Asia Pac. J. Math. 2024 **11**:95

REGULATOR PROBLEM FOR LINEAR DISCRETE-TIME DELAY SYSTEMS WITH SYMMETRICAL CONSTRAINTS

OU-AZZOU RACHID*, ABDELHAK AHMED

[1]Department of Mathematics, Laboratory: Partial Differential Equations, Algebra, and Spectral Geometry, Faculty of Sciences, Ibn Tofail University, Kenitra, Morocco

*Corresponding author: ouazzourachid1994@gmail.com

Received Jun. 23, 2024

Abstract. In this paper, we investigate the Linear Constrained Regulation Problem LCRP for discrete-time delay dynamical systems, with the symmetrical constraints on the control variable and origin of the input space positioned on the boundary of the constraint's domain. Our primary aim is to examine the conditions under which a state feedback law can be established, ensuring both constraints satisfaction on the control variable and asymptotic stability of the state variable. In our investigation, we consider both delay-independent and delay-dependent conditions.

2020 Mathematics Subject Classification. 93C05.

Key words and phrases. linear discrete-time systems; constrained control; positive invariance; asymptotic stability.

1. Introduction

The regulator problem for linear discrete-time delay dynamical systems with non-symmetrical constrained control was extensively investigated by many authors, such as Hmamed [24], Benzaouia and El Faiz [7], Bensalah and Baron [3], and Dorea and Olaru [19]. In all these publications have considered the origin on the interior of the domain of constraints. So, for the origin situated on the boundary of the domain of constraints results are needed. Recently, Benzaouia [5], Bistoris and Olaru [11] and Bistoris et al [12] have considered the linear constrained regulation problem, for discrete system and continuous system with the origin on the boundary of the domain of constraints. Ou-azzou and Abdelhak [36] have considered the linear constrained regulation problem, for linear continuous-time delay system with the origin on the boundary of a symmetrical domain of constraints. In this paper, we investigate the linear constrained regulation problem for a discrete-time system with delay and the origin on the boundary of a symmetrical domain of constraints.To resolve this problem we

DOI: 10.28924/APJM/11-95

choose the postive invariance concept, proposed by Gutman and Hagander [21].

We consider linear discrete-time systems with time delay described by the difference equation:

$$\begin{cases} x(k+1) = A_0 x(k) + A_1 x(k-r) + Bu(k), \ k > 0 \\ x(\theta) = \varphi(\theta), \ \theta \in [-r, 0] \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $k \in T$ is the time variable and $r \in \mathbb{N}$ is the time delay.

Matrices $A_0 \in \mathbb{R}^{n\times n}$, $A_1 \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are constant and satisfy the following condition:

(a) (A_0, B) is stabilizable in the independent delay case.

(b) $(A_0 + A_1, B)$ is stabilizable in the dependent delay case.

The control vector $u(.)$ has to satisfy a constraints of the form

$$u(k) \in \Omega\{v \in \mathbb{R}^m | - q \leqslant v \leqslant q\}, \ k \geq 0 \tag{2}$$

where $q \in \mathbb{R}^m_+$.

The linear constrained regulation problem (LCRP) consists of finding a linear state feedback control law $u(.) = Fx(.)$, where $F \in \mathbb{R}^{m\times n}$. Such that the constraints (2) are satisfied and the state of system (1) converge asymptotically to the equilibrium $x_e = 0$, thise is only possible if the domaine

$$D(F, q, q) = \{x \in \mathbb{R}^n | - q \leqslant Fx \leqslant q\} \tag{3}$$

is positively invariant with respect to the system

$$x(k+1) = (A_0 + BF)x(k) + A_1 x(k-r), \ k > 0 \tag{4}$$

In the most of the preceeding papers, the hypothesis $q > 0$ is made, that is at least one component of q is null, or equivalently at least one component of control variable has to satisfy $u_i \geq 0$ or $u_i \leq 0$, then the results of theses papers can't apply. The objective of our paper is to give a solution of the regulation problem in this case. So consider the domaine of constraints (3) defined with a vecteur $q \in \mathbb{R}^m_+$ which has at least one null component. Define the integer $'s'$ as the numbre of non null components of q. Then $0 < s < m$. Without loss of generality, we assume that

$$q_j > 0, \ j = 1, ..., s \ and \ q_j = 0 \ ; \ j = s+1, ..., m \tag{5}$$

In the following, we will denote $q = \begin{pmatrix} q^* \\ 0 \end{pmatrix}$, with $q^* = \begin{pmatrix} q_1 \\ \vdots \\ q_s \end{pmatrix} \in \mathbb{R}^s$ and $q^* > 0$.

The paper is organized as follows: In section 2, we present some definitions and useful results for the following. In section 3, we establish sufficient conditions for $u(.) = Fx(.)$, with $F \in \mathbb{R}^{m\times n}$ and

$rank(F) = m$, to be a solution to the linear constrained regulation problem. Finally, an example is given in section 4.

Notation

In this paper, we employ distinct notations and symbols to denote different mathematical objects and concepts. Below is an overview of the notations used:

✓ The capital letters represent real matrices.

✓ The lowercase letters denote column vectors or scalars.

✓ T represents the discrete time set $T = \{0, 1, 2, \ldots\}$.

✓ $\mathbb{R}^n$ represents the real n-space.

✓ $\mathbb{R}^n_+$ denotes the nonnegative orthant of the real n-space.

✓ $\mathbb{R}^{n\times p}$ refers to the set of real $n \times p$ matrices.

✓ $x = [x_1\ x_2\ ...\ x_n]^T$ denotes a real vector.

✓ $x < y$ $(x \leqslant y)$ is equivalent to $x_i < y_i$ $(x_i \leqslant y_i)$, $i = 1, 2, ..., n$..

✓ $H = (h_{ij})_{1\leqslant i,j\leqslant n}$ denotes a real matrix.

✓ $|H|$ denotes the matrix $|H| = (|h_{ij}|)_{1\leqslant i,j\leqslant n}$.

✓ $H < G$ $(H \leqslant G)$ is equivalent to $h_{i,j} < g_{i,j}$ $(h_{i,j} \leqslant g_{i,j})$, $i, j = 1, 2, ..., n$..

✓ $\rho(H)$ is the spectral radius of H.

2. Conditions of positive invariance

In this section we will establish results on positive invarience with respect to autonomous systems described by:

$$\begin{cases} z(k+1) = Hz(k) + Gz(k-r),\ k > 0 \\ z(\theta) = \psi(\theta),\ \theta \in [-r, 0] \end{cases} \tag{6}$$

with $z \in \mathbb{R}^m$, $H \in \mathbb{R}^{m\times n}$ and $G \in \mathbb{R}^{m\times n}$.

Let us define the domain

$$D(I_m, q, q) = \{z \in \mathbb{R}^m | -q \leqslant z \leqslant q\}$$

with $q = \begin{pmatrix} q^* \\ 0 \end{pmatrix}$, $q^* > 0$ and $q^* \in \mathbb{R}^s$.

Definition 2.1. A set D of $\mathbb{R}^n$ is said to be positively invariant with respect to motions of system (6), if for every $\psi(\theta) \in D(\theta \in [-r, 0])$ the motion $z(k; \psi) \in D$ for every $k \geqslant 0$.

Throughout this paper, if A is a matrix of $\mathbb{R}^{m\times m}$ we will denote by $A_{11} \in \mathbb{R}^{s\times s}$, $A_{12} \in \mathbb{R}^{s\times(m-s)}$, $A_{21} \in \mathbb{R}^{(m-s)\times s}$ and $A_{22} \in \mathbb{R}^{(m-s)\times(m-s)}$ the matrices such that

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{7}$$

2.1. Conditions for positive invariance dependent of delay.

In [24], the author has considered the case where $x_e = 0$ is in the interior of $D(I_m, q, q)$, that is $q > 0$ and has proved that $D(I_m, q, q)$ is positively invariant with respect to system (6) if and only if

$$(|H| + |G|)q \leqslant q \tag{8}$$

we will use the following decomposition

Theorem 2.2. *The polyhedral set $D(I_m, q, q)$ with $q = \begin{pmatrix} q^* \\ 0 \end{pmatrix}$, $q^* > 0$ and $q^* \in \mathbb{R}^s$ is positively invariant independent of delay with respect to system (6) if and only if*

$$(|H_{11}| + |G_{11}|)q^* \leqslant q^* \textit{ and } H_{21} = G_{21} = 0$$

where H_{11}, H_{21}, G_{11} and G_{21} are given by decomposition (7).

Proof.

If:) Let assume that $(|H_{11}| + |G_{11}|)q^* \leqslant q^*$ and $H_{21} = G_{21} = 0$. Let $z(.)$ be the solution of system (6) with $z(k) \in D(I_m, q, q)$, $\forall k \in [-r, 0]$, that means

$$-\begin{pmatrix} q^* \\ 0 \end{pmatrix} \leqslant z(k) \leqslant \begin{pmatrix} q^* \\ 0 \end{pmatrix}, \ \forall k \in [-r, 0] \tag{9}$$

If we decompose $z(k)$ as $z(k) = \begin{pmatrix} z_1(k) \\ z_2(k) \end{pmatrix}$, where $z_1(k) \in \mathbb{R}^s$ and $z_2(k) \in \mathbb{R}^{m-s}$, then it follows from (9) that $-q^* \leqslant z_1(k) \leqslant q^*$, $\forall k \in [-r, 0]$ and $z_2(k) = 0$, $\forall k \in [-r, 0]$.

According to the above, using the decomposition (7) of matrices H and G, we obtain

$$\begin{cases} z_1(k+1) = H_{11}z_1(k) + H_{12}z_2(k) + G_{11}z_1(k-r) + G_{12}z_2(k-r), \ k > 0 \\ z_2(k+1) = H_{21}z_1(k) + H_{22}z_2(k) + G_{21}z_1(k) + G_{22}z_2(k-r), \ k > 0 \\ z_1(\theta) = \psi_1(\theta) \textit{ and } z_2(\theta) = \psi_2(\theta), \ \theta \in [-r, 0] \end{cases} \tag{10}$$

Since $H_{21} = G_{21} = 0$, then

$$\begin{cases} z_1(k+1) = H_{11}z_1(k) + H_{12}z_2(k) + G_{11}z_1(k-r) + G_{21}z_2(k-r), \ k > 0 \\ z_1(\theta) = \psi_1(\theta), \ \theta \in [-r, 0] \end{cases} \tag{11}$$

and

$$\begin{cases} z_2(k+1) = H_{22}z_2(k) + G_{21}z_2(k-r), \ k > 0 \\ z_2(\theta) = \psi_2(\theta), \ \theta \in [-r, 0] \end{cases} \tag{12}$$

From (12) and $z_2(k) = 0$ for all $k \in [-r, 0]$ we deduce that $z_2(k) = 0$ for all $k \geqslant -r$.

Using $z_2(k) = 0, \forall k \geqslant -r$, system (11), can be written as

$$\begin{cases} z_1(k+1) = H_{11} z_1(k) + G_{11} z_1(k-r), \ k > 0 \\ z_1(\theta) = \psi_1(\theta), \ \theta \in [-r, 0] \end{cases} \tag{13}$$

with $-q^* \leqslant z_1(k) \leqslant q^*$, $\forall k \geqslant -r$ and $q^* > 0$.

By replacing q by q^* in (8), and system (6) by system (13), we deduce from [24] that $D(I_s, q^*, q^*)$ is positively invariant with respect to system (13), that is

$$-q^* \leqslant z_1(k) \leqslant q^*, \ \forall k \geqslant -r$$

finally

$$-\begin{pmatrix} q^* \\ 0 \end{pmatrix} \leqslant z(k) \leqslant \begin{pmatrix} q^* \\ 0 \end{pmatrix}, \ \forall k \geqslant -r$$

this implies that the polyhedral set $D(I_m, q, q)$ is positively invariant independent of delay with respect to system (6).

Only If:) Assume that the polyhedral set $D(I_m, q, q)$ is positively invariant with respect to system (6).

Let $z(.)$ be the solution of system (6), with

$$-q \leqslant z(k) \leqslant q, \ \forall k \in [-r, 0]$$

The positive invariance of the set $D(I_m, q, q)$ implies that

$$-q \leqslant z(k) \leqslant q, \ \forall k \geqslant -r$$

Therefore

$$-\begin{pmatrix} q^* \\ 0 \end{pmatrix} \leqslant \begin{pmatrix} z_1(k) \\ z_2(k) \end{pmatrix} \leqslant \begin{pmatrix} q^* \\ 0 \end{pmatrix}, \ \forall k \geqslant -r$$

then $z_2(k) = 0$ for all $k \geqslant -r$.

From system (10), we obtain

$$z_2(k+1) = H_{21} z_1(k) + G_{21} z_1(k-r) = 0 \ , \ \forall k \geqslant 0$$

for $k = 0$, we have

$$z_2(1) = H_{21} z_1(0) + G_{21} z_1(-r) = 0 \ , \ \forall z_1(0), z_1(-r) \in D(I_s, q_1, q_1)$$

which implies $H_{21} = G_{21} = 0$.

From system (10) and $H_{21} = G_{21} = 0$, we deduce that

$$\begin{cases} z_1(k+1) = H_{11} z_1(k) + G_{11} z_1(k-r), \ k > 0 \\ z_1(\theta) = \psi_1(\theta), \ \theta \in [-r, 0] \end{cases} \tag{14}$$

with

$$-q^* \leqslant z_1(k) \leqslant q^* \ , \ \forall k \geqslant -r, \ q^* > 0$$

this implies that the domain $D(I_s, q^*, q^*)$, with $q^* > 0$, is a positively invariant set respectively to system (14). As mentioned at the beginning of the subsection we deduce from [24] that $(|H_{11}| + |G_{11}|)q^* \leqslant q^*$. □

2.2. Conditions for positive invariance independent of delay.

In this subsection, we will use the following Lemma.

Lemma 2.3. [24, *Lemma 3.1*] *The solution of system* (6) *satisfies the following relation:*

$$z(k-r) = z(k) - \sum_{i=0}^{r-1} [z(i+k+1-r) - z(i+k-r)] \tag{15}$$

We use a similar reasoning as in [Halle [22]]. System (15) can be written as

$$z(k-r) = z(k) - \sum_{i=0}^{r-1} [(H-I)z(i+k-r) - Gz(i+k-2r)] \tag{16}$$

If we substitute this expression for $z(k-r)$ back into system (6), we obtain the equation:

$$z(k+1) = (H+G)z(k) - G\sum_{i=0}^{r-1} [(H-I)z(i+k-r) + Gz(i+k-2r)] \tag{17}$$

Then

$$\begin{cases} z(k+1) = Mz(k) + \sum_{i=0}^{r-1} [Vz(i+k-r) + Wz(i+k-2r)], \ k > 0 \\ z(\theta) = \phi(\theta), \ \theta \in [-2r, 0] \end{cases} \tag{18}$$

with $M = H + G$, $V = G(I - H)$, $W = -G^2$.

If the zero solution of (18) is asymptotically stable for arbitrary initial data on $[-2r, 0]$, then the zero solution of (6) is also asymptotically stable, as (6) is a specific case of (18). To simplify the analysis, we will use the system dynamics described in (18) to obtain stability or positive invariance conditions for system (6).

In the forthcoming, we will provide a set of conditions that are both necessary and sufficient to ensure the positive invariance of $D(I_m, q, q)$ with respect to the motions of system (6), accounting for delay dependence.

Remark 1. In [24] the author has proved that $D(I_m, q, q)$, with $q > 0$, is positively invariant dependent of delay with respect to system (6) if and only if

$$(|M| + r(|V| + |W|))q \leqslant q \tag{19}$$

In this paper, where $q = \begin{pmatrix} q^* \\ 0 \end{pmatrix}$, $q^* > 0$, we prove the following result.

Theorem 2.4. *The polyhedral set* $D(I_m, q, q)$ *with* $q = \begin{pmatrix} q^* \\ 0 \end{pmatrix}$, $q^* > 0$ *and* $q^* \in \mathbb{R}^s$ *is positively invariant dependent of delay with respect to system* (6) *if and only if*

$$(|M_{11}| + r(|V_{11}| + |W_{11}|))q^* \leqslant q^* \text{ and } M_{21} = V_{21} = W_{21} = 0$$

where M_{11}, M_{21}, V_{11}, V_{21}, W_{11} *and* W_{21} *are given by decomposition* (7).

Proof.

If:) Let assume that $(|M_{11}| + r(|V_{11}| + |W_{11}|))q^* \leqslant q^*$ *and* $M_{21} = V_{21} = W_{21} = 0$.

Let $z(.)$ be a solution of system (18) with

$$-\begin{pmatrix} q^* \\ 0 \end{pmatrix} \leqslant z(k) \leqslant \begin{pmatrix} q^* \\ 0 \end{pmatrix}, \ \forall k \in [-2r, 0] \tag{20}$$

Let $z(k) = \begin{pmatrix} z_1(k) \\ z_2(k) \end{pmatrix}$, where $z_1(k) \in \mathbb{R}^s$ and $z_2(k) \in \mathbb{R}^{m-s}$. Then equation (20) implies that $-q^* \leqslant z_1(k) \leqslant q^*$ and $z_2(k) = 0$ for all $k \in [-2r, 0]$.

According to the above, using the decomposition (7) of matrices M, V and W, we have

$$\begin{cases} z_1(k+1) &= M_{11}z_1(k) + M_{12}z_2(k) + \sum_{i=0}^{r-1}[V_{11}z_1(i+k-r) + V_{12}z_2(i+k-r) \\ & \quad +W_{11}z_1(i+k-2r) + W_{12}z_2(i+k-2r)], \ k > 0 \\ z_1(\theta) &= \phi_1(\theta), \ \theta \in [-2r, 0] \end{cases} \tag{21}$$

and

$$\begin{cases} z_2(k+1) &= M_{21}z_1(k) + M_{22}z_2(k) + \sum_{i=0}^{r-1}[V_{21}z_1(i+k-r) + V_{22}z_2(i+k-r) \\ & \quad +W_{21}z_1(i+k-2r) + W_{22}z_2(i+k-2r)], \ k > 0 \\ z_2(\theta) &= \phi_2(\theta), \ \theta \in [-2r, 0] \end{cases} \tag{22}$$

Using $M_{21} = V_{21} = W_{21} = 0$, we obtain

$$\begin{cases} z_2(k+1) = M_{22}z_2(k) + \sum_{i=0}^{r-1}[V_{22}z_2(i+k-r) + W_{22}z_2(i+k-2r)], \ k > 0 \\ z_2(\theta) = \phi_2(\theta), \ \theta \in [-2r, 0] \end{cases} \tag{23}$$

By $z_2(k) = 0$, $\forall k \in [-2r, 0]$, we can deduce that $z_2(k) = 0$, $\forall k \geqslant -2r$. System (21) can be written as

$$\begin{cases} z_1(k+1) = M_{11}z_1(k) + \sum_{i=0}^{r-1}[V_{11}z_1(i+k-r) + W_{11}z_1(i+k-2r)], \ k > 0 \\ z_1(\theta) = \phi_1(\theta), \ \theta \in [-2r, 0] \end{cases} \tag{24}$$

with $-q^* \leqslant z_1(k) \leqslant q^*$, $\forall k \geqslant -2r$ and $q^* > 0$.

By virtue of remark 1, we deduce that

$$-q^* \leqslant z_1(k) \leqslant q^*, \ \forall k \geqslant -2r$$

finally, we obtain $-q \leqslant z(k) \leqslant q, \ \forall k \geqslant -2r$. This implies that the polyhedral set $D(I_m, q, q)$ is positively invariant dependent of delay with respect to system (6).

Only If:) Assume that the polyhedral set $D(I_m, q, q)$ is positively invariant dependent of delay with respect to system (6). Let $z(.)$ the solution of system (18) with

$$-q \leqslant z(k) \leqslant q, \ \forall k \in [-2r, 0]$$

The positive invariance of the set $D(I_m, q, q)$ implies that

$$-q \leqslant z(k) \leqslant q, \ \forall k \geqslant -2r$$

Therefore

$$-\begin{pmatrix} q^* \\ 0 \end{pmatrix} \leqslant \begin{pmatrix} z_1(k) \\ z_2(k) \end{pmatrix} \leqslant \begin{pmatrix} q^* \\ 0 \end{pmatrix}, \ \forall k \geqslant -2r$$

then $z_2(k) = 0, \forall k \geqslant -2r$. From system (22), we obtain

$$z_2(k+1) = H_{21} z_1(k) + \sum_{i=0}^{r-1} [V_{21} z_1(i+k-r) + W_{21} z_1(i+k-2r)] = 0, \ \forall k \geqslant -2r$$

Therefore, we have

$$z_2(1) = H_{21} z_1(0) + \sum_{i=0}^{r-1} [V_{21} z_1(i-r) + W_{21} z_1(i-2r)] = 0$$

for all $z_1(0)$, $z_1(i-r)$ and $z_1(i-2r)$ in $D(I_s, q^*, q^*)$ with $i = 1, ..., r-1$, this implies that $M_{21} = V_{21} = W_{21} = 0$.

From system (21) and $M_{21} = V_{21} = W_{21} = 0$, we deduce that

$$\begin{cases} z_1(k+1) = M_{11} z_1(k) + \sum_{i=0}^{r-1} [V_{11} z_1(i+k-r) + W_{11} z_1(i+k-2r)], \ k > 0 \\ z_1(\theta) = \phi_1(\theta), \ \theta \in [-2r, 0] \end{cases} \tag{25}$$

with

$$-q^* \leqslant z_1(k) \leqslant q^*, \ \forall k \geqslant -2r$$

By virtue of remark 1, we obtain $(|M_{11}| + r(|V_{11}| + |W_{11}|))q^* \leqslant q^*$. □

3. Main results

In this section, we will establish sufficient conditions for a linear state feedback control law $u(.) = Fx(.)$, with $F \in \mathbb{R}^{m \times n}$ and rank$F = m$ to be a solution to the linear constrained regulation problem. For that, we need the two lemmas below.

Lemma 3.1. [24, *Lemma 4.1*] *The set* $KerF$ *with* $F \in \mathbb{R}^{m\times n}$*, and* $rankF = m$ *is positively invariant with respect to motions of system* (4) *if and only if there exist matrices* $H,\ G \in \mathbb{R}^{m\times n}$ *satisfying:*

$$\begin{cases} F(A_0 + BF) = HF & (a1) \\ FA_1 = GF & (a2) \end{cases}$$

Lemma 3.2. [24, *Lemma 4.2*] *If domain* $D(F, q, q)$ *is positively invariant with respect to system* (4)*, then* $kerF$ *is also positively invariant with respect to system* (4)*.*

Remark 2. The strict positivity of q in Lemma 3.2 is not necessary.

In the following, we apply the results established in section 2 and the results of Lemma 3.1 and Lemma 3.2 to the problem of the constrained regulator described in section 1, we obtain the following results.

3.1. **Independent of delay case.**

Theorem 3.3. *The polyhedral set* $D(F, q, q)$ *with* $F \in \mathbb{R}^{m\times n}$*, and* $rankF = m$ *is positively invariant independent of delay with respect to system* (4) *if and only if there exist matrices* $H,\ G \in \mathbb{R}^{m\times n}$ *satisfying:*

$$\begin{cases} F(A_0 + BF) = HF & (b1) \\ FA_1 = GF & (b2) \\ (|H_{11}| + |G_{11}|)q^* \leqslant q^* \text{ and } H_{21} = G_{21} = 0 & (b3) \end{cases}$$

where H_{11}*,* H_{21}*,* G_{11} *and* G_{21} *are given by the decomposition* (7)*.*

Proof.

Necessity: Suppose the domain $D(F, q, q)$ is positively invariant with respect to system (4). According to lemma 3.2, Ker F, is also positively invariant with respect to system (4), and from lemma 3.1, we deduce that there exist matrices H and $G \in \mathbb{R}^{m\times n}$ that satisfy:

$$\begin{cases} F(A_0 + BF) = HF & (b1) \\ FA_1 = GF & (b2) \end{cases}$$

Consider the change in variables $y(k) = Fx(k)$. By conditions $(b1)$ and $(b2)$, system (4), can be transformed to system (6), and $D(F, q, q)$ to domain $D(I_m, q, q)$ which is also positively invariant with respect to system (6) and by virtue of theorem 2.2 this is equivalent to condition $(b3)$.

Sufficiency: By introducing the change in variables $y(k) = Fx(k)$ and considering the conditions $(b1)$ and $(b2)$, we can transforme system (4) to system (6), and domain $D(F, q, q)$ to domain $D(I_m, q, q)$.

Using theorem 2.2, condition $(b3)$ guarantees the positive invariance of domain $D(I_m, q, q)$ with respect to system (6), which is equivalent to the positive invariance of $D(F, q, q)$ with respect to system (4). □

We are now in a position to establish conditions for a linear state feedback control law $u(.) = Fx(.)$ to be a solution to the linear constrained regulation problem.

Theorem 3.4. *For a matrix $F \in \mathbb{R}^{m\times n}$, with rank$F = m$ if there exist matrices $H,\ G \in \mathbb{R}^{m\times n}$ satisfying:*

$$\begin{cases} F(A_0 + BF) = HF & (c1) \\ FA_1 = GF & (c2) \\ (|H_{11}| + |G_{11}|)q^* < q^* \text{ and } H_{21} = G_{21} = 0 & (c3) \end{cases}$$

where H_{11}, H_{21}, G_{11} and G_{21} are given by decomposition (7), then $u(.) = Fx(.)$ assymptotically stabilizy the system (4) when the initial data $x(\theta) \in D(F, q, q)$, $\forall\theta \in [-r, 0]$ and also satisfies the constraints 2.

Proof.
By virtue of theorem 3.3, the conditions $(c1)$, $(c2)$ and $(c3)$ imply the positive invariance of the set $D(F, q, q)$. To complete the proof we shall prove that $u(.) = Fx(.)$ is a stabilizing control in $D(F, q, q)$). Let us make the change in variables $z(k) = Fx(k)$, with rank$F = m$. From conditions $(c1)$ and $(c2)$ it follows that $z(.)$ is a solution of a system described by (6) and the domain $D(I_m, q, q)$ is positively invariant independent of delay with respect to system (6).
Let $x(.)$ a solution of a system (4), such that $x(\theta) \in D(F, q, q)$ for $\theta \in [-r, 0]$, by positive invariance of $D(F, q, q)$, we deduce $x(k) \in D(F, q, q)$, $\forall k > 0$ thus, we have $z(k) \in D(I_m, q, q)$, or equivalently $z_1(k) \in D(I_s, q^*, q^*)$ and $z_2(k) = 0$, $\forall k > 0$, where $z_1 \in \mathbb{R}^s$ and $z_2 \in \mathbb{R}^{m-s}$. From system (6) and decomposition (7) of matrices H and G it follows that $z_1(.)$ verify system (13). Consider the norm on $\mathbb{R}^s$ defined by

$$\|v\| = \max_{0\le i\le s} \frac{|\ v_i\ |}{q_i^*}$$

where q_i^* is the i-th component of q^* and v_i the i-th component of v. The subordinate matrix norm based on the norm $\|.\|$ is given by

$$\|L\| = \max_{\|\omega\|\neq 0} \frac{\|L\omega\|}{\|\omega\|}$$

with L a matrix of order s.
put $L = (|H_{11}| + |G_{11}|) \geqslant 0$, by $(|H_{11}| + |G_{11}|)q^* < q^*$, we have

$$\|L\| < 1$$

On the other hand, it can be observed that the necessary and sufficient condition to ensure that all eigenvalues of system (6) are located within the unit disk $D(0; 1)$.

All the solutions of the characteristic equation

$$det(tI_m - H_{11} - t^{-r}G_{11}) = 0 \tag{26}$$

verify $|t| < 1$. Suppose that there exists a solution of the characteristic equation (26), with $|t| \geqslant 1$. We have (See [31])

$$\begin{aligned} |t| &\leqslant \rho(H_{11} + t^{-r}G_{11}) \\ &\leqslant \rho(|H_{11}| + |t^{-r}||G_{11}|) \\ &\leqslant \rho(|H_{11}| + |G_{11}|) \;\; because \;\; |t^{-r}| \leqslant 1 \\ &\leqslant \|L\| \end{aligned}$$

This is a contradiction.

Then $\lim_{k\to+\infty}(z_1(k;\psi_1(\theta))) = 0$. From $z_2(k) = 0,\ \forall k > 0$, we deduce that $\lim_{k\to+\infty}(z(k;\psi(\theta))) = 0$. By rank $F = m$, we have $\lim_{k\to+\infty}(x(k;\varphi(\theta))) = 0$. □

3.2. Dependent of delay case.

Theorem 3.5. *The polyhedral set $D(F, q, q)$ with $F \in \mathbb{R}^{m\times n}$ and $rankF = m$ is positively invariant dependent of delay with respect to system (4) if and only if there exist matrices H and $G \in \mathbb{R}^{m\times n}$ satisfying:*

$$\begin{cases} F(A_0 + BF) = HF & (d1) \\ FA_1 = GF & (d2) \\ (|M_{11}| + r(|V_{11}| + |W_{11}|))q^* \leqslant q^* \; and \; M_{21} = V_{21} = W_{21} = 0 & (d3) \end{cases}$$

where M_{11}, M_{21}, V_{11}, V_{21}, W_{11} and W_{21} are given by decomposition (7).

Proof. This follows readily from theorem 2.4 and theorem 3.3. □

Theorem 3.6. *For a matrix $F \in \mathbb{R}^{m\times n}$, with $rankF = m$, if there exist matrices $H,\ G \in \mathbb{R}^{m\times n}$ satisfying:*

$$\begin{cases} F(A_0 + BF) = HF & (e1) \\ FA_1 = GF & (e2) \\ (|M_{11}| + r(|V_{11}| + |W_{11}|))q^* < q^* \; and \; M_{21} = V_{21} = W_{21} = 0 & (e3) \end{cases}$$

where M_{11}, M_{21}, V_{11}, V_{21}, W_{11} and W_{21} are given by decomposition (7), then $u(.) = Fx(.)$ assymptotically stabilizy the system (4) when the initial data $x(\theta) \in D(F, q, q)$, $\forall\theta \in [-r, 0]$ and also satisfies the constraints 2.

Proof.

By virtue of theorem 3.5, the conditions $(e1)$, $(e2)$ and $(e3)$ imply the positive invariance of the set $D(F, q, q)$. To complete the proof, we shall prove that $u(.) = Fx(.)$ is a stabilizing control in $D(F, q, q)$. The change in variable $z(k) = Fx(k)$ transform the system (4) to system (6). The usual scheme used in the literature for obtaining delay-dependent stability is to use system (18) instead of system (6), the asymptotic stability of (18) guarantees the asymptotic stability of (6) .

Let $x(.)$ a solution of a system (4), such that $x(\theta) \in D(F, q, q)$ for $\theta \in [-2r, 0]$, by positive invariance of $D(F, q, q)$, we deduce that $x(k) \in D(F, q, q)$, $\forall k \geqslant 0$ thus, we have $z(k) \in D(I_m, q, q)$, $\forall k > 0$ with $q = \begin{pmatrix} q^* \\ 0 \end{pmatrix}$, then by using the decomposition (7) of matrices M, V and W, we obtain $z_2(k) = 0$ and the system (24), with $z_1(k) \in D(I_s, q^*, q^*)$, and $z_1 \in \mathbb{R}^s$.

Consider the norm

$$\|\upsilon\| = \max_{0 \leqslant i \leqslant s} \frac{|\upsilon_i|}{q_i^*}$$

with q_i^* the i-th component of q^* and υ_i the i-th component of υ. The subordinate matrix norm based on the norm $\|.\|$ is given by

$$\|L\| = \max_{\|\omega\| \neq 0} \frac{\|L\omega\|}{\|\omega\|}$$

Put $L = (|M_{11}| + r(|V_{11}| + |W_{11}|)) \geqslant 0$, by $(|M_{11}| + r(|V_{11}| + |W_{11}|))q^* < q^*$, we have

$$\|L\| < 1$$

On the other hand, it is seen that (See [29]) the necessary and sufficient condition for the asymptotic stability of (24) is that all the solutions of the characteristic equation

$$det\left[tI_m - M_{11} - \sum_{i=0}^{r-1}[V_{11}t^{i-r} + W_{11}t^{i-2r}]\right] = 0 \tag{27}$$

verify $|t| < 1$. By contradiction, suppose that there exists a solution of (27) with $|t| \geqslant 1$, then

$$\begin{aligned} |t| &\leqslant \rho(M_{11} + \textstyle\sum_{i=0}^{r-1}\left[V_{11}t^{i-r} + W_{11}t^{i-2r}\right]) \\ &\leqslant \rho(|M_{11}| + \textstyle\sum_{i=0}^{r-1}[|V_{11}||t^{i-r}| + |W_{11}||t^{i-2r}|]) \\ &\leqslant \rho((|M_{11}| + r(|V_{11}| + |W_{11}|)),\ because\ \ |t^{i-r}| \leqslant 1\ and\ |t^{i-2r}| \leqslant 1 \\ &\leqslant \|L\| \end{aligned}$$

which is a contradiction.

Then $\lim_{k \to +\infty}(z_1(k; \phi_1(\theta))) = 0$ and $z_2(k) = 0,\ \forall k \geqslant -2r$, hence $\lim_{k \to +\infty}(z(k; \phi(\theta))) = 0$. By $rankF = m$, we deduce that $\lim_{k \to +\infty}(x(k; (\theta))) = 0$. □

4. Algorithm

All the results that we have indicated in section 4 are based on the existence of the matrices H and G. It is obvious that there existence depends on the matrices A_0, A_1,B and F. For this we have the following lemmas.

Lemma 4.1 (Hmamed et al [25]). *There exists a matrix $H \in \mathbb{R}^{m\times m}$ and $G \in \mathbb{R}^{m\times m}$ solution of*

$$\begin{cases} F(A_0 + BF) = HF \\ FA_1 = GF \end{cases} \tag{28}$$

where $F \in \mathbb{R}^{m\times n}$ and $rankF = m$, $m \leqslant n$ if and only if

$$rank \begin{bmatrix} F(A_0+BF) \\ F \end{bmatrix} = m \text{ and } rank \begin{bmatrix} FA_1 \\ F \end{bmatrix} = m \tag{29}$$

Corollary 4.2. *If condition (29) is satisfied, then the solution of (28) is given by*

$$H = [F_1((A_0)_{11} + B_1F_1) + F_2((A_0)_{21} + B_2F_2)]F_1^{-1} \tag{30}$$

and

$$G = [F_1(A_1)_{11} + F_2(A_1)_{21}]F_1^{-1} \tag{31}$$

with $F = \begin{bmatrix} F_1 & F_2 \end{bmatrix}$, $F_1 \in \mathbb{R}^{m\times m}$, $F_2 \in \mathbb{R}^{m\times n-m}$, $rankF_1 = m$, $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, $B_1 \in \mathbb{R}^{m\times m}$, $B_2 \in \mathbb{R}^{n-m\times m}$, $A_0 = \begin{bmatrix} (A_0)_{11} & (A_0)_{12} \\ (A_0)_{21} & (A_0)_{22} \end{bmatrix}$, $A_1 = \begin{bmatrix} (A_1)_{11} & (A_1)_{12} \\ (A_1)_{21} & (A_1)_{22} \end{bmatrix}$, $(A_0)_{11}, (A_1)_{11} \in \mathbb{R}^{m\times m}$, $(A_0)_{12}, (A_1)_{12} \in \mathbb{R}^{m\times(n-m)}$, $(A_0)_{21}, (A_1)_{21} \in \mathbb{R}^{(n-m)\times m}$ *and* $(A_0)_{22}, (A_1)_{22} \in \mathbb{R}^{(n-m)\times(n-m)}$.

The search of a matrix F solution of the LCRP problem can be done according to the following algorithms:

Algorithm 1: (Independent of delay case)

Step1: Choose $F \in \mathbb{R}^{m\times n}$ such that:

$$rankF = m,\ rank \begin{bmatrix} F(A_0+BF) \\ F \end{bmatrix} = m \text{ and } rank \begin{bmatrix} FA_1 \\ F \end{bmatrix} = m$$

Step2: Compute H and G by equations (30) and (30).

Step3: Using the decomposition (7) of matrices H and G, compute $|H_{11}|$, $|G_{11}|$, $|H_{21}|$ and $|G_{21}|$.

Step4: If $|H_{11}|$ and $|G_{11}|$ satisfy the condition $c3$ of theorem 3.4 then stop, else go to Step1 and change the matrix F.

Algorithm 2: (Dependent of delay case)

Step1: Choose $F \in \mathbb{R}^{m\times n}$ such that:

$$rankF = m,\ rank \begin{bmatrix} F(A_0+BF) \\ F \end{bmatrix} = m \text{ and } rank \begin{bmatrix} FA_1 \\ F \end{bmatrix} = m$$

Step2: Compute H and G by equations (30) and (30).

Step3: Compute M, V and W by equations $M = H + G$, $V = GH$ and $W = G^2$.

Step4: Using the decomposition (7) of matrices M, V and W, compute $|M_{11}|$, $|V_{11}|$, $|W_{11}|$, $|M_{21}|$, $|V_{21}|$

and $|W_{21}|$.

Step5: If $|M_{11}|$, $|V_{11}|$, $|W_{11}|$, $|M_{21}|$, $|V_{21}|$ and $|W_{21}|$ satisfy the condition $c3$ of theorem 3.6 then stop, else go to Step1 and change the matrix F.

5. Example

Consider the system (1) with

$$A_0 = \begin{pmatrix} 2 & -\frac{1}{10} & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{32} & \frac{1}{2} \end{pmatrix}, A_1 = \begin{pmatrix} \frac{1}{4} & \frac{1}{84} & \frac{1}{84} \\ 0 & \frac{3}{4} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \ and \ B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}$$

The control vector $u(.) = \begin{pmatrix} u_1(.) \\ u_2(.) \\ u_3(.) \end{pmatrix} \in \mathbb{R}^3$ is subject to constraints

$$-q \leq u(k) \leq q, \ where \ q = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

or equivalently

$$-3 \leqslant u_1(k) \leqslant 3, \ -1 \leqslant u_2(k) \leqslant 1 \ and \ u_3(k) = 0, \ \forall k \geqslant 0$$

Note that A_0 is unstable. The eigenvalues of A_0 are $\lambda_1 = 2$, $\lambda_2 = 0.59$ and $\lambda_3 = 0.41$.

Let

$$F = \begin{pmatrix} -3 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

then

$$H = \begin{pmatrix} \frac{1}{2} & \frac{-3}{10} & 0 \\ 0 & \frac{7}{16} & \frac{1}{4} \\ 0 & 0 & 0 \end{pmatrix} \ and \ G = \begin{pmatrix} \frac{1}{4} & -\frac{1}{14} & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & \frac{3}{4} \end{pmatrix}$$

by using the decomposition (7), we have

$$H_{11} = \begin{pmatrix} \frac{1}{2} & \frac{-3}{10} \\ 0 & \frac{7}{16} \end{pmatrix}, \ G_{11} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{14} \\ 0 & \frac{1}{2} \end{pmatrix}, \ H_{21} = \begin{pmatrix} 0 & 0 \end{pmatrix} \ and \ G_{21} = 0$$

therefore

$$|H_{11}| = \begin{pmatrix} \frac{1}{2} & \frac{3}{10} \\ 0 & \frac{7}{16} \end{pmatrix} \ and \ |G_{11}| = \begin{pmatrix} \frac{1}{4} & \frac{1}{14} \\ 0 & \frac{1}{2} \end{pmatrix}$$

We can verify that $|H_{11}|$ and $|G_{11}|$ satisfy the hypothesis of theorem 2.2. Then $u_1(.) = -3x_1$, $u_2(.) = \frac{1}{4}x_2$ and $u_3(.) = \frac{1}{4}x_2 + \frac{1}{2}x_3$ stabilizes the system on

$$D(F, q, q) = \{x \in \mathbb{R}^3 \mid \ -1 \leqslant x_1 \leqslant 1 \ ; \ -4 \leqslant x_2 \leqslant 4 \ ; \ x_2 + 2x_3 = 0\}$$

In Figure 1, we plot the trajectory of system (1), for the initial conditions $\varphi_2 = [-1, 4, -2]^T$ in $D(F, q, q)$ with a delay $r = 1$. We notice that the trajectory of our system does not leave the domain $D(F, q, q)$ for any instant $k \geqslant 0$, moreover it converge asymptotically to the origin $x_e = 0$. The same results are obtained for arbitrarily initial conditions $\varphi \in D(F, q, q)$.

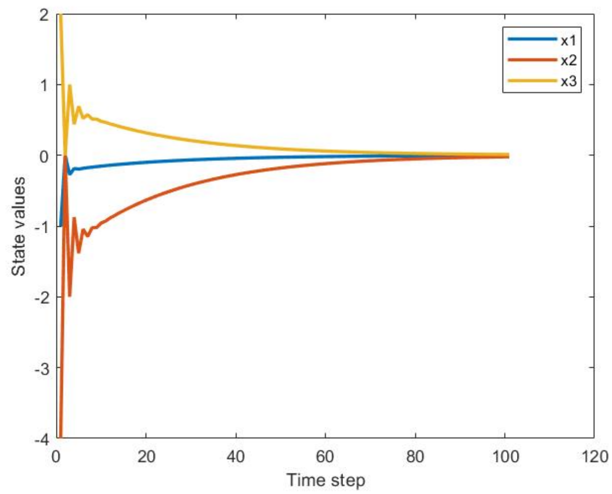

FIGURE 1. The asymptotic stability of the closed-loop system for initial state $\varphi_2 = [-1, 4, -2]^T$ with $r = 1$.

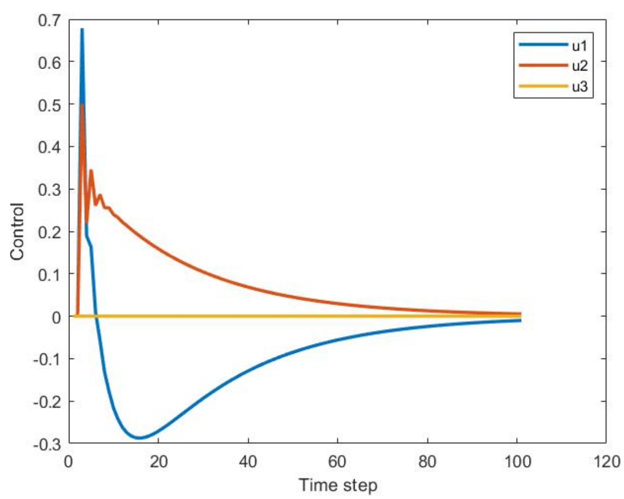

FIGURE 2. The evolution of the control u.

6. Conclusion

The symmetrical constrained regulation problem for discrete-time delay dynamical systems with origin on the boundary of the domain of constraints is studied.
Two cases are considered: delay-independent and delay-dependent. In each case we use properties of positive invariance to give sufficient condition for a state feedback control law $u(.) = Fx(.)$ to be a solution to the linear constrained regulation problem. Finally, an example of application of the results is given.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] A. Abdelhak, R. Ou-azzou, Further results on the regulation problem for linear systems with constraints on control and its increment, Math. Model. Comput. 10 (2023), 1063–1070. https://doi.org/10.23939/mmc2023.04.1063.

[2] A. Belhouari, E. Tissir, A. Hmamed, Stability of interval matrix polynomial in continuous and discrete cases, Syst. Control Lett. 18 (1992), 183–189. https://doi.org/10.1016/0167-6911(92)90004-c.

[3] H. Bensalah, L. Baron, Positive invariance of constrained linear continuous-time delay system with delay dependence, in: Proceedings of the International Conference of Control, Dynamic Systems, and Robotics Ottawa, Ontario, Canada, 2015.

[4] A. Benzaouia, The regulator problem for linear discrete-time systems with nonsymmetrical constrained control, in: Proceedings of the 30th IEEE Conference on Decision and Control, Brighton, pp. 1742–1743, 1991.

[5] A. Benzaouia, Further results on the saturated controller design for linear continuous-time systems, in: Proceedings of the 10th Mediterranean Conference on Control and Automation - MED2002, Lisbon, Portugal, 2002.

[6] A. Benzaouia, C. Burgat, Regulator problem for linear discrete-time systems with non-symmetrical constrained control, Int. J. Control. 48 (1988), 2441–2451. https://doi.org/10.1080/00207178808906339.

[7] A. Benzaouia, S. El Faiz, The regulator problem for linear systems with constrained control: an LMI approach, IMA J. Math. Control Inf. 23 (2006), 335–345. https://doi.org/10.1093/imamci/dni062.

[8] A. Benzaouia, A. Hmamed, Regulator problem for linear continuous-time systems with nonsymmetrical constrained control, IEEE Trans. Autom. Control. 38 (1993), 1556–1560. https://doi.org/10.1109/9.241576.

[9] A. Benzaouia, A. Hmamed, F. Tadeo, Stabilisation of controlled positive delayed continuous-time systems, Int. J. Syst. Sci. 41 (2010), 1473–1479. https://doi.org/10.1080/00207720903353641.

[10] G. Bitsoris, Positively invariant polyhedral sets of discrete-time linear systems, Int. J. Control. 47 (1988), 1713–1726. https://doi.org/10.1080/00207178808906131.

[11] G. Bitsoris, S. Olaru, Further results on the linear constrained regulation problem, in: 21st Mediterranean Conference on Control and Automation, IEEE, Platanias, Chania - Crete, Greece, 2013: pp. 824-830. https://doi.org/10.1109/MED.2013.6608818.

[12] G. Bitsoris, S. Olaru, M. Vassilaki, On the linear constrained regulation problem for continuous-time systems, IFAC Proc. Vol. 47 (2014), 4004–4009. https://doi.org/10.3182/20140824-6-za-1003.02558.

[13] Tadeusz Kaczorek Mikołaj Busłowicz, Robust stability of positive discrete-time interval systems with time-delays, Bull. Polish Acad. Sci. Techn. Sci. 52 (2004), 99–102.

[14] M. Busłowicz, Simple conditions for robust stability of linear positive discrete-time systems with one delay, J. Autom. Mob. Robot. Intell. Syst. 2 (2008), 18–22.

[15] M. Busłowicz, Robust stability of positive continuous-time linear systems with delays, Int. J. Appl. Math. Comp. Sci. 20 (2010), 665–670. https://doi.org/10.2478/v10006-010-0049-8.

[16] D.W. Clarke, C. Mohtadi, P.S. Tuffs, Generalized predictive control–Part I. The basic algorithm, Automatica 23 (1987), 137–148. https://doi.org/10.1016/0005-1098(87)90087-2.

[17] D.W. Clarke, C. Mohtadi, P.S. Tuffs, Generalized predictive control–Part II extensions and interpretations, Automatica 23 (1987), 149–160. https://doi.org/10.1016/0005-1098(87)90088-4.

[18] M. Dambrine, J.P. Richard, P. Borne, Feedback control of time-delay systems with bounded control and state, Math. Probl. Eng. 1 (1995), 77–87. https://doi.org/10.1155/s1024123x95000081.

[19] C.E.T. Dórea, S. Olaru, S.I. Niculescu, Delay-dependent polyhedral invariant sets for continuous-time linear systems, IFAC-PapersOnLine 55 (2022), 108–113. https://doi.org/10.1016/j.ifacol.2022.11.316.

[20] E.G. Gilbert, K.T. Tan, Linear systems with state and control constraints: the theory and application of maximal output admissible sets, IEEE Trans. Autom. Control 36 (1991), 1008–1020. https://doi.org/10.1109/9.83532.

[21] P.-O. Gutman, P. Hagander, A new design of constrained controllers for linear systems, IEEE Trans. Autom. Control 30 (1985), 22–33. https://doi.org/10.1109/tac.1985.1103785.

[22] J.K. Hale, Theory of functional differential equations, Springer, New York, 1977. https://doi.org/10.1007/978-1-4612-9892-2.

[23] J.K. Hale, S.M.V. Lunel, Introduction to functional differential equations, Springer, New York, 1993. https://doi.org/10.1007/978-1-4612-4342-7.

[24] A. Hmamed, Constrained regulation of linear discrete-time systems with time delay: Delay-dependent and delay-independent conditions, Int. J. Syst. Sci. 31 (2000), 529–536. https://doi.org/10.1080/002077200291109.

[25] A. Hmamed, A. Benzaouia, H. Bensalah, Regulator problem for linear continuous-time delay systems with nonsymmetrical constrained control, IEEE Trans. Autom. Control 40 (1995), 1615–1619. https://doi.org/10.1109/9.412630.

[26] A. Hmamed, A. Benzaouia, M. Ait Rami, F. Tadeo, Positive stabilization of discrete-time systems with unknown delay and bounded controls, in: 2007 European Control Conference (ECC), IEEE, Kos, 2007: pp. 5616–5622. https://doi.org/10.23919/ECC.2007.7068251.

[27] A. Hmamed, M. Ait Rami, A. Benzaouia, F. Tadeo, Stabilization under constrained states and controls of positive systems with time delays, Eur. J. Control 18 (2012), 182–190. https://doi.org/10.3166/ejc.18.182-190.

[28] A. Hmamed, A. Benzaouia, M.A. Rami, F. Tadeo, Memoryless control to drive states of delayed continuous-time systems within the nonnegative orthant, IFAC Proc. Vol. 41 (2008), 3934–3939. https://doi.org/10.3182/20080706-5-KR-1001.00662.

[29] A. Hmamed, E. Tissir, Further results on the stability of discrete-time matrix polynomials, Int. J. Syst. Sci. 29 (1998), 819–821. https://doi.org/10.1080/00207729808929574.

[30] T. Kaczorek, Stability of positive continuous-time linear systems with delays, in: 2009 European Control Conference (ECC), IEEE, Budapest, 2009: pp. 1610–1613. https://doi.org/10.23919/ECC.2009.7074637.

[31] P. Lancaster, M. Tismenetsky, The theory of matrices: with applications, 2nd ed, Academic Press, Orlando, 1985.

[32] M. Mansour, Robust stability of interval matrices, in: Proceedings of the 28th IEEE Conference on Decision and Control, IEEE, Tampa, FL, USA, 1989: pp. 46–51. https://doi.org/10.1109/CDC.1989.70071.

[33] J.C. Hennet, S. Tarbouriech, Stability and stabilization of delay differential systems, Automatica 33 (1997), 347–354. https://doi.org/10.1016/s0005-1098(96)00185-9.

[34] N. Stanković, S. Olaru, S.-I. Niculescu, Further remarks on asymptotic stability and set invariance for linear delay-difference equations, Automatica 50 (2014), 2191–2195. https://doi.org/10.1016/j.automatica.2014.05.019.

[35] M.T. Laraba, S. Olaru, S.I. Niculescu, F. Blanchini, G. Giordano, D. Casagrande, S. Miani, Guide on set invariance for delay difference equations, Ann. Rev. Control 41 (2016), 13–23. https://doi.org/10.1016/j.arcontrol.2016.04.020.

[36] O.A. Rachid, A. Ahmed, Regulator problem for linear continuous-time delay dynamical systems with symmetrical constrained control, Asia Pac. J. Math. 10 (2023), 37. https://doi.org/10.28924/APJM/10-37.

[37] B. Porter, Eigenvalue assignment in linear multivariable systems by output feedback, Int. J. Control 25 (1977), 483–490. https://doi.org/10.1080/00207177708922246.

[38] B.S. Stojanovic, Lj.D. Debeljkovic, Delay dependent stability of linear time-delay systems, Theor. Appl. Mech. 40 (2013), 223–245. https://doi.org/10.2298/tam1302223s.

[39] M. Vassilaki, G. Bitsoris, Constrained regulation of linear continuous-time dynamical systems, Syst. Control Lett. 13 (1989), 247–252. https://doi.org/10.1016/0167-6911(89)90071-6.

[40] W. Wonham, On pole assignment in multi-input controllable linear systems, IEEE Trans. Autom. Control 12 (1967), 660–665. https://doi.org/10.1109/tac.1967.1098739.

[41] X. Liu, W. Yu, L. Wang, Stability analysis of positive systems with bounded time-varying delays, IEEE Trans. Circuits Syst. II 56 (2009), 600–604. https://doi.org/10.1109/tcsii.2009.2023305.