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BIPOLAR FUZZY INTERIOR IDEALS OF Γ-SEMIRINGS

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Abstract. This paper delves into the investigation of bipolar fuzzy interior ideals within the framework of Γ-semirings, uncovering essential properties such as the characteristic function criterion and the level set criterion. Additionally, it offers a novel characterization of regular Γ-semirings, drawing connections between bipolar fuzzy interior ideals and bipolar fuzzy ideals.

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1. Introduction

The concept of Γ-rings, which serves as a generalization of traditional rings, was first introduced by Nobusawa [13], marking a significant advancement in algebraic structures. Semirings, another key mathematical concept, were explored in depth by Vandiver [14], who laid the groundwork for their study. Building upon these foundational ideas, Rao [12] introduced Γ-semirings, a broader and more versatile framework that unifies and extends the principles of rings, Γ-rings, and semirings. This progression of concepts reflects the evolving nature of algebraic theory, opening new pathways for research and application. Zadeh first introduced the concept of fuzzy sets of a set [16] in 1965. Several extensions of fuzzy sets have been developed since then, such as intuitionistic fuzzy sets, fuzzy sets with interval values, vague sets, neutrosophic sets, etc. Mandal [11] investigated fuzzy ideals and fuzzy interior ideals in ordered semirings. Zhang [17] introduced the concept of bipolar-valued fuzzy sets in 1994, which was a significant extension of fuzzy sets whose membership degree interval is extended

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from $[0,1]$ to $[-1,1]$. Numerous creators like Bhargavi [1] and Eswarlal explored and developed fuzzy notions in Γ-semiring. Madhulatha et al. [8–10] studied bipolar fuzzy Γ-semirings, bipolar fuzzy ideals, bipolar fuzzy bi-ideals of Γ-semirings. Vijay Kumar et al. [15] introduced the concepts of bipolar fuzzy quasi ideals and bipolar N subgroups of near rings. Bhargavi et al. [2–5] studied vague bi-ideals, vague quasi ideals, vague interior ideals, vague bi-quasi-interior ideals, and vague bi-interior ideals of Γ-semirings. Khamrot et al. [7] introduced the concept of bipolar complex fuzzy interior ideals in semigroups.

In this paper, we introduce the notion of bipolar fuzzy interior ideals within the structure of Γ-semirings and explore their fundamental properties. We establish that the homomorphic image and inverse image of a bipolar fuzzy interior ideal in Γ-semirings are themselves bipolar fuzzy interior ideals. Moreover, we demonstrate that within regular Γ-semirings, the concepts of bipolar fuzzy ideals and bipolar fuzzy interior ideals converge, highlighting a significant intersection between these two concepts. This convergence not only enriches the theoretical landscape of Γ-semirings but also provides deeper insights into the structural integrity of these algebraic systems.

2. Preliminaries

In this section, we recall some of the basic concepts and definitions that we need in the sequel.

Definition 2.1. [1] Let $\vee$ and Γ be two additive commutative semigroups. Then $\vee$ is called a Γ-semiring if there exists a mapping $\vee \times \Gamma \times \vee \to \vee$ image denoted by $\ddot{c}\alpha\ddot{p}$ for $\ddot{c}, \ddot{p} \in \vee$ and $\alpha \in \Gamma$, satisfying the following conditions: for all $\ddot{c}, \ddot{p}, \ddot{u} \in \vee$ and $\alpha, \beta \in \Gamma$,

(i) $\ddot{c}\alpha(\ddot{p} + \ddot{u}) = \ddot{c}\alpha\ddot{p} + \ddot{c}\alpha\ddot{u}$,

(ii) $(\ddot{c} + \ddot{p})\alpha\ddot{u} = \ddot{c}\alpha\ddot{u} + \ddot{p}\alpha\ddot{u}$,

(iii) $\ddot{c}(\alpha + \beta)\ddot{u} = \ddot{c}\alpha\ddot{u} + \ddot{c}\beta\ddot{u}$,

(iv) $\ddot{c}\alpha(\ddot{p}\beta\ddot{u}) = (\ddot{c}\alpha\ddot{p})\beta\ddot{u}$.

Definition 2.2. [16] Let $\vee$ be any non-empty set. A mapping $\xi : \vee \to [0,1]$ is called a fuzzy set of $\vee$.

Definition 2.3. [17] Let $\vee$ be the universe of discourse. A bipolar fuzzy set (BFS) ξ in $\vee$ is an object having the form $\xi := \{(\ddot{v}, \xi^-(\ddot{v}), \xi^+(\ddot{v})) : \ddot{v} \in \vee\}$, where $\xi^- : \vee \to [-1,0]$ and $\xi^+ : \vee \to [0,1]$ are mappings.

For the sake of simplicity, we shall use the symbol $\xi = (\vee; \xi^-, \xi^+)$ for the BFS $\xi := \{(\ddot{v}, \xi^-(\ddot{v}), \xi^+(\ddot{v})) : \ddot{v} \in \vee\}$.

Definition 2.4. [17] Let $\xi = (\vee; \xi^-, \xi^+)$ be a BFS and $s \times t \in [-1,0] \times [0,1]$, the sets $\xi^-_s = \{\ddot{v} \in \vee : \xi^-(\ddot{v}) \leq s\}$ and $\xi^+_t = \{\ddot{v} \in \vee : \xi^+(\ddot{v}) \geq t\}$ are called negative s-cut and positive t-cut, respectively. For $s \times t \in [-1,0] \times [0,1]$, the set $\xi_{(s,t)} = \xi^-_s \cap \xi^+_t$ is called the (s,t)-set of $\xi = (\vee; \xi^-, \xi^+)$.

Definition 2.5. [17] Let $\xi = (\vee; \xi^-, \xi^+)$ and $\eta = (\vee; \eta^-, \eta^+)$ be two BFSs in a universe of discourse $\vee$. The intersection of ξ and η is defined as

$$(\xi^- \cap \eta^-)(\ddot{v}) = \min\{\xi^-(\ddot{v}), \eta^-(\ddot{v})\} \text{ and } (\xi^+ \cap \eta^+)(\ddot{v}) = \min\{\xi^+(\ddot{v}), \eta^+(\ddot{v})\}, \forall \ddot{v} \in \vee.$$

The union of ξ and η is defined as

$$(\xi^- \cup \eta^-)(\ddot{v}) = \max\{\xi^-(\ddot{v}), \eta^-(\ddot{v})\} \text{ and } (\xi^+ \cup \eta^+)(\ddot{v}) = \max\{\xi^+(\ddot{v}), \eta^+(\ddot{v})\}, \forall \ddot{v} \in \vee.$$

A BFS ξ is contained in another bipolar fuzzy set η, written with $\xi \subseteq \eta$ if

$$\xi^-(\ddot{v}) \geq \eta^-(\ddot{v}) \text{ and } \xi^+(\ddot{v}) \leq \eta^+(\ddot{v}), \forall \ddot{v} \in \vee.$$

Definition 2.6. [6] Let $J : K \to L$ be a homomorphism from a set K onto a set L and let $\xi = (K; \xi^-, \xi^+)$ be a BFS of K and $\eta = (L; \eta^-, \eta^+)$ be a BFS of L, then the homomorphic image $J(\xi)$ of ξ is $J(\xi) = (L; (J(\xi))^-, (J(\xi))^+)$ defined as for all $\ddot{v} \in L$,

$$(J(\xi))^-(\ddot{v}) = \begin{cases} \inf\{\xi^-(\ddot{u}) : \ddot{u} \in J^{-1}(\ddot{v})\}, & \text{if } J^{-1}(\ddot{v}) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

and

$$(J(\xi))^+(\ddot{v}) = \begin{cases} \sup\{\xi^+(\ddot{u}) : \ddot{u} \in J^{-1}(\ddot{v})\}, & \text{if } J^{-1}(\ddot{v}) \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

The inverse image $J^{-1}(\eta)$ of η under J is a BFS defined as

$$\left(J^{-1}(\eta)\right)^-(\ddot{u}) = \eta^-(J(\ddot{u})) \text{ and } \left(J^{-1}(\eta)\right)^+(\ddot{u}) = \eta^+(J(\ddot{u})), \forall \ddot{u} \in K.$$

Definition 2.7. [1] Let D be a subset of a Γ-semiring $\vee$. The characteristic function of D taking values in $[0, 1]$ is a fuzzy set δ_D given by

$$\delta_D(\ddot{v}) = \begin{cases} 1, & \text{if } \ddot{v} \in D \\ 0, & \text{otherwise.} \end{cases}$$

Then δ_D is a fuzzy characteristic function of D.

Definition 2.8. [8] Let D be a subset of a Γ-semiring $\vee$. The bipolar fuzzy characteristic function δ_D of D is given by

$$\delta_D^+(\ddot{v}) = \begin{cases} 1, & \text{if } \ddot{v} \in D \\ 0, & \text{otherwise} \end{cases} \text{ and } \delta_D^-(\ddot{v}) = \begin{cases} -1, & \text{if } \ddot{v} \in D \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.9. [1] A BFS $\xi = (\vee; \xi^-, \xi^+)$ in $\vee$ is called a bipolar fuzzy Γ-semiring (BFGSR) of $\vee$ if it satisfies the following properties: for all $\ddot{c}, \ddot{p} \in \vee$ and $\gamma \in \Gamma$,

(i) $\xi^-(\ddot{c} + \ddot{p}) \leq \max\{\xi^-(\ddot{c}), \xi^-(\ddot{p})\}$,

(ii) $\xi^-(\ddot{c}\gamma\ddot{p}) \leq \max\{\xi^-(\ddot{c}), \xi^-(\ddot{p})\}$,

(iii) $\xi^+(\ddot{c} + \ddot{p}) \geq \min\{\xi^+(\ddot{c}), \xi^+(\ddot{p})\}$,

(iv) $\xi^+(\ddot{c}\gamma\ddot{p}) \geq \min\{\xi^+(\ddot{c}), \xi^+(\ddot{p})\}$.

Definition 2.10. [3] A BFS $\xi = (\vee; \xi^-, \xi^+)$ in $\vee$ is called a bipolar fuzzy left (resp., right) ideal of $\vee$ if it satisfies the following properties: for any $\ddot{e}, \ddot{o} \in \vee$ and $\varrho \in \Gamma$,
(i) $\xi^-(\ddot{e} + \ddot{o}) \leq \max\{\xi^-(\ddot{e}), \xi^-(\ddot{o})\}$,
(ii) $\xi^-(\ddot{e}\varrho\ddot{o}) \leq \xi^-(\ddot{o})$ (resp., $\leq \xi^-(\ddot{e})$),
(iii) $\xi^+(\ddot{e} + \ddot{o}) \geq \min\{\xi^+(\ddot{e}), \xi^+(\ddot{o})\}$,
(iv) $\xi^+(\ddot{e}\varrho\ddot{o}) \geq \xi^+(\ddot{o})$ (resp., $\geq \xi^+(\ddot{e})$).
Also, ξ is a bipolar fuzzy ideal (BFI) of $\vee$ if it is both a bipolar fuzzy left ideal and a bipolar fuzzy right ideal of $\vee$.

Definition 2.11. [3] An element v of a Γ-semiring $\vee$ is said to be regular if $v \in v\Gamma \vee \Gamma v$. If all the elements of a Γ-semiring $\vee$ are regular, then $\vee$ is known as a regular Γ-semiring.

Definition 2.12. [3] A non-empty subset I of a Γ-semiring $\vee$ is called an interior ideal of $\vee$ if I is an additive subsemigroup of $\vee$ and $\vee\Gamma I\Gamma\vee \subseteq I$.

3. Main Results

In this section, we introduce and investigate the concept of bipolar fuzzy interior ideals, delving into their distinctive properties and characteristics. Following this, we explore the implications of replacing the join operation $\vee$ with a Γ-semiring, examining how this substitution influences the structure and behaviour of these ideals within the broader algebraic framework.

Definition 3.1. A BFGSR $\xi = (\vee; \xi^-, \xi^+)$ of $\vee$ is called a bipolar fuzzy interior ideal (BFII) of $\vee$ if it satisfies the following properties: for any $\ddot{e}, \ddot{o}, \ddot{v} \in \vee$ and $\varrho, \tau \in \Gamma$,
(i) $\xi^-(\ddot{e}\varrho\ddot{o}\tau\ddot{v}) \leq \xi^-(\ddot{o})$,
(ii) $\xi^+(\ddot{e}\varrho\ddot{o}\tau\ddot{v}) \geq \xi^+(\ddot{o})$.

Example 3.2. Let $\vee$ be the set of all negative integers and Γ be the set of all negative even integers. Then $\vee$ and Γ are additive commutative semigroups. Define the product $\ddot{e}\varrho\ddot{o}$ as usual product of $\ddot{e}, \ddot{o} \in \vee$ and $\varrho \in \Gamma$. Then $\vee$ is a Γ-semiring. Define a BFS $\xi = (\vee; \xi^-, \xi^+)$ of $\vee$ as follows:

$$\xi^-(\psi) = \begin{cases} -0.5 & \text{if } \psi = -1 \\ -0.6 & \text{if } \psi = -2 \\ -0.8 & \text{otherwise} \end{cases} \text{ and } \xi^+(\psi) = \begin{cases} 0.5 & \text{if } \psi = -1 \\ 0.6 & \text{if } \psi = -2 \\ 0.7 & \text{otherwise} \end{cases}$$

Then ξ is a BFII of $\vee$.

Theorem 3.3. *A BFS $\xi = (\vee; \xi^-, \xi^+)$ in $\vee$ is a BFII of $\vee$ if and only if for all $s \times t \in [-1, 0] \times [0, 1]$, $\emptyset \neq \xi^-_s$ and $\emptyset \neq \xi^+_t$ are interior ideals of $\vee$.*

Proof. Suppose $\xi = (\vee; \xi^-, \xi^+)$ is a BFII of $\vee$. From Theorem 3.4 of [8], level cuts are sub Γ-semirings of $\vee$. Let $\ddot{e}, \ddot{o} \in \vee$, $\ddot{k} \in \xi^-_s$, $\ddot{l} \in \xi^+_t$, and $\varrho, \tau \in \Gamma$. Then $\xi^-(\ddot{k}) \leq s$ and $\xi^+(\ddot{l}) \geq t$. Now, $\xi^-(\ddot{e}\varrho\ddot{k}\tau\ddot{o}) \leq \xi^-(\ddot{k}) \leq s$

and $\xi^+(\ddot{e}\varrho\ddot{l}\tau\ddot{o}) \leq \xi^+(\ddot{l}) \leq t$. Thus, $\ddot{e}\varrho\ddot{k}\tau\ddot{o} \in \xi_s^-$ and $\ddot{e}\varrho\ddot{l}\tau\ddot{o} \in \xi_t^+$. Hence, level cuts ξ_s^- and ξ_t^+ are interior ideals of $\vee$.

Conversely, suppose that ξ_s^- and ξ_t^+ are interior ideals of $\vee$. From Theorem 3.4 of [8], ξ is a BFGSR of $\vee$. Let $\ddot{e}, \ddot{o} \in \vee$ and $\varrho, \tau \in \Gamma$. Suppose $s = \xi^-(\ddot{q})$. Then $\ddot{q} \in \xi_s^-$. Thus, $\ddot{e}\varrho\ddot{q}\tau\ddot{o} \in \xi_s^-$, that is, $\xi^-(\ddot{e}\varrho\ddot{q}\tau\ddot{o}) \leq s = \xi^-(\ddot{q})$. Suppose $t = \xi^+(\ddot{q})$. Then $\ddot{q} \in \xi_t^+$. Thus, $\ddot{e}\varrho\ddot{q}\tau\ddot{o} \in \xi_t^+$, that is, $\xi^+(\ddot{e}\varrho\ddot{q}\tau\ddot{o}) \leq t = \xi^+(\ddot{q})$. Hence, ξ is a BFII of $\vee$. □

Note 3.4. Let κ be a nonempty subset of $\vee$. Then δ_κ is a BFGSR of $\vee$ if and only if κ is a sub Γ-semiring of $\vee$.

Theorem 3.5. *Let κ be a non-empty subset of $\vee$. Then δ_κ is a BFII of $\vee$ if and only if κ is an interior ideal of $\vee$.*

Proof. Suppose δ_κ is a BFII of $\vee$. As noted above, κ is a sub Γ-semiring of $\vee$. Let $\ddot{e}, \ddot{o} \in \vee, \ddot{k} \in \kappa$, and $\varrho, \tau \in \Gamma$. Then $\delta_\kappa^+(\ddot{e}\varrho\ddot{k}\tau\ddot{o}) \geq \delta_\kappa^+(\ddot{k}) = 1$, so $\ddot{e}\varrho\ddot{k}\tau\ddot{o} \in \kappa$. Now, $\delta_\kappa^-(\ddot{e}\varrho\ddot{k}\tau\ddot{o}) \geq \delta_\kappa^-(\ddot{k}) = -1$, so $\ddot{e}\varrho\ddot{k}\tau\ddot{o} \in \kappa$. Hence, κ is an interior ideal of $\vee$.

Conversely, suppose κ is an interior ideal of $\vee$. As noted above, δ_κ is a BFGSR of $\vee$. Let $\ddot{e}, \ddot{o}, \ddot{k} \in \vee$ and $\varrho, \tau \in \Gamma$.

If $\ddot{k} \in \kappa$, then $\ddot{e}\varrho\ddot{k}\tau\ddot{o} \in \kappa$. Thus, $\delta_\kappa^+(\ddot{e}\varrho\ddot{k}\tau\ddot{o}) = 1 = \delta_\kappa^+(\ddot{k})$ and $\delta_\kappa^-(\ddot{e}\varrho\ddot{k}\tau\ddot{o}) = -1 = \delta_\kappa^-(\ddot{k})$.

If $\ddot{k} \notin \kappa$, then $\delta_\kappa^-(\ddot{k}) = 0$ and $\delta_\kappa^+(\ddot{k}) = 0$. Thus, $\delta_\kappa^+(\ddot{e}\varrho\ddot{k}\tau\ddot{o}) \geq \delta_\kappa^+(\ddot{k})$ and $\delta_\kappa^-(\ddot{e}\varrho\ddot{k}\tau\ddot{o}) \leq \delta_\kappa^-(\ddot{k})$.

Hence, δ_κ is a BFII of $\vee$. □

Note 3.6. Let $\vee$ and κ be two Γ-semirings and let f be a homomorphism from $\vee$ to κ. If ξ is a BFGSR of κ, then the inverse image of ξ, $f^{-1}(\xi)$ is a BFGSR of $\vee$.

Theorem 3.7. *Suppose $\vee$ and κ are two Γ-semirings and let f be a homomorphism of $\vee$ onto κ. If ξ is a BFII of κ, then the inverse image of ξ, $f^{-1}(\xi)$ is a BFII of $\vee$.*

Proof. Given ξ is a BFII of κ. As noted above, $f^{-1}(\xi)$ is a BFGSR of $\vee$. Let $\ddot{e}, \ddot{o}, \ddot{k} \in \vee$ and $\varrho, \tau \in \Gamma$. Then

$$\begin{aligned}(f^{-1}(\xi))^-(\ddot{e}\varrho\ddot{k}\tau\ddot{o}) &= \xi^-(f(\ddot{e}\varrho\ddot{k}\tau\ddot{o})) \\ &= \xi^-(f(\ddot{e})\varrho f(\ddot{k})\tau f(\ddot{o})) \\ &\leq \xi^-(f(\ddot{k})) \\ &= (f^{-1}(\xi))^-(\ddot{k}),\end{aligned}$$

$$\begin{aligned}(f^{-1}(\xi))^+(\ddot{e}\varrho\ddot{k}\tau\ddot{o}) &= \xi^+(f(\ddot{e}\varrho\ddot{k}\tau\ddot{o})) \\ &= \xi^+(f(\ddot{e})\varrho f(\ddot{k})\tau f(\ddot{o})) \\ &\geq \xi^+(f(\ddot{k})) \\ &= (f^{-1}(\xi))^+(\ddot{k}).\end{aligned}$$

Hence, $f^{-1}(\xi)$ is a BFII of $\vee$. □

Note 3.8. Let $\vee$ and κ be two Γ-semirings and let f be a homomorphism from $\vee$ to κ. If ξ is a BFGSR of $\vee$, then the homomorphic image of ξ, $f(\xi)$ is a BFGSR of κ.

Theorem 3.9. *Suppose $\vee$ and κ are two Γ-semirings and let f be a homomorphism of $\vee$ to κ. If ξ is a BFII of $\vee$, then the homomorphic image of ξ, $f(\xi)$ is a BFII of κ.*

Proof. Given ξ is a BFII of $\vee$. As noted, $f(\xi)$ is a BFGSR of $\vee$. Suppose $f^{-1}(\ddot{e}), f^{-1}(\ddot{o})$, and $f^{-1}(\ddot{v})$ are non-empty. Then there exist $\hat{e}, \hat{o}$, and $\hat{v} \in \vee$ such that $f(\hat{e}) = \ddot{e}, f(\hat{o}) = \ddot{o}$, and $f(\hat{v}) = \ddot{v}$. Then $\hat{e} + \hat{o} \in f^{-1}(\ddot{e} + \ddot{o})$ and $\hat{e}\varrho\hat{o}\tau\hat{v} \in f^{-1}(\ddot{e}\varrho\ddot{o}\tau\ddot{v})$. Now,

$$\begin{aligned}(f(\xi))^-(\ddot{e}\varrho\ddot{o}\tau\ddot{v}) &= \inf\{\xi^-(\hat{s}) : \hat{s} \in f^{-1}(\ddot{e}\varrho\ddot{o}\tau\ddot{v})\} \\ &= \inf\{\xi^-(\hat{e}\varrho\hat{o}\tau\hat{v}) : \hat{e} \in f^{-1}(\ddot{e}), \hat{o} \in f^{-1}(\ddot{o}), \hat{v} \in f^{-1}(\ddot{v})\} \\ &\leq \inf\{\xi^-(\hat{o}) : \hat{o} \in f^{-1}(\ddot{o})\} \\ &= (f(\xi))^-(\ddot{o}),\end{aligned}$$

$$\begin{aligned}(f(\xi))^+(\ddot{e}\varrho\ddot{o}\tau\ddot{v}) &= \sup\{\xi^+(\hat{s})) : \hat{s} \in f^{-1}(\ddot{e}\varrho\ddot{o}\tau\ddot{v})\} \\ &= \sup\{\xi^+(\hat{e}\varrho\hat{o}\tau\hat{v}) : \hat{e} \in f^{-1}(\ddot{e}), \hat{o} \in f^{-1}(\ddot{o}), \hat{v} \in f^{-1}(\ddot{v})\} \\ &\geq \sup\{\xi^+(\hat{o}) : \hat{o} \in f^{-1}(\ddot{o})\} \\ &= (f(\xi))^+(\ddot{o}).\end{aligned}$$

Hence, $f(\xi)$ is a BFII of κ. □

Theorem 3.10. *If $\xi = (\vee; \xi^-, \xi^+)$ and $\varkappa = (\vee; \varkappa^-, \varkappa^+)$ are two BFIIs of $\vee$, then $\xi \cap \varkappa$ is a BFII of $\vee$.*

Proof. Given $\xi = (\vee; \xi^-, \xi^+)$ and $\varkappa = (\vee; \varkappa^-, \varkappa^+)$ are two BFIIs of $\vee$. From Theorem 3.6 of [8], the intersection of an arbitrary family of BFGSRs is a BFGSR. Let $\ddot{e}, \ddot{o}, \ddot{v} \in \vee$ and $\varrho, \tau \in \Gamma$. Then

$$\begin{aligned}(\xi^- \cap \varkappa^-)(\ddot{e}\varrho\ddot{o}\tau\ddot{v}) &= \min\{\xi^-(\ddot{e}\varrho\ddot{o}\tau\ddot{v}), \varkappa^-(\ddot{e}\varrho\ddot{o}\tau\ddot{v})\} \\ &\leq \min\{\xi^-(\ddot{o}), \varkappa^-(\ddot{o})\} \\ &= (\xi^- \cap \varkappa^-)(\ddot{o}),\end{aligned}$$

$$\begin{aligned}(\xi^+ \cap \varkappa^+)(\ddot{e}\varrho\ddot{o}\tau\ddot{v}) &= \min\{\xi^+(\ddot{e}\varrho\ddot{o}\tau\ddot{v}), \varkappa^+(\ddot{e}\varrho\ddot{o}\tau\ddot{v})\} \\ &\geq \min\{\xi^+(\ddot{o}), \varkappa^+(\ddot{o})\} \\ &= (\xi^+ \cap \varkappa^+)(\ddot{o}).\end{aligned}$$

Hence, $\xi \cap \varkappa$ is BFII of $\vee$. □

Corollary 3.11. *The intersection of an arbitrary family of BFIIs is a BFII.*

In general, the union of two BFIIs is not a BFII.

Example 3.12. Let ξ and $\varkappa$ be two BFIIs of $\vee$ defined by

$$\xi^-(\psi) = \begin{cases} -0.64 & \text{if } \psi = 0 \\ -0.51 & \text{if } \psi > 0 \\ -0.42 & \text{if } \psi < 0 \end{cases}, \xi^+(\psi) = \begin{cases} 0.72 & \text{if } \psi = 0 \\ 0.63 & \text{if } \psi > 0 \\ 0.2 & \text{if } \psi < 0 \end{cases}$$

and

$$\varkappa^-(\psi) = \begin{cases} -0.73 & \text{if } \psi = 0 \\ -0.47 & \text{if } \psi > 0 \\ -0.35 & \text{if } \psi < 0 \end{cases}, \varkappa^+(\psi) = \begin{cases} 0.83 & \text{if } \psi = 0 \\ 0.54 & \text{if } \psi > 0 \\ 0.19 & \text{if } \psi < 0 \end{cases}$$

Then $\xi \cup \varkappa$ is not a BFII of $\vee$.

Theorem 3.13. *If ξ and $\varkappa$ are two BFIIs of $\vee$, then $\xi \cup \varkappa$ is a BFII of $\vee$ if and only if $\xi \subseteq \varkappa$ or $\varkappa \subseteq \xi$.*

Proof. Given ξ and $\varkappa$ are two BFIIs of $\vee$. Suppose $\xi \subseteq \varkappa$. From Theorem 3.9 of [8], the union of an arbitrary family of BFGSRs is a BFGSR. Let $\ddot{e}, \ddot{o}, \ddot{v} \in \vee$ and $\varrho, \tau \in \Gamma$. Then

$$\begin{aligned}(\xi^- \cup \varkappa^-)(\ddot{e}\varrho\ddot{o}\tau\ddot{v}) &= \max\{\xi^-(\ddot{e}\varrho\ddot{o}\tau\ddot{v}), \varkappa^-(\ddot{e}\varrho\ddot{o}\tau\ddot{v})\} \\ &= \xi^-(\ddot{e}\varrho\ddot{o}\tau\ddot{v}) \\ &\leq \xi^-(\ddot{o}) \\ &= \max\{\xi^-(\ddot{o}), \varkappa^-(\ddot{o})\} \\ &= (\xi^- \cup \varkappa^-)(\ddot{o}),\end{aligned}$$

$$\begin{aligned}(\xi^+ \cup \varkappa^+)(\ddot{e}\varrho\ddot{o}\tau\ddot{v}) &= \max\{\xi^+(\ddot{e}\varrho\ddot{o}\tau\ddot{v}), \varkappa^+(\ddot{e}\varrho\ddot{o}\tau\ddot{v})\} \\ &= \varkappa^+(\ddot{e}\varrho\ddot{o}\tau\ddot{v}) \\ &\geq \varkappa^+(\ddot{o}) \\ &= \max\{\xi^+(\ddot{o}), \varkappa^+(\ddot{o})\} \\ &= (\xi^+ \cup \varkappa^+)(\ddot{o}).\end{aligned}$$

Hence, $\xi \cup \varkappa$ is a BFII of $\vee$. Similarly, if $\varkappa \subseteq \xi$, then $\xi \cup \varkappa$ is a BFII of $\vee$.

The converse is obvious. □

Theorem 3.14. *If ξ is a BFI of $\vee$, then ξ is a BFII of $\vee$.*

Proof. Given ξ is a BFI of $\vee$. Let $\ddot{e}, \ddot{o}, \ddot{v} \in \vee$ and $\varrho, \tau \in \Gamma$. Since ξ is a BFI of $\vee$, we have $\xi^-(\ddot{e}\varrho\ddot{o}\ddot{o}) \leq \xi^-(\ddot{e})$ and $\xi^-(\ddot{e}\varrho\ddot{o}) \leq \xi^-(\ddot{o})$. Thus, $\xi^-(\ddot{e}\varrho\ddot{o}) \leq \max\{\xi^-(\ddot{e}), \xi^-(\ddot{o})\}$. Also, $\xi^+(\ddot{e}\varrho\ddot{o}) \geq \xi^+(\ddot{e})$ and $\xi^+(\ddot{e}\varrho\ddot{o}) \geq \xi^+(\ddot{o})$. Thus, $\xi^+(\ddot{e}\varrho\ddot{o}) \geq \min\{\xi^+(\ddot{e}), \xi^+(\ddot{o})\}$. Thus, ξ is a BFGSR of $\vee$. Now,

$$\begin{aligned}\xi^-(\ddot{e}\varrho\ddot{o}\tau\ddot{v}) &= \xi^-(\ddot{e}\varrho(\ddot{o}\tau\ddot{v}))\\ &\leq \xi^-(\ddot{o}\tau\ddot{v})\\ &\leq \xi^-(\ddot{o}),\end{aligned}$$

$$\begin{aligned}\xi^+(\ddot{e}\varrho\ddot{o}\tau\ddot{v}) &= \xi^+(\ddot{e}\varrho(\ddot{o}\tau\ddot{v}))\\ &\geq \xi^+(\ddot{o}\tau\ddot{v})\\ &\geq \xi^+(\ddot{o}).\end{aligned}$$

Therefore, ξ is a BFII of $\vee$. □

Theorem 3.15. *If $\vee$ is regular, then every BFII is a BFI of $\vee$.*

Proof. Given ξ is a BFII of $\vee$. Let $\ddot{e}, \ddot{o}, \ddot{v} \in \vee$ and $\varrho, \tau \in \Gamma$. Then $\xi^-(\ddot{e}\varrho\ddot{o}) \leq \max\{\xi^-(\ddot{e}), \xi^-(\ddot{o})\}$ and $\xi^+(\ddot{e}\varrho\ddot{o}) \geq \min\{\xi^+(\ddot{e}), \xi^+(\ddot{o})\}$. Since $\vee$ is regular, we have $\ddot{o}$=$\ddot{o}\varrho\ddot{m}\tau\ddot{o}$ for some $\ddot{m} \in \vee$ and $\varrho, \tau \in \Gamma$. Now,

$$\begin{aligned}\xi^-(\ddot{e}\gamma\ddot{o}) &= \xi^-(\ddot{e}\gamma(\ddot{o}\varrho\ddot{m}\tau\ddot{o}))\\ &= \xi^-(\ddot{e}\gamma\ddot{o}\varrho(\ddot{m}\tau\ddot{o}))\\ &\leq \xi^-(\ddot{o}),\end{aligned}$$

$$\begin{aligned}\xi^+(\ddot{e}\gamma\ddot{o}) &= \xi^+(\ddot{e}\gamma(\ddot{o}\varrho\ddot{m}\tau\ddot{o}))\\ &= \xi^+(\ddot{e}\gamma\ddot{o}\varrho(\ddot{m}\tau\ddot{o}))\\ &\geq \xi^+(\ddot{o}).\end{aligned}$$

Therefore, ξ is a BFI of $\vee$. □

4. Conclusion

In this article, we introduced the concept of bipolar fuzzy interior ideals (BFII) within the framework of Γ-semirings and thoroughly examined their key properties. Our investigation revealed that in regular Γ-semirings, every BFII naturally aligns with the broader category of bipolar fuzzy ideals (BFI). This finding not only solidifies the relationship between these two classes of ideals but also offers new perspectives on the structural characteristics of Γ-semirings. Our results open avenues for further exploration into the algebraic properties and applications of these ideals in more complex semiring structures.

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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