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CHARACTERIZING ε-ŁUKASIEWICZ FUZZY UP (BCC)-FILTERS IN UP (BCC)-ALGEBRAS

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Abstract. This paper presents the development of ε-Łukasiewicz fuzzy sets using the Łukasiewicz t-norm derived from a given fuzzy set. These ε-Łukasiewicz fuzzy sets are subsequently applied to UP (BCC)-algebras. Additionally, the paper introduces the concept of ε-Łukasiewicz fuzzy UP (BCC)-filters and examines their various properties. It is evident that ε-Łukasiewicz fuzzy UP (BCC)-filters represent a broader generalization of ε-Łukasiewicz fuzzy UP (BCC)-ideals. Three specific subsets, termed the $\in$-set, q-set, and O-set, are constructed, with an exploration of the conditions under which these subsets qualify as UP (BCC)-filters.

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1. Introduction

Zadeh [21] initially proposed the concept of fuzzy sets, which have found numerous real-world applications, leading to extensive exploration of their principles by researchers. Subsequent to the introduction of fuzzy sets, considerable effort has been dedicated to their generalization. The intersection of fuzzy sets with other uncertainty models, such as soft sets and rough sets, has been investigated in [1–3]. Advancements in technology have enabled sophisticated inference and problem-solving capabilities, particularly in adapting to varying themes through programming. Łukasiewicz logic, defined by the Łukasiewicz t-norm, represents a non-classical, multi-valued logic originally formulated in the early 20th century with three truth values. An important extension in fuzzy set theory is the ε-Łukasiewicz

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fuzzy set, which originates from Łukasiewicz logic, a non-classical, many-valued logic. This type of fuzzy set utilizes the Łukasiewicz t-norm and t-conorm to define operations such as intersection, union, and complement. The parameter ε is introduced to enhance flexibility and control over the degree of fuzziness within the set. Foundational works on the ε-Łukasiewicz fuzzy set can be explored in [5,6,14].

Iampan [9] introduced UP-algebras as an innovative algebraic structure. Subsequently, Somjanta et al. [19] and Guntasow et al. [7] integrated fuzzy set theory into the framework of UP-algebras. Dokkhamdang et al. [4] introduced the concept of fuzzy UP-subalgebras with thresholds in UP-algebras. Tanamoon et al. [20] pioneered the concepts of Q-fuzzy UP-ideals and Q-fuzzy UP-subalgebras within the framework of UP-algebras. Building on this foundation, Sripaeng et al. [18] expanded these ideas by introducing anti Q-fuzzy UP-ideals and anti Q-fuzzy UP-subalgebras, offering further insights into the structure and properties of UP-algebras through the lens of fuzzy set theory. Furthermore, Poungsumpao et al. [16] explored fuzzy UP-subalgebras and fuzzy UP-ideals of UP-algebras using upper t-(strong) level subsets and lower t-(strong) level subsets. UP-algebras (see [9]) and BCC-algebras (see [15]) have been identified as synonymous concepts, as demonstrated by Jun et al. [13] in 2022. In alignment with Komori's initial characterization in 1984, our research team will henceforth adopt the term BCC rather than UP in future investigations and publications for consistency.

In this paper, we leverage the Łukasiewicz t-norm to introduce the concept of ε-Łukasiewicz fuzzy sets, derived from a given fuzzy set, and apply this innovative framework to BCC-algebras. We define ε-Łukasiewicz fuzzy BCC-filters and delve into their properties. By establishing conditions for an ε-Łukasiewicz fuzzy set to qualify as an ε-Łukasiewicz fuzzy BCC-subalgebra, we provide a comprehensive characterization of these structures. Additionally, we introduce and examine three specific subsets—termed the $\in$-set, q-set, and O-set—identifying the conditions under which they can function as BCC-filters.

2. Preliminaries

The concept of BCC-algebras (originally discussed in [15]) can be redefined without the requirement of condition (2.6) as follows:

An algebra $X = (X, *, 0)$ of type $(2, 0)$ is called a *BCC-algebra* (see [8]) if it satisfies the following conditions:

$$(\forall x, y, z \in X)((y * z) * ((x * y) * (x * z)) = 0) \tag{2.1}$$

$$(\forall x \in X)(0 * x = x) \tag{2.2}$$

$$(\forall x \in X)(x * 0 = 0) \tag{2.3}$$

$$(\forall x, y \in X)(x * y = 0, y * x = 0 \Rightarrow x = y) \tag{2.4}$$

Following this, we will denote X as a BCC-algebra $(X, *, 0)$ unless stated otherwise.

We define a binary relation $\leq$ on X as follows:

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x * y = 0) \tag{2.5}$$

In X, the following assertions are valid (see [9]).

$$(\forall x \in X)(x \leq x) \tag{2.6}$$

$$(\forall x, y, z \in X)(x \leq y, y \leq z \Rightarrow x \leq z) \tag{2.7}$$

$$(\forall x, y, z \in X)(x \leq y \Rightarrow z * x \leq z * y) \tag{2.8}$$

$$(\forall x, y, z \in X)(x \leq y \Rightarrow y * z \leq x * z) \tag{2.9}$$

$$(\forall x, y, z \in X)(x \leq y * x, \text{ in particular, } y * z \leq x * (y * z)) \tag{2.10}$$

$$(\forall x, y \in X)(y * x \leq x \Leftrightarrow x = y * x) \tag{2.11}$$

$$(\forall x, y \in X)(x \leq y * y) \tag{2.12}$$

$$(\forall a, x, y, z \in X)(x * (y * z) \leq x * ((a * y) * (a * z))) \tag{2.13}$$

$$(\forall a, x, y, z \in X)(((a * x) * (a * y)) * z \leq (x * y) * z) \tag{2.14}$$

$$(\forall x, y, z \in X)((x * y) * z \leq y * z) \tag{2.15}$$

$$(\forall x, y, z \in X)(x \leq y \Rightarrow x \leq z * y) \tag{2.16}$$

$$(\forall x, y, z \in X)((x * y) * z \leq x * (y * z)) \tag{2.17}$$

$$(\forall a, x, y, z \in X)((x * y) * z \leq y * (a * z)) \tag{2.18}$$

Definition 2.1. [9] A nonempty subset S of X is called

(1) a *BCC-subalgebra* of X if it satisfies the following property:

$$(\forall x, y \in S)(x * y \in S) \tag{2.19}$$

(2) a *BCC-ideal* of X if it satisfies the following properties:

$$0 \in S \tag{2.20}$$

$$(\forall x, y, z \in X)(x * (y * z), y \in S \Rightarrow x * z \in S) \tag{2.21}$$

(3) a *BCC-filter* of X if it satisfies (2.20) and the following property:

$$(\forall x, y \in X)(x * y, x \in S \Rightarrow y \in S) \tag{2.22}$$

A *fuzzy set* [21] in a nonempty set X is defined to be a function $\mu : X \to [0, 1]$, where $[0, 1]$ is the unit closed interval of real numbers.

Definition 2.2. [19] A fuzzy set μ in X is said to be

(1) a *fuzzy BCC-subalgebra* of X if it satisfies the following property:

$$(\forall x, y \in X)(\mu(x * y) \geq \min\{\mu(x), \mu(y)\}) \tag{2.23}$$

(2) a *fuzzy BCC-ideal* of X if it satisfies the following properties:

$$(\forall x \in X)(\mu(0) \geq \mu(x)) \tag{2.24}$$

$$(\forall x, y, z \in X)(\mu(x * z) \geq \min\{\mu(x * (y * z)), \mu(y)\}) \tag{2.25}$$

(3) a *fuzzy BCC-filter* of X if it satisfies (2.24) and the following property:

$$(\forall x, y \in X)(\mu(y) \geq \min\{\mu(x * y), \mu(x)\}) \tag{2.26}$$

A fuzzy set μ in a nonempty set X of the form

$$\mu(x) = \begin{cases} t \in (0,1] & \text{if } x = a \\ 0 & \text{otherwise,} \end{cases}$$

is said to be a *fuzzy point* with support a and value t and is denoted by $[a/t]$.

For a fuzzy set μ in a set X, we say that a fuzzy point $[a/t]$ is

(1) contained in μ, denoted by $[a/t] \in \mu$, (see [17]) if $\mu(a) \geq t$,
(2) quasi-coincident with μ, denoted by $[a/t]q\mu$, (see [17]) if $\mu(a) + t > 1$.

Proposition 2.3. *If μ is a fuzzy set in a set X and $\varepsilon \in (0,1)$, then its ε-Łukasiewicz fuzzy set L^ε_μ satisfies the following property:*

(1) $(\forall x, y \in X)(\mu(x) \geq \mu(y) \Rightarrow L^\varepsilon_\mu(x) \geq L^\varepsilon_\mu(y))$
(2) $(\forall x \in X)([x/\varepsilon]q\mu \Rightarrow L^\varepsilon_\mu(x) = \mu(x) + \varepsilon - 1)$
(3) $(\forall x \in X, \forall \delta \in (0,1))(\varepsilon \geq \delta \Rightarrow L^\varepsilon_\mu(x) \geq L^\delta_\mu(x))$

3. Main results

In this section, we revisit ε-Łukasiewicz fuzzy sets and introduce ε-Łukasiewicz fuzzy BCC-filters, a novel extension for BCC-algebras.

Definition 3.1. Let μ be a fuzzy set in a set X and let $\varepsilon \in [0,1]$. A function $L^\varepsilon_\mu : X \to [0,1]; x \mapsto \max\{0, \mu(x) + \varepsilon - 1\}$ is called an *ε-Łukasiewicz fuzzy set* of μ in X.

Definition 3.2. [10] Let μ be a fuzzy set in X. Then its ε-Łukasiewicz fuzzy set L^ε_μ in X is called an *ε-Łukasiewicz fuzzy BCC-subalgebra* of X if it satisfies the following property:

$$(\forall x, y \in X, \forall t_a, t_b \in (0,1]) \left(\begin{array}{l} [x/t_a] \in L^\varepsilon_\mu, [y/t_b] \in L^\varepsilon_\mu \\ \Rightarrow [(x * y)/\min\{t_a, t_b\}] \in L^\varepsilon_\mu \end{array} \right) \tag{3.1}$$

Definition 3.3. [11] Let μ be a fuzzy set in X. Then its ε-Łukasiewicz fuzzy set L_μ^ε in X is called an *ε-Łukasiewicz fuzzy BCC-ideal* of X if it satisfies the following properties:

$$(\forall x \in X, \forall t_a \in (0,1])([x/t_a] \in L_\mu^\varepsilon \Rightarrow [0/t_a] \in L_\mu^\varepsilon) \tag{3.2}$$

$$(\forall x,y,z \in X, \forall t_a,t_b \in (0,1]) \left(\begin{array}{l} [(x*(y*z))/t_a] \in L_\mu^\varepsilon, [y/t_b] \in L_\mu^\varepsilon \\ \Rightarrow [(x*z)/\min\{t_a,t_b\}] \in L_\mu^\varepsilon \end{array} \right) \tag{3.3}$$

Theorem 3.4. [11] *Let μ be a fuzzy set in X. Then its ε-Łukasiewicz fuzzy set L_μ^ε in X is an ε-Łukasiewicz fuzzy BCC-ideal of X if and only if it satisfies the following properties:*

$$(\forall x \in X)(L_\mu^\varepsilon(0) \geq L_\mu^\varepsilon(x)) \tag{3.4}$$

$$(\forall x,y,z \in X)(L_\mu^\varepsilon(x*z) \geq \min\{L_\mu^\varepsilon(x*(y*z)), L_\mu^\varepsilon(y)\}) \tag{3.5}$$

Definition 3.5. Let μ be a fuzzy set in X. Then its ε-Łukasiewicz fuzzy set L_μ^ε in X is called an ε-Łukasiewicz fuzzy BCC-filter of X if it satisfies (3.2) and the following properties:

$$(\forall x,y \in X, \forall t_a,t_b \in (0,1]) \left(\begin{array}{l} [(x*y)/t_a] \in L_\mu^\varepsilon, [x/t_b] \in L_\mu^\varepsilon \\ \Rightarrow [y/\min\{t_a,t_b\}] \in L_\mu^\varepsilon \end{array} \right) \tag{3.6}$$

Theorem 3.6. *Let μ be a fuzzy set in X. Then its ε-Łukasiewicz fuzzy set L_μ^ε in X is an ε-Łukasiewicz fuzzy BCC-filter of X if and only if it satisfies the following properties:*

$$(\forall x \in X)(L_\mu^\varepsilon(0) \geq L_\mu^\varepsilon(x)) \tag{3.7}$$

$$(\forall x,y \in X)(L_\mu^\varepsilon(y) \geq \min\{L_\mu^\varepsilon(x*y), L_\mu^\varepsilon(x)\}) \tag{3.8}$$

Proof. Assume that L_μ^ε is an ε-Łukasiewicz fuzzy BCC-filter of X. Let $x,y \in X$. Since $[x/L_\mu^\varepsilon(x)] \in L_\mu^\varepsilon$, we have $[0/L_\mu^\varepsilon(x)] \in L_\mu^\varepsilon$ by (3.2), and so $L_\mu^\varepsilon(0) \geq L_\mu^\varepsilon(x)$. Note that $[(x*y)/L_\mu^\varepsilon(x*y)] \in L_\mu^\varepsilon, [x/L_\mu^\varepsilon(y)] \in L_\mu^\varepsilon$ for all $x,y \in X$. It follows from (3.6) that $[L_\mu^\varepsilon(x*y)/\min\{L_\mu^\varepsilon(x*y), L_\mu^\varepsilon(x)\}] \in L_\mu^\varepsilon$, that is, $L_\mu^\varepsilon(x*y) \geq \min\{L_\mu^\varepsilon(x*y), L_\mu^\varepsilon(x)\}$ for all $x,y \in X$.

Conversely, let L_μ^ε be an ε-Łukasiewicz fuzzy set satisfying (3.7) and (3.8). If $[x/t] \in L_\mu^\varepsilon$ for all $x \in X$ and $t \in (0,1]$, then $L_\mu^\varepsilon(0) \geq L_\mu^\varepsilon(x) \geq t$ for all $x \in X$ by (3.7). Hence, $[0/t] \in L_\mu^\varepsilon$. Let $x,y \in X$ and $t_a,t_b \in (0,1]$ be such that $[(x*y)/t_a] \in L_\mu^\varepsilon, [x/t_b] \in L_\mu^\varepsilon$. Then $L_\mu^\varepsilon(x*y) \geq t_a$ and $L_\mu^\varepsilon(x) \geq t_b$. It follows from (3.8) that $L_\mu^\varepsilon(x*y) \geq \min\{L_\mu^\varepsilon(x*y), L_\mu^\varepsilon(x)\} \geq \min\{t_a,t_b\}$. Hence, $[(x*y)/\min\{t_a,t_b\}] \in L_\mu^\varepsilon$. Therefore, L_μ^ε is an ε-Łukasiewicz fuzzy BCC-filter of X. □

Proposition 3.7. *Every ε-Łukasiewicz fuzzy BCC-ideal of X is an ε-Łukasiewicz fuzzy BCC-filter of X.*

Proof. Let L^{ε}_{μ} be an ε-Łukasiewicz fuzzy BCC-filter of X and let $x, y \in X$. Then

$$\begin{aligned} L^{\varepsilon}_{\mu}(y) &= L^{\varepsilon}_{\mu}(0 * y) \\ &\geq \min\{L^{\varepsilon}_{\mu}(0 * (x * y)), L^{\varepsilon}_{\mu}(x)\} \\ &= \min\{L^{\varepsilon}_{\mu}(x * y), L^{\varepsilon}_{\mu}(x)\}. \end{aligned}$$

By Theorems 3.4 and 3.6, we have L^{ε}_{μ} is an ε-Łukasiewicz fuzzy BCC-filter of X. □

The following example illustrates that the converse of Proposition 3.7 does not hold in general.

Example 3.8. [7] Let $X = \{0, 1, 2, 3\}$ with the following Cayley table:

$*$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	2
3	0	1	0	0

Then X is a BCC-algebra. Define a fuzzy set μ as follows:

$$\mu : X \to [0,1], x \mapsto \begin{cases} 0.95 & \text{if } x = 0 \\ 0.35 & \text{if } x = 1 \\ 0.25 & \text{if } x = 2 \\ 0.25 & \text{if } x = 3 \end{cases}$$

Given $\varepsilon = 0.85$, the ε-Łukasiewicz fuzzy set L^{ε}_{μ} of μ in X is given as follows:

$$L^{\varepsilon}_{\mu} : X \to [0,1], x \mapsto \begin{cases} 0.8 & \text{if } x = 0 \\ 0.2 & \text{if } x = 1 \\ 0.1 & \text{if } x = 2 \\ 0.1 & \text{if } x = 3 \end{cases}$$

Hence, L^{ε}_{μ} is an ε-Łukasiewicz fuzzy BCC-filter of X but it is not an ε-Łukasiewicz fuzzy BCC-ideal of X because $L^{\varepsilon}_{\mu}(2 * 3) = L^{\varepsilon}_{\mu}(2) = 0.1 \ngeq 0.2 = \min\{0.8, 0.2\} = \min\{L^{\varepsilon}_{\mu}(0), L^{\varepsilon}_{\mu}(1)\} = \min\{L^{\varepsilon}_{\mu}(2 * (1 * 3)), L^{\varepsilon}_{\mu}(1)\}$ by Theorem 3.4.

Proposition 3.9. *If L^{ε}_{μ} is an ε-Łukasiewicz fuzzy BCC-filter of X, then*

$$(\forall x, y \in X)(y \leq x \Rightarrow L^{\varepsilon}_{\mu}(x) \leq L^{\varepsilon}_{\mu}(y)). \tag{3.9}$$

Proof. Let L_μ^ε be an ε-Łukasiewicz fuzzy BCC-filter of X and let $x, y \in X$ be such that $y \leq x$. Then

$$\begin{aligned} L_\mu^\varepsilon(y) &= \max\{0, \mu(y) + \varepsilon - 1\} \\ &\geq \max\{0, \min\{\mu(x * y), \mu(x)\} + \varepsilon - 1\} \\ &= \max\{0, \min\{\mu(0), \mu(x)\} + \varepsilon - 1\} \\ &= \max\{0, \mu(x) + \varepsilon - 1\} \\ &= L_\mu^\varepsilon(x). \end{aligned}$$

□

Theorem 3.10. *If μ is a fuzzy BCC-filter of X, then its ε-Łukasiewicz fuzzy set L_μ^ε in X is an ε-Łukasiewicz fuzzy BCC-filter of X.*

Proof. Assume that μ is a fuzzy BCC-filter of X. Let $x \in X$ and $t_a \in (0, 1]$ be such that $[x/t_a] \in L_\mu^\varepsilon$. Then $L_\mu^\varepsilon(x) \geq t_a$. Thus,

$$\begin{aligned} L_\mu^\varepsilon(0) &= \max\{0, \mu(0) + \varepsilon - 1\} \\ &\geq \max\{0, \mu(x) + \varepsilon - 1\} \\ &= L_\mu^\varepsilon(x) \\ &\geq t_a. \end{aligned}$$

Then $[0/t_a] \in L_\mu^\varepsilon$. Let $x, y \in X$ and $t_a, t_b \in (0, 1]$ be such that $[(x * y)/t_a] \in L_\mu^\varepsilon$ and $[x/t_b] \in L_\mu^\varepsilon$. Then $L_\mu^\varepsilon(x * y) \geq t_a$ and $L_\mu^\varepsilon(x) \geq t_b$. Thus,

$$\begin{aligned} L_\mu^\varepsilon(y) &= \max\{0, \mu(y) + \varepsilon - 1\} \\ &\geq \max\{0, \min\{\mu(x * y), \mu(x)\} + \varepsilon - 1\} \\ &= \max\{0, \min\{\mu(x * y) + \varepsilon - 1, \mu(x) + \varepsilon - 1\}\} \\ &= \min\{\max\{\mu(x * y) + \varepsilon - 1\}, \max\{\mu(x) + \varepsilon - 1\}\} \\ &= \min\{L_\mu^\varepsilon(x * y), L_\mu^\varepsilon(x)\} \\ &\geq \min\{t_a, t_b\}. \end{aligned}$$

Then $[y/\min\{t_a, t_b\}] \in L_\mu^\varepsilon$. Hence, L_μ^ε is an ε-Łukasiewicz fuzzy BCC-filter of X. □

Let μ be a fuzzy set in X. For an ε-Łukasiewicz fuzzy set L_μ^ε of μ in X and $t \in (0, 1]$, consider the sets

$$\begin{aligned} (L_\mu^\varepsilon, t)_\in &= \{x \in X : [x/t] \in L_\mu^\varepsilon\}, \\ (L_\mu^\varepsilon, t)_q &= \{x \in X : [x/t] q L_\mu^\varepsilon\}, \end{aligned}$$

which are called the $\in$-set and q-set, respectively, of L_μ^ε (with value t).

We investigate the conditions under which the $\in$-set and q-set of ε-Łukasiewicz fuzzy sets can function as BCC-filters.

Theorem 3.11. *Let L_μ^ε be an ε-Łukasiewicz fuzzy set of a fuzzy set μ in X. Then the $\in$-set $(L_\mu^\varepsilon, t)_\in$ of L_μ^ε with value $t \in (0.5, 1]$ is a BCC-filter of X if and only if the following assertion is valid:*

$$(\forall x \in X)(\max\{L_\mu^\varepsilon(0), 0.5\} \geq L_\mu^\varepsilon(x)) \tag{3.10}$$

$$(\forall x, y \in X)(\max\{L_\mu^\varepsilon(y), 0.5\} \geq \min\{L_\mu^\varepsilon(x * y), L_\mu^\varepsilon(x)\}) \tag{3.11}$$

Proof. Assume that the $\in$-set $(L_\mu^\varepsilon, t)_\in$ of L_μ^ε with value $t \in (0.5, 1]$ is a BCC-filter of X. If (3.10) is not valid, then there exists $a \in X$ such that $\max\{L_\mu^\varepsilon(0), 0.5\} < L_\mu^\varepsilon(a)$. Thus, $L_\mu^\varepsilon(a) \in (0.5, 1]$ and $L_\mu^\varepsilon(a) > L_\mu^\varepsilon(0)$. If we take $t = L_\mu^\varepsilon(a)$, then $[a/t] \in L_\mu^\varepsilon$, that is, $a \in (L_\mu^\varepsilon, s)_\in$ and $0 \notin (L_\mu^\varepsilon, t)_\in$. This is a contradiction and so $L_\mu^\varepsilon(x) \leq \max\{L_\mu^\varepsilon(0), 0.5\}$ for all $x \in X$. Now, if (3.11) is not valid, then there exist $a, b \in X$ such that $\max\{L_\mu^\varepsilon(b), 0.5\} < \min\{L_\mu^\varepsilon(a*b), L_\mu^\varepsilon(a)\}$. If we take $s = \min\{L_\mu^\varepsilon(a*b), L_\mu^\varepsilon(a)\}$, then $s \in (0.5, 1]$ and $[(a*b)/s], [a/s] \in L_\mu^\varepsilon$, that is, $a*b, a \in (L_\mu^\varepsilon, s)_\in$. Since $(L_\mu^\varepsilon, s)_\in$ is a BCC-filter of X, we have $b \in (L_\mu^\varepsilon, s)_\in$. But $[b/s]_{\in} L_\mu^\varepsilon$ implies $b \notin (L_\mu^\varepsilon, s)_\in$, a contradiction. Thus, $\max\{L_\mu^\varepsilon(y), 0.5\} \geq \min\{L_\mu^\varepsilon(x * y), L_\mu^\varepsilon(x)\}$ for all $x, y \in X$.

Conversely, suppose that L_μ^ε satisfies (3.10) and (3.11). For every $t \in (0.5, 1]$, we have $0.5 < t \leq L_\mu^\varepsilon(x) \leq \max\{L_\mu^\varepsilon(0), 0.5\}$ for all $x \in (L_\mu^\varepsilon, t)_\in$ by (3.10). Then $0 \in (L_\mu^\varepsilon, t)_\in$. Let $t \in (0.5, 1]$ and $x, y \in X$ be such that $x * y \in (L_\mu^\varepsilon, t)_\in$ and $x \in (L_\mu^\varepsilon, t)_\in$. Then $L_\mu^\varepsilon(x * y) \geq t$ and $L_\mu^\varepsilon(x) \geq t$, which imply from (3.11) that $0.5 < t \leq \min\{L_\mu^\varepsilon(x * y), L_\mu^\varepsilon(x)\} \leq \max\{L_\mu^\varepsilon(x * y), 0.5\}$. Thus, $[(x * y)/t] \in L_\mu^\varepsilon$, that is, $x * y \in (L_\mu^\varepsilon, t)_\in$. Hence, $(L_\mu^\varepsilon, t)_\in$ is a BCC-filter of X for $t \in (0.5, 1]$. □

Theorem 3.12. *Let L_μ^ε be an ε-Łukasiewicz fuzzy set of a fuzzy set μ in X. If μ is a fuzzy BCC-filter of X, then the q-set $(L_\mu^\varepsilon, t)_q$ of L_μ^ε with value $t \in (0, 1]$ is a BCC-filter of X.*

Proof. Assume that the L_μ^ε is an ε-Łukasiewicz fuzzy BCC-filter of X and let $t \in (0, 1]$. If $0 \notin (L_\mu^\varepsilon, t)_q$, then $[0/t]\overline{q}L_\mu^\varepsilon$, that is, $L_\mu^\varepsilon(0) + t \leq 1$. Since $L_\mu^\varepsilon(0) \geq L_\mu^\varepsilon(x)$ for $x \in (L_\mu^\varepsilon, t)_q$, it follows that $L_\mu^\varepsilon(x) \leq L_\mu^\varepsilon(0) \leq 1 - t$. Hence, $[x/t]\overline{q}L_\mu^\varepsilon$, and so $x \notin (L_\mu^\varepsilon, t)_q$. This is a contradiction, so $0 \in (L_\mu^\varepsilon, t)_q$. Let $t \in (0, 1]$ and $x, y \in (L_\mu^\varepsilon, t)_q$ be such that $x * y \in (L_\mu^\varepsilon, t)_q$ and $x \in (L_\mu^\varepsilon, t)_q$. Then $[x * y/t]_q L_\mu^\varepsilon$ and $[x/t]_q L_\mu^\varepsilon$, that is, $L_\mu^\varepsilon(x * y) + t > 1$ and $L_\mu^\varepsilon(x) + t > 1$. It follows from Theorems 3.6 and 3.10 that $L_\mu^\varepsilon(y) + t \geq \min\{L_\mu^\varepsilon(x * y), L_\mu^\varepsilon(x)\} + t = \min\{L_\mu^\varepsilon(x * y) + t, L_\mu^\varepsilon(x) + t\} > 1$. Thus, $[y/t]qL_\mu^\varepsilon$. So $y \in (L_\mu^\varepsilon, t)_q$. Hence, $(L_\mu^\varepsilon, t)_q$ is a BCC-filter of X. □

Theorem 3.13. *Let μ be a fuzzy set in X. For an ε-Łukasiewicz fuzzy set L_μ^ε of μ in X, if the q-set $(L_\mu^\varepsilon, t)_q$ is a BCC-filter of X, then L_μ^ε satisfies the following properties:*

$$(\forall t_a \in (0, 0.5])(0 \in (L^\varepsilon_\mu, t_a)_\in) \tag{3.12}$$

$$(\forall x, y \in X, \forall t_a, t_b \in (0, 0.5]) \left(\begin{array}{l} [x * y/t_a]qL^\varepsilon_\mu, [x/t_b]qL^\varepsilon_\mu \\ \Rightarrow [y/\max\{t_a, t_b\}] \in (L^\varepsilon_\mu, \max\{t_a, t_b\})_\in \end{array} \right) \tag{3.13}$$

Proof. Let $x, y, z \in X$ and $t_a, t_b \in (0, 0.5]$. If $0 \notin (L^\varepsilon_\mu, t_a)_\in$, then $[0/t_a]\overline{\in}L^\varepsilon_\mu$ and so $L^\varepsilon_\mu(0) < t_a \leq 1 - t_a$ since $t_a \leq 0.5$. Hence, $[0/t_a]\overline{q}L^\varepsilon_\mu$. This is a contradiction, so $0 \in (L^\varepsilon_\mu, t_a)_\in$. Let $x, y \in X$ be such that $x * y \in (L^\varepsilon_\mu, t)_q$ and $x \in (L^\varepsilon_\mu, t)_q$. Then $[x * y/t]qL^\varepsilon_\mu$ and $[x/t]qL^\varepsilon_\mu$, that is, $L^\varepsilon_\mu(x * y) > 1 - t$ and $L^\varepsilon_\mu(x) > 1-t$. It follows that $L^\varepsilon_\mu(y) \geq \min\{L^\varepsilon_\mu(x*y), L^\varepsilon_\mu(x)\} > 1-t$. Thus, $[y/t]qL^\varepsilon_\mu$ and so $y \in (L^\varepsilon_\mu, t)_q$. Hence, $(L^\varepsilon_\mu, t)_q$ is a BCC-filter of X. □

Theorem 3.14. *If an ε-Łukasiewicz fuzzy set L^ε_μ in X satisfies the following properties:*

$$(\forall x \in X, \forall t \in (0.5, 1])([x/t]qL^\varepsilon_\mu \Rightarrow [0/t] \in L^\varepsilon_\mu) \tag{3.14}$$

$$(\forall x, y \in X, \forall t_a, t_b \in (0.5, 1]) \left(\begin{array}{l} [x * y/t_a]qL^\varepsilon_\mu, [x/t_b]qL^\varepsilon_\mu \\ \Rightarrow [y/\max\{t_a, t_b\}] \in L^\varepsilon_\mu \end{array} \right) \tag{3.15}$$

then the nonempty $\in$-set $(L^\varepsilon_\mu, \max\{t_a, t_b\})_\in$ of L^ε_μ is a BCC-filter of X for all $t_a, t_b \in (0.5, 1]$.

Proof. Let $t_a, t_b \in (0.5, 1]$ and assume that the $\in$-set $(L^\varepsilon_\mu, \max\{t_a, t_b\})_\in$ of L^ε_μ is nonempty. Then there exists $x \in (L^\varepsilon_\mu, \max\{t_a, t_b\})_\in$, and so $L^\varepsilon_\mu(x) \geq \max\{t_a, t_b\} > 1 - \max\{t_a, t_b\}$, that is, $[x/\max\{t_a, t_b\}]qL^\varepsilon_\mu$. Hence, $[0/\max\{t_a, t_b\}] \in L^\varepsilon_\mu$ by (3.14), and so $0 \in (L^\varepsilon_\mu, \max\{t_a, t_b\})_\in$. Let $x, y \in X$ be such that $x * y \in (L^\varepsilon_\mu, \max\{t_a, t_b\})_\in$ and $x \in (L^\varepsilon_\mu, \max\{t_a, t_b\})_\in$. Then $L^\varepsilon_\mu(x * y) \geq \max\{t_a, t_b\} > 1 - \max\{t_a, t_b\}$ and $L^\varepsilon_\mu(x) \geq \max\{t_a, t_b\} > 1 - \max\{t_a, t_b\}$, that is, $[(x * y)/\max\{t_a, t_b\}]qL^\varepsilon_\mu$ and $[x/\max\{t_a, t_b\}]qL^\varepsilon_\mu$. It follows from (3.15) that $[y/\max\{t_a, t_b\}] \in L^\varepsilon_\mu$, so $y \in (L^\varepsilon_\mu, \max\{t_a, t_b\})_\in$. Hence, $(L^\varepsilon_\mu, \max\{t_a, t_b\})_\in$ is a BCC-filter of X for all $t_a, t_b \in (0.5, 1]$. □

Theorem 3.15. *If an ε-Łukasiewicz fuzzy set L^ε_μ in X satisfies (3.14) and*

$$(\forall x, y \in X, \forall t_a, t_b \in (0.5, 1]) \left(\begin{array}{l} [x * y/t_a]qL^\varepsilon_\mu, [x/t_b]qL^\varepsilon_\mu \\ \Rightarrow [y/\min\{t_a, t_b\}] \in L^\varepsilon_\mu \end{array} \right) \tag{3.16}$$

then the nonempty $\in$-set $(L^\varepsilon_\mu, \min\{t_a, t_b\})_\in$ of L^ε_μ is a BCC-filter of X for all $t_a, t_b \in (0.5, 1]$.

Proof. Let $t_a, t_b \in (0.5, 1]$ and assume that the $\in$-set $(L^\varepsilon_\mu, \min\{t_a, t_b\})_\in$ of L^ε_μ is nonempty. Then there exists $x \in (L^\varepsilon_\mu, \min\{t_a, t_b\})_\in$, and so $L^\varepsilon_\mu(x) \geq \min\{t_a, t_b\} > 1 - \min\{t_a, t_b\}$, that is, $[x/\min\{t_a, t_b\}]qL^\varepsilon_\mu$. Hence, $[0/\min\{t_a, t_b\}] \in L^\varepsilon_\mu$ by (3.14), and so $0 \in (L^\varepsilon_\mu, \min\{t_a, t_b\})_\in$. Let $x, y \in X$ be such that $x * y \in (L^\varepsilon_\mu, \min\{t_a, t_b\})_\in$ and $x \in (L^\varepsilon_\mu, \min\{t_a, t_b\})_\in$. Then $L^\varepsilon_\mu(x * y) \geq \min\{t_a, t_b\} > 1 - \min\{t_a, t_b\}$ and $L^\varepsilon_\mu(x) \geq \min\{t_a, t_b\} > 1 - \min\{t_a, t_b\}$, that is, $[(x * y)/\min\{t_a, t_b\}]qL^\varepsilon_\mu$ and $[x/\min\{t_a, t_b\}]qL^\varepsilon_\mu$. It follows from (3.16) that $[y/\min\{t_a, t_b\}] \in L^\varepsilon_\mu$, so $y \in (L^\varepsilon_\mu, \min\{t_a, t_b\})_\in$. Hence, $(L^\varepsilon_\mu, \min\{t_a, t_b\})_\in$ is a BCC-filter of X for all $t_a, t_b \in (0.5, 1]$. □

Theorem 3.16. *If an ε-Łukasiewicz fuzzy set L^{ε}_{μ} in X satisfies (3.14) and*

$$(\forall x \in X, \forall t \in (0.5, 1])([x/t]qL^{\varepsilon}_{\mu} \Rightarrow [0/t] \in L^{\varepsilon}_{\mu}) \tag{3.17}$$

$$(\forall x, y \in X, \forall t_a, t_b \in (0.5, 1]) \left(\begin{array}{l} [x * y/t_a]qL^{\varepsilon}_{\mu}, [x/t_b]qL^{\varepsilon}_{\mu} \\ \Rightarrow [y/\min\{t_a, t_b\}] \in L^{\varepsilon}_{\mu} \end{array} \right) \tag{3.18}$$

then the nonempty $\in$-set $(L^{\varepsilon}_{\mu}, \max\{t_a, t_b\})_{\in}$ of L^{ε}_{μ} is a BCC-filter of X for all $t_a, t_b \in (0.5, 1]$.

Proof. Let $x \in (L^{\varepsilon}_{\mu}, \max\{t_a, t_b\})_{\in}$ for $t_a, t_b \in (0.5, 1]$. Then $L^{\varepsilon}_{\mu}(x) \geq \max\{t_a, t_b\} > 1-\max\{t_a, t_b\}$, and so $[x/\max\{t_a, t_b\}]qL^{\varepsilon}_{\mu}$. Hence, $[0/\max\{t_a, t_b\}] \in L^{\varepsilon}_{\mu}$ by (3.17), which implies that $0 \in (L^{\varepsilon}_{\mu}, \max\{t_a, t_b\})_{\in}$. Let $x, y \in (L^{\varepsilon}_{\mu}, \max\{t_a, t_b\})_{\in}$ for $t_a, t_b \in (0.5, 1]$. Then $L^{\varepsilon}_{\mu}(x * y) \geq \max\{t_a, t_b\} > 1 - \max\{t_a, t_b\}$ and $L^{\varepsilon}_{\mu}(x) \geq \max\{t_a, t_b\} > 1-\max\{t_a, t_b\}$, that is, $[x*y/\max\{t_a, t_b\}]qL^{\varepsilon}_{\mu}$ and $[x/\max\{t_a, t_b\}]qL^{\varepsilon}_{\mu}$. It follows from (3.18) that $[y/\max\{t_a, t_b\}] \in L^{\varepsilon}_{\mu}$. Hence, $y \in (L^{\varepsilon}_{\mu}, \max\{t_a, t_b\})_{\in}$. Therefore, $(L^{\varepsilon}_{\mu}, \max\{t_a, t_b\})_{\in}$ is a BCC-filter of X for all $t_a, t_b \in (0.5, 1]$. □

Theorem 3.17. *If an ε-Łukasiewicz fuzzy set L^{ε}_{μ} in X satisfies the following properties:*

$$(\forall x \in X, \forall t \in (0.5, 1])([x/t]qL^{\varepsilon}_{\mu} \Rightarrow [0/t] \in (L^{\varepsilon}_{\mu})) \tag{3.19}$$

$$(\forall x, y \in X, \forall t_a, t_b \in (0.5, 1]) \left(\begin{array}{l} [x * y/t_a] \in L^{\varepsilon}_{\mu}, [x/t_b] \in L^{\varepsilon}_{\mu} \\ \Rightarrow [y/\min\{t_a, t_b\}]qL^{\varepsilon}_{\mu} \end{array} \right) \tag{3.20}$$

then the nonempty q-set $(L^{\varepsilon}_{\mu}, \min\{t_a, t_b\})_q$ of L^{ε}_{μ} is a BCC-filter of X for all $t_a, t_b \in (0, 0.5]$.

Proof. Let $t_a, t_b \in (0, 0.5]$. If $(L^{\varepsilon}_{\mu}, \min\{t_a, t_b\})_q$ is nonempty, then there exists $x \in (L^{\varepsilon}_{\mu}, \min\{t_a, t_b\})_q$. Hence, $L^{\varepsilon}_{\mu}(x) > 1 - \min\{t_a, t_b\} \geq \min\{t_a, t_b\}$, which shows that $[x/\min\{t_a, t_b\}] \in L^{\varepsilon}_{\mu}$. It follows from (3.10) that $[0/\min\{t_a, t_b\}]qL^{\varepsilon}_{\mu}$. Thus, $0 \in (L^{\varepsilon}_{\mu}, \min\{t_a, t_b\})_q$. Let $x, y \in X$ be such that $x * y \in (L^{\varepsilon}_{\mu}, \min\{t_a, t_b\})_q$ and $x \in (L^{\varepsilon}_{\mu}, \min\{t_a, t_b\})_q$. Then $L^{\varepsilon}_{\mu}(x * y) > 1 - \min\{t_a, t_b\} \geq \min\{t_a, t_b\}$ and $L^{\varepsilon}_{\mu}(y) > 1 - \min\{t_a, t_b\} \geq \min\{t_a, t_b\}$. Thus, $[x * y/\min\{t_a, t_b\}] \in L^{\varepsilon}_{\mu}$ and $[x/\min\{t_a, t_b\}] \in L^{\varepsilon}_{\mu}$. It follows from (3.11) that $[y/\min\{t_a, t_b\}]qL^{\varepsilon}_{\mu}$, that is, $y \in (L^{\varepsilon}_{\mu}, \min\{t_a, t_b\})_q$. Hence, $(L^{\varepsilon}_{\mu}, \min\{t_a, t_b\})_q$ is a BCC-filter of X. □

Theorem 3.18. *If an ε-Łukasiewicz fuzzy set L^{ε}_{μ} in X satisfies (3.12) and (3.13), then the q-set $(L^{\varepsilon}_{\mu}, t)_q$ of L^{ε}_{μ} is a BCC-filter of X for all $t \in (0.5, 1]$.*

Proof. Assume that L^{ε}_{μ} satisfies (3.12) and (3.13). The condition (3.12) induces $L^{\varepsilon}_{\mu}(0) + t \geq 2t > 1$, that is, $[0/t]qL^{\varepsilon}_{\mu}$. Hence, $0 \in (L^{\varepsilon}_{\mu}, t)_q$. Let $x, y \in X$ be such that $x * y \in (L^{\varepsilon}_{\mu}, t)_q$ and $x \in (L^{\varepsilon}_{\mu}, t)_q$. Then $[(x * y)/t]qL^{\varepsilon}_{\mu}$ and $[x/t]qL^{\varepsilon}_{\mu}$. It follows from (3.13) that $y \in (L^{\varepsilon}_{\mu}, \min\{t, t\})_{\in} = (L^{\varepsilon}_{\mu}, t)_{\in}$. Hence, $L^{\varepsilon}_{\mu}(y) \geq t > 1 - t$, that is, $y \in (L^{\varepsilon}_{\mu}, t)_q$. Therefore, $(L^{\varepsilon}_{\mu}, t)_q$ is a BCC-filter of X for all $t \in (0.5, 1]$. □

Let μ be a fuzzy set in X. Consider an ε-Łukasiewicz fuzzy set L_μ^ε associated with μ in X. Define the set $O(L_\mu^\varepsilon) = \{x \in X : L_\mu^\varepsilon(x) > 0\}$, known as the O-set of L_μ^ε. It is noted that $O(L_\mu^\varepsilon)$ can be expressed as $O(L_\mu^\varepsilon) = \{x \in X : \mu(x) + \varepsilon - 1 > 0\}$.

Theorem 3.19. *Let L_μ^ε be an ε-Łukasiewicz fuzzy set of a fuzzy set μ in X. If μ is a fuzzy BCC-filter of X, then the O-set $O(L_\mu^\varepsilon)$ of L_μ^ε is a BCC-filter of X.*

Proof. Assume that μ is a fuzzy BCC-filter of X. Then L_μ^ε is an ε-Łukasiewicz fuzzy BCC-filter of X by Theorem 3.10. It is clear that $0 \in L_\mu^\varepsilon$. Let $x, y \in O(L_\mu^\varepsilon)$ be such that $\mu(x*y)+\varepsilon-1 > 0$ and $\mu(x)+\varepsilon-1 > 0$. It follows from (3.8) that $L_\mu^\varepsilon(y) \geq \min\{L_\mu^\varepsilon(x * y), L_\mu^\varepsilon(x)\} = \min\{\mu(x * y) + \varepsilon - 1, \mu(x) + \varepsilon - 1\} > 0$. Thus, $y \in O(L_\mu^\varepsilon)$. Hence, $O(L_\mu^\varepsilon)$ is a BCC-filter of X. □

Theorem 3.20. *Let μ be a fuzzy set in X. If an ε-Łukasiewicz fuzzy set L_μ^ε of μ in X satisfies the following properties:*

$$(\forall x \in X, \forall t \in (0,1])([x/t]qL_\mu^\varepsilon \Rightarrow [0/t]qL_\mu^\varepsilon) \tag{3.21}$$

$$(\forall x, y \in X, \forall t_a, t_b \in (0.5,1]) \left(\begin{array}{l} [x * y/t_a] \in L_\mu^\varepsilon, [x/t_b] \in L_\mu^\varepsilon \\ \Rightarrow [y/\min\{t_a, t_b\}]qL_\mu^\varepsilon \end{array} \right) \tag{3.22}$$

then the O-set $O(L_\mu^\varepsilon)$ of L_μ^ε is a BCC-filter of X.

Proof. If $x \in O(L_\mu^\varepsilon)$, then $\mu(x) > 1 - \varepsilon$, that is, $[x/(1-\varepsilon)] \in \mu$. Hence, $[0/(1-\varepsilon)]qL_\mu^\varepsilon$ by (3.21), and so $L_\mu^\varepsilon(0) + 1 - \varepsilon > 1$. Thus, $L_\mu^\varepsilon(0) > \varepsilon > 0$, which shows that $0 \in O(L_\mu^\varepsilon)$. Let $x, y \in X$ be such that $x*y \in O(L_\mu^\varepsilon)$ and $x \in O(L_\mu^\varepsilon)$. Then $\mu(x*y)+\varepsilon-1 > 0$ and $\mu(x)+\varepsilon-1 > 0$. Since $[x*y/L_\mu^\varepsilon(x*y)] \in L_\mu^\varepsilon$ and $[x/L_\mu^\varepsilon(x)] \in L_\mu^\varepsilon$. It follows from (3.22) that

$$[y/\max\{L_\mu^\varepsilon(x * y), L_\mu^\varepsilon(x)\}]qL_\mu^\varepsilon. \tag{3.23}$$

If $y \notin O(L_\mu^\varepsilon)$, then $L_\mu^\varepsilon(y) = 0$. Thus,

$$\begin{aligned}
&L_\mu^\varepsilon(y) + \max\{L_\mu^\varepsilon(x * y), L_\mu^\varepsilon(x)\} \\
&= \max\{L_\mu^\varepsilon(x * y), L_\mu^\varepsilon(x)\} \\
&= \max\{\max\{0, \mu(x * y) + \varepsilon - 1\}, \max\{0, \mu(x) + \varepsilon - 1\}\} \\
&= \max\{\mu(x * y) + \varepsilon - 1, \mu(x) + \varepsilon - 1\} \\
&= \max\{\mu(x * y), \mu(x)\} + \varepsilon - 1 \\
&\leq 1 + \varepsilon - 1 \\
&= \varepsilon \\
&\leq 1,
\end{aligned}$$

which shows that (3.23) is not valid. This is a contradiction. So $y \in O(L_\mu^\varepsilon)$. Hence, $O(L_\mu^\varepsilon)$ is a BCC-filter of X. □

Theorem 3.21. *Let μ be a fuzzy set in X. If an ε-Łukasiewicz fuzzy set L_μ^ε of μ in X satisfies the following properties:*

$$[0/\varepsilon]q\mu \tag{3.24}$$

$$(\forall x, y \in X)\left(\begin{array}{l}[x * y/\varepsilon]qL_\mu^\varepsilon, [x/\varepsilon]qL_\mu^\varepsilon \\ \Rightarrow [y/\varepsilon]qL_\mu^\varepsilon\end{array}\right) \tag{3.25}$$

then the O-set $O(L_\mu^\varepsilon)$ of L_μ^ε is a BCC-filter of X.

Proof. By (3.24), we have $\mu(0) + \varepsilon > 1$ and so $L_\mu^\varepsilon(0) = \max\{0, \mu(0) + \varepsilon - 1\} = \mu(0) + \varepsilon - 1 > 0$. Hence, $0 \in O(L_\mu^\varepsilon)$. Let $x, y \in X$ be such that $x * y \in O(L_\mu^\varepsilon)$ and $x \in O(L_\mu^\varepsilon)$. Then $\mu(x * y) + \varepsilon - 1 > 0$ and $\mu(x) + \varepsilon - 1 > 0$. Hence, $L_\mu^\varepsilon(x*y) + 1 = \max\{0, \mu(x*y) + \varepsilon - 1\} + 1 = \mu(x*y) + \varepsilon - 1 + 1 = \mu(x*y) + \varepsilon > 1$ and $L_\mu^\varepsilon(x) + 1 = \max\{0, \mu(x) + \varepsilon - 1\} + 1 = \mu(x) + \varepsilon - 1 + 1 = \mu(x) + \varepsilon > 1$, that is, $[x * y/\varepsilon]qL_\mu^\varepsilon$ and $[x/\varepsilon]qL_\mu^\varepsilon$. It follows from (3.25) that $[y/\varepsilon] \in L_\mu^\varepsilon$, which shows that $L_\mu^\varepsilon(y) \geq \varepsilon > 0$. Hence, $y \in O(L_\mu^\varepsilon)$. Hence, $O(L_\mu^\varepsilon)$ is a BCC-filter of X. □

We use BCC-filter to create an ε-Łukasiewicz fuzzy BCC-filter.

Theorem 3.22. *Let F be a BCC-filter of X and let $\alpha, \beta \in (0, 1]$ with $\alpha \geq \beta$. For every ε, define the ε-Łukasiewicz fuzzy set L_μ^ε of μ in X as follows:*

$$L_\mu^\varepsilon : X \to [0, 1], x \mapsto \begin{cases} \alpha & \textit{if } x \in F \\ \beta & \textit{otherwise} \end{cases}$$

Then L_μ^ε is an ε-Łukasiewicz fuzzy BCC-filter of X.

Proof. Since $0 \in F$, we have $L_\mu^\varepsilon(0) = \alpha \geq L_\mu^\varepsilon(x)$ for all $x \in X$. Let $x, y \in X$. If $y \in F$, then $L_\mu^\varepsilon(y) = \alpha \geq \min\{L_\mu^\varepsilon(x * y), L_\mu^\varepsilon(x)\}$. If $y \notin F$, then $x * y \notin F$ or $x \notin F$. Hence, $\min\{L_\mu^\varepsilon(x * y), L_\mu^\varepsilon(x)\} = \beta = L_\mu^\varepsilon(y)$. Therefore, L_μ^ε is an ε-Łukasiewicz fuzzy BCC-filter of X. □

4. Conclusions

Jun [12] introduced ε-Łukasiewicz fuzzy sets using the Łukasiewicz t-norm. This paper applies these sets to BCC-filters in BCC-algebras, proposing ε-Łukasiewicz fuzzy BCC-filters and examining their properties. It is evident that ε-Łukasiewicz fuzzy UP (BCC)-filters represent a broader generalization of ε-Łukasiewicz fuzzy UP (BCC)-ideals. It discusses the characterization of ε-Łukasiewicz fuzzy BCC-filters and explores their relationship with traditional fuzzy BCC-filters. The study establishes conditions for ε-Łukasiewicz fuzzy sets to qualify as ε-Łukasiewicz fuzzy BCC-filters, and investigates when subsets—$\in$-set, q-set, and O-set—can function as BCC-filters.

The insights and findings from this study are expected to influence future research in various algebraic systems, potentially serving as mathematical tools for decision theory, medical diagnosis systems, automation, and other fields.

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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