapsto \eta(\theta,.) and \theta \mapsto f(\theta,.) are usc$, then follows that

$$\begin{aligned}
\langle \beta\left(\theta_0, v\right), \eta\left(\vartheta, \theta_0\right)\rangle + J^{\circ}\left(\theta_0, \vartheta - \theta_0\right) &\geq \limsup_n \langle \beta\left(\theta_k, v\right), \eta\left(\vartheta, \theta_k\right)\rangle + \limsup_n J^{\circ}\left(\theta_k, \vartheta - \theta_k\right) \\
&\geq \limsup_n \left(\langle \beta\left(\theta_k, v\right), \eta\left(\vartheta, \theta_k\right)\rangle + J^{\circ}\left(\theta_k, \vartheta - \theta_k\right)\right) \\
&\geq \liminf_n \left(\langle \beta\left(\theta_k, v\right), \eta\left(\vartheta, \theta_k\right)\rangle + J^{\circ}\left(\theta_k, \vartheta - \theta_k\right)\right) \\
&\geq \liminf_n \sigma\left(\vartheta - \theta_k\right) + \psi\left(\theta_k, \vartheta\right) \\
&\geq \sigma\left(\vartheta - \theta_0\right) + \psi\left(\theta_0, \vartheta\right).
\end{aligned}$$

The fact that $\theta_0 \in G(\vartheta)$ demonstrates that $G(\vartheta)$ is a weakly closed set for all $\vartheta \in K$. Due to the fact that K is closed, bounded and convex, it is weakly compact. Therefore, $G(\vartheta)$ is weakly compact in K for every $\vartheta \in K$. After considering Fan-KKM Theorem and Theorem 2.1, it can be concluded that

$$\bigcap_{\vartheta \in K} F(\vartheta) = \bigcap_{\vartheta \in K} G(\vartheta) \neq \emptyset.$$

At least one solution exists for HVI$(\beta\eta)$. □

Theorem 2.3. *For the 2-arg of a mapping $\beta : K \times \Gamma^* \to \Gamma^*$, Considering T is a set-valued mapping and be $\eta -$ H.C, $(\eta\kappa\sigma\psi - p\ )$-monotone operator from K to 2^{Γ^*}. Let K be a nonempty, unbounded, closed, and convex set of a reflexive B.S. Γ. Assuming the following are true,*

i. $\eta(\theta, \theta) = 0$ *for all* $\theta, \in K$*;*
ii. $\eta(\theta, \vartheta) + \eta(\vartheta, \theta) = 0$ *for all* $\theta, \vartheta \in K$*;*
iii. $\theta \mapsto \eta(\theta, \cdot)$ *is usc and convex in 1-arg;*
iv. $\sigma : \Gamma \to \mathbb{R}, \psi(., \theta)$*, is weakly lsc;*
v. T *is* η*-coercive* $w.r.t$ κ*.*

Therefore, there is at least one solve of $HVI(\beta\eta)$.

Proof. For $r > 0$, let

$$B_r = \{\vartheta \in \Gamma : \|\vartheta\| \leq r\}.$$

Considering the preceding issue, determine the value of $\theta_r \in K \cap B_r$ and $u_r \in T(\theta_r)$. where

$$\langle \beta(\theta_r, u_r), \eta(\vartheta, \theta_r) \rangle + J^\circ(\theta_r, \vartheta - \theta_r) \geq 0 \text{ for } \vartheta \in K \cap B_r. \tag{2.6}$$

Via Theorem 2.2, we know that problem (2.6) has unique solution $\theta_r \in K \cap B_r$. In the coercive circumstances, take θ_0 whose norm $\|\theta_0\|$ is less than r. Following that $\theta_0 \in K \cap B_r$ and

$$\langle \beta(\theta_r, u_r), \eta(\theta_0, \theta_r) \rangle + J^\circ(\theta_r, \theta_0 - \theta_r) \geq 0 \text{ for } u_r \in T(\theta_r). \tag{2.7}$$

The result of condition (i) and for $u_r \in T(\theta_r)$ and $u_0 \in T(\theta_0)$ is that

$$\begin{aligned}
&\langle \beta(\theta_r, u_r), \eta(\theta_0, \theta_r) \rangle + J^\circ(\theta_r, \theta_0 - \theta_r) \\
=& -\langle \beta(\theta_r, u_r) - \beta(\theta_r, u_0), \eta(\theta_r, \theta_0) \rangle + J^\circ(\theta_r, \theta_0 - \theta_r) + \langle \beta(\theta_r, u_0), \eta(\theta_0, \theta_r) \rangle \\
\leq& -\langle \beta(\theta_r, u_r) - \beta(\theta_r, u_0), \eta(\theta_r, \theta_0) \rangle + J^\circ(\theta_r, \theta_0 - \theta_r) + \|\beta(\theta_r, u_0)\| \cdot \|\eta(\theta_r, \theta_0)\| \\
\leq& \|\eta(\theta_r, \theta_0)\| \left(-\frac{\langle \beta(\theta_r, u_r) - \beta(\theta_r, u_0), \eta(\theta_r, \theta_0) \rangle + J^\circ(\theta_r, \theta_0 - \theta_r)}{\|\eta(\theta_r, \theta_0)\|} + \|\beta(\theta_r, u_0)\| \right).
\end{aligned}$$

We can choose r large enough so that the inequality and the η-coercivity of T with regard to κ involve that, if $\|\theta_r\| = r$ for every r, then

$$\langle \beta(\theta_r, u_r), \eta(\theta_0, \theta_r) \rangle + J^\circ(\theta_r, \theta_0 - \theta_r) < 0.$$

It contradicts (2.7). There is r where $\|\theta_r\| < r$. A small enough $0 < \epsilon < 1$ can be chosen for each $\vartheta \in K$.

$$\theta_r + \epsilon(\vartheta - \theta_r) \in K \cap B_r.$$

As a result of the fact that (2.6), thus,

$$\begin{aligned}
0 \leq& \langle \beta(\theta_r, u_r), \eta(\theta_r + \epsilon(\vartheta - \theta_r), \theta_r) \rangle + J^\circ((\theta_r + \epsilon(\vartheta - \theta_r), \theta_r - \theta_r + \epsilon(\vartheta - \theta_r)) \\
\leq& (1-\epsilon)\langle \beta(\theta_r, u_r), \eta(\theta_r, \theta_r) \rangle + (1-\epsilon) f(\theta_r, \theta_r) + \epsilon \langle \beta(\theta_r, u_r), \eta(\vartheta, \theta_r) \rangle \\
&+ \epsilon f(\vartheta, \theta_r) \\
=& \epsilon \left[\langle \beta(\theta_r, u_r), \eta(\vartheta, \theta_r) \rangle + J^\circ(\theta_r, \vartheta - \theta_r) \right].
\end{aligned}$$

This suggests that

$$\langle \beta(\theta_r, u_r), \eta(\vartheta, \theta_r) \rangle + J^\circ(\theta_r, \vartheta - \theta_r) \geq 0$$

for $\vartheta \in K$ and $u_r \in T(\theta_r)$. With this, the proof is finished. □

Remark 2.2. (*i*) *In the case where T is continuous on subspaces with finite dimensions, theorems 2.2-2.5 also hold.*

(*ii*) *The known results of P. Hartman and G. Stampacchia* (*1966*) *are made better and more general by theorems 2.2-2.5.*

Question 2.1. *Can anyone investigate the concept of* $(\eta\sigma\psi - p)$*-monotone operators using a different method instead of KKM-mapping?*

References

[1] A.E. Hashoosh, M. Alimohammady, $B_{\alpha,\beta}$-operator and Fitzpatrick functions, Jordan J. Math. Stat. 10 (2017), 33–52.

[2] U. Kamraksa, R. Wangkeeree, Generalized equilibrium problems and fixed point problems for nonexpansive semigroups in Hilbert spaces, J. Glob. Optim. 51 (2011), 689–714. https://doi.org/10.1007/s10898-011-9654-9.

[3] A.E. Hashoosh, M. Alimohammady, Existence and uniqueness results for a nonstandard variational-hemivariational inequalities with application, Int. J. Ind. Math. 9 (2017), 241–250.

[4] A.E. Hashoosh, M. Alimohammady, On well-posed of generalized equilibrium problems involving α monotone bifunction, J. Hyperstruct. 5 (2016), 151–168.

[5] A.E. Hashoosh, M. Alimohammady, G.A. Almusawi, Existence results for nonlinear quasi-hemivariational inequality systems, Univ. Thi-Qar J. 11 (2016), 90–109. https://doi.org/10.32792/utq/utj/vol11/4/7.

[6] A.E. Hashoosh, M. Alimohammady, M.K. Kalleji, Existence results for some equilibrium problems involving α-monotone bifunction, Int. J. Math. Math. Sci. 2016 (2016), 2093026. https://doi.org/10.1155/2016/2093026.

[7] I. Andrei, N. Costea, Nonlinear hemivariational inequalities and applications to nonsmooth mechanics, Adv. Nonlinear Var. Ineq. 13 (2010), 1–17.

[8] K. Fan, A generalization of Tychonoff's fixed point theorem, Math. Ann. 142 (1961), 305–310. https://doi.org/10.1007/bf01353421.

[9] P. Hartman, G. Stampacchia, On some non-linear elliptic differential-functional equations, Acta Math. 115 (1966), 271–310. https://doi.org/10.1007/bf02392210.

[10] A.E. Hashoosh, Existence and uniqueness solutions for a class of hemivariational inequalities, J. Math. Ineq. 11 (2017), 565–576. https://doi.org/10.7153/jmi-11-46.

[11] A. Tada, W. Takahashi, Weak and strong convergence theorems for a nonexpansive mapping and an equilibrium problem, J. Optim. Theory Appl. 133 (2007), 359–370. https://doi.org/10.1007/s10957-007-9187-z.

[12] R.U. Verma, On monotone nonlinear variational inequality problems, Comment. Math. Univ. Carolinae, 39 (1998), 91–98. http://dml.cz/dmlcz/118988.

[13] R.U. Verma, On generalized variational inequalities involving relaxed lipschitz and relaxed monotone operators, J. Math. Anal. Appl. 213 (1997), 387–392. https://doi.org/10.1006/jmaa.1997.5556.

[14] R.U. Verma, A-monotonicity and its role in nonlinear variational inclusions, J. Optim. Theory Appl. 129 (2006), 457–467. https://doi.org/10.1007/s10957-006-9079-7.

[15] P. Cardaliaguet, M. Cirant, A. Porretta, Remarks on Nash equilibria in mean field game models with a major player, Proc. Amer. Math. Soc. 148 (2020), 4241–4255. https://doi.org/10.1090/proc/15135.

[16] A.A. Alwan, H.A. Naser, A.E. Hashoosh, The solutions of mixed hemiequilibrium problem with application in Sobolev space, IOP Conf. Ser.: Mater. Sci. Eng. 571 (2019), 012019. https://doi.org/10.1088/1757-899x/571/1/012019.

[17] N.V. Tran, L.T.T. Hai, T.V. An, P.T. Vuong, A fixed-time stable forward–backward dynamical system for solving generalized monotone inclusions, J. Appl. Math. Comput. (2024). https://doi.org/10.1007/s12190-024-02186-1.

[18] P.V. Hai, P.T. Vuong, Third order dynamical systems for the sum of two generalized monotone operators, J. Optim. Theory Appl. 202 (2024), 519–553. https://doi.org/10.1007/s10957-024-02437-y.

[19] S. Migórski, A. Ochal, M. Sofonea, Weak solvability of antiplane frictional contact problems for elastic cylinders, Nonlinear Anal.: Real World Appl. 11 (2010), 172–183. https://doi.org/10.1016/j.nonrwa.2008.10.045.

[20] F.H. Clarke, A new approach to Lagrange multipliers, Math. Oper. Res. 1 (1976), 165–174. https://doi.org/10.1287/moor.1.2.165.

[21] F.H. Clarke, Generalized gradients of Lipschitz functionals, Adv. Math. 40 (1981), 52–67. https://doi.org/10.1016/0001-8708(81)90032-3.

[22] B. Lee, M.F. Khan, Salahuddin, Hybrid-type set-valued variational-like inequalities in reflexive Banach spaces, J. Appl. Math. Inf. 27 (2009), 1371–1379.

[23] Z. Liu, S. Zeng, Equilibrium problems with generalized monotone mapping and its applications, Math. Methods Appl. Sci. 39 (2015), 152–163. https://doi.org/10.1002/mma.3471.

[24] M.R. Bai, S.Z. Zhou, G.Y. Ni, Variational-like inequalities with relaxed $\eta - \alpha$ pseudomonotone mappings in Banach spaces, Appl. Math. Lett. 19 (2006), 547–554. https://doi.org/10.1016/j.aml.2005.07.010.

[25] N.K. Mahato, C. Nahak, Weakly relaxed α-pseudomonotonicity and equilibrium problem in Banach spaces, J. Appl. Math. Comp. 40 (2012), 499–509. https://doi.org/10.1007/s12190-012-0584-6.

[26] D.T. Luc, Existence results for densely pseudomonotone variational inequalities, J. Math. Anal. Appl. 254 (2001), 291–308. https://doi.org/10.1006/jmaa.2000.7278.