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THE EXISTENCE, UNIQUENESS AND STABILITY ANALYSIS OF LEVI

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Abstract. In this article, we study the existence, uniqueness, and stability of Linear Evaluation Variational Inequalities LEVI. The new results are provided on the existence and uniqueness of the solution in a closed and convex set in the real Hilbert space satisfying LEVI. A positive semi-definite matrix is constructed to discuss the stability and instability analysis of LEVI. The novel results are provided for the asymptotic stability of LEVI. Furthermore, an optimization problem is presented to study the behaviour of the spectrum of matrices corresponding to LEVI.

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1. Introduction

In applied mathematics and engineering the study of stability analysis of a dynamical system is a very important and interesting topic. The computation of spectrum for linearized operators for non-linear differential equations determine the Lyapunov stability, see [1] for more and comprehensive details. The evolution variational inequalities [2] can be used to represent a class of unilaterally constrained dynamical systems. The evolution variational inequalities are a special type of dynamical system where the state of the system is forced to remain in a closed and convex set $K \subseteq \mathbf{R}^n$ in a real Hilbert space H.

The evolution variational inequalities are widely used in applied mathematics and many other research areas of science and has many very important applications across urban transportation, networking, networking in traffic problems, agricultural and energy markets, for more details see [3–7] and the references therein.

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The stability analysis of evolution variational inequalities has been studied by many authors; see, e.g., [8–12]. The evolution variational inequalities show the instability when $K \subset \mathbf{R}^n$ while stability for $K = \mathbf{R}^n$. Furthermore, one must need the computation of spectrum of Jacobian for unconstrained dynamical system in order to discuss the stability analysis of the evolution variational inequalities. But, this isn't practically achievable because the linearization of an unconstrained dynamical system maybe exponentially stable while evolution variational inequalities remains unstable. On the other side, its also possible that the tangent linearization of unconstrained dynamical system maybe of exponential unstable but the constrained dynamical system is asymptotically stable.

In this paper, we have presented new results for the stability analysis of unilaterally constrained dynamical systems. For this purpose, we have shown the existence of a positive semi-definite matrix G such that the spectrum of $(GA + A^tG)$ is non-negative for a given positive definite matrix $A \in \mathbf{R}^{n,n}$. For the existence of G, we make use of the famous and well-known linear algebra tool, the singular value decomposition and Schur complement lemma. For the asymptotically stable solutions of dynamical system under consideration, we aim to show the existence of a positive definite matrix G such that the spectrum corresponding to $(GA + A^tG)$ is strictly positive.

For the instbility of linear evolution variational inequalities, we have presented the existence of a positive definite matrix G such that $((G + G^t)A + \alpha G)$ is strictly negative for $\alpha > 0$. For this purpose, we proposed and then solve an optimization problem subject to constrained on the matrix-valued functions and their time derivatives.

The variational inequalities unveil the unified framework for consideration and study of problems lying in equilibrium systems, dynamical systems, complementarity problems and fixed point problems [13]. Recently many iterative methods has been noticed in terms of rich theoretical and algorithmic development in the wake of solutions to equilibrium problems in nonlinear analysis and optimization. Variational inequalities problems are native framework for formulating and analyzing equilibrium problems in regard to existence and uniqueness of their solutions, sensitivity and stability analysis of the obtained recovered solutions in terms of numerical analysis formalism.

The ingenuity of variational inequalities implied in larger areas like social sciences, weather forecasting, dam-construction engineering and dynamical systems in particular. The variational inequalities and fixed point formulation are concomitant problems. The equivalence between variational inequalities and fixed point formulation plays vital role in developing and analyzing a class of dynamical systems. The salient view of this mechanism expresses that set of stationary points of dynamical system correspond to the corresponding variational inequalities. Considered dynamical system [14] related to congested transportation system and problems of human migration with core interests of existence of equilibria. In [15], authors have shown that and established unique solution of dynamical system by utilizing the Lipchitz condition in equilibrium systems.

The stability issues of equilibria of projected dynamical systems are addressed in [16]. They supported two particular approaches named as monotonicity and regularity during the stability analysis, leading prosperous results in price equilibrium problems and market strategic equilibrium problems. The use of evolution variational inequalities for social dynamical applications like a human migration, which is more interacting problem globally and of greater humanitarian attentions were studied [16]. One of the main remark is that the linear growth condition has been evoked in the sequential variational inequalities with a non-homogeneous Markov chain. Moreover this formulation works under general conservation laws for population flows. A special type of variational inequalities, known as quasi variational inequalities, which are usually studied in impulse control system, operation research and economics, more as a programing resource in transportation and social sciences.

Overview of article: In Section 2 of manuscript, the statement of the problem under consideration is provided. In Section 3, we present some new results to study the existence and uniqueness of the solution of given LVEI. For existence and uniqueness, a closed and convex set in the real Hilbert space is considered while the operators used are of the non-linear nature. In Section 4, some novel results are provided based on optimization technique to discuss the stability and instability of LEVI. Finally, the conclusions are provided in Section 5 of the manuscript.

2. Problem Statement

Let $K \subset \mathbf{R}^n$ be a non-empty closed and convex set. Let $0 \in K$. The problem under consideration is to find $x(t) \in C^0\left([t_0, \infty); \mathbf{R}^n]\right)$ such that for $x(t)$, $\frac{dx(t)}{dt} \in L^{\infty}_{loc}(t_0, \infty; \mathbf{R}^n)$ and

$$(P) \longleftarrow \begin{cases} \langle \frac{dx(t)}{dt} + Ax(t) + Fx(t), v - x(t) \rangle \geq 0, \quad \forall\, v \in K,\ t \geq t_0 \quad (a) \\ x(t) \in K, \qquad\qquad t \geq t_0 \quad (b) \\ x(t_0) = x_0 \quad (c). \end{cases}$$

3. Existence and Uniqueness of Linear Evolution Variational Inequalities

Theorem 1 shows the uniqueness and existence of variational inequality (a).

Theorem 1. *Let $K \subset \mathbf{R}^n$ be a closed and convex set in real Hilbert Space H, and let $\frac{d}{dt}, F$ be the non-linear operators. Let the operators $\frac{d}{dt}, F$ are strongly monotone with constants $\alpha \geq 0, \theta \geq 0, \omega \geq 0$, and Lipschitz continuous with constants $\beta \geq 0, \phi \geq 0, \upsilon \geq 0$, and there exist a constant $\rho \geq 0, \eta \geq 0, \xi \geq 0$ such that*

$$0 < \rho < \frac{2\alpha}{\beta^2},\ \ 0 < \eta < \frac{2\theta}{\phi^2},\ \ 0 < \xi < \frac{2\omega}{\upsilon^2}.$$

Then there exists a unique solution $x(t) \in K$ satisfying the variational inequality (a) in problem (P).

Proof. **Existence:** The variational inequality

$$\langle \frac{dx(t)}{dt} + Ax(t) + Fx(t), v - x(t) \rangle \geq 0,\ \forall\, v \in K,\ t \geq t_0$$

is equivalent to fixed point problem

$$x(t) = P_k[x(t) - \rho\frac{d}{dt}x(t)] + Q_K[x(t) - \eta Ax(t)] + R_k[x(t) - \xi Fx(t)].$$

Thus we can associate a mapping $E(x(t))$ with the variational inequality (a) as

$$E(x(t)) = P_k[x(t) - \rho\frac{d}{dt}x(t)] + Q_K[x(t) - \eta Ax(t)] + R_k[x(t) - \xi Fx(t)]. \tag{1}$$

It is enough to show that the mapping $E(x(t))$ defined by Equ. 1 has a fixed point. For $x_1(t) \neq x_2(t)$ in K, consider that

$$\left\|E(x_1(t)) - E(x_2(t))\right\| = A + B + C$$

with

$$A = \left\|P_K[x_1(t) - \rho\tfrac{d}{dt}x_1(t)] - P_K[x_2(t) - \rho\tfrac{d}{dt}x_2(t)]\right\|,$$

$$B = \left\|Q_K[x_1(t) - \eta Ax_1(t)] - Q_K[x_2(t)\eta - Ax_2(t)]\right\|,$$

and

$$C = \left\|R_k[x_1(t) - \xi Fx_1(t)] - R_k[x_2(t) - \xi Fx_2(t)]\right\|.$$

Since, P_K, Q_K, R_K are non-expansive mapping. In turn this implies that the express for $\left\|E(x_1(t)) - E(x_2(t))\right\|$ is

$$\leq \left\|x_1 - x_2 - \rho(\tfrac{d}{dt}x_1 - \tfrac{d}{dt}x_2)\right\| + \left\|x_1 - x_2 - \eta(Ax_1 - Ax_2)\right\| + \left\|x_1 - x_2 - \xi(Fx_1 - Fx_2)\right\|. \tag{2}$$

In Equ. 2, we have omitted the dependency on t. Furthermore, consider that

$$\begin{aligned}
&\left\|x_1 - x_2\rho(\tfrac{d}{dt}x_1 - \tfrac{d}{dt}x_2)\right\|^2 \\
&= \langle x_1 - x_2\rho(\frac{d}{dt}x_1 - \frac{d}{dt}x_2), x_1 - x_2\rho(\frac{d}{dt}x_1 - \frac{d}{dt}x_2)\rangle \\
&= \langle x_1 - x_2, x_1 - x_2\rangle - 2\rho\langle x_1 - x_2, \frac{d}{dt}(x_1 - x_2)\rangle + \rho^2\langle\frac{d}{dt}x_1 - \frac{d}{dt}x_2, \frac{d}{dt}x_1 - \frac{d}{dt}x_2\rangle \\
&= \left\|x_1 - x_2\right\|^2 - 2\rho\alpha\left\|x_1 - x_2\right\|^2 + \rho^2\beta^2\left\|x_1 - x_2\right\|^2 = (1 - 2\rho\alpha + \rho^2\beta^2)\left\|x_1 - x_2\right\|^2.
\end{aligned} \tag{3}$$

Here we have used the strongly monotonicity of operator $\frac{d}{dt}$ with constants $\alpha > 0$ and Lipschitz continuity with constant $\beta > 0$. Again consider that

$$\begin{aligned}
&\left\|x_1 - x_2\eta(Ax_1 - Ax_2)\right\|^2 \\
&= \langle x_1 - x_2 - \eta(Ax_1 - Ax_2), x_1 - x_2 - \eta(Ax_1 - Ax_2)\rangle \\
&= \langle x_1 - x_2, x_1 - x_2\rangle - 2\eta\langle x_1 - x_2, Ax_1 - Ax_2\rangle + \eta^2\langle Ax_1 - Ax_2, Ax_1 - Ax_2\rangle \\
&= \left\|x_1 - x_2\right\|^2 - 2\eta\theta\left\|x_1 - x_2\right\|^2 + \eta^2\phi^2\left\|x_1 - x_2\right\|^2 = (1 - 2\eta\theta + \eta^2\phi^2)\left\|x_1 - x_2\right\|^2.
\end{aligned} \tag{4}$$

Here we have used strongly monotonicity of operator A with constant $\theta > 0$ and Lipschitz continuity with constant $\phi > 0$. Consider that

$$\begin{aligned}
&\left\|x_1 - x_2\xi(Fx_1 - Fx_2)\right\|^2 \\
&= \langle x_1 - x_2 - \xi(Fx_1 - Fx_2), x_1 - x_2 - \xi(Fx_1 - Fx_2)\rangle \\
&= \langle x_1 - x_2, x_1 - x_2\rangle - 2\xi\langle x_1 - x_2, Fx_1 - Fx_2\rangle + \xi^2\langle Fx_1 - Fx_2, Fx_1 - Fx_2\rangle \\
&= \left\|x_1 - x_2\right\|^2 - 2\xi\omega\left\|x_1 - x_2\right\|^2 + \xi^2\omega^2\left\|x_1 - x_2\right\|^2 = (1 - 2\xi\omega + \xi^2\omega^2)\left\|x_1 - x_2\right\|^2. \qquad (5)
\end{aligned}$$

Here we have used strongly monotonicity of operator F with constant $\omega > 0$ and Lipschitz continuity with constant $v > 0$. The inequality 2 in view of Equ. 3 to Equ. 5 implies the following inequality for $\|E(x_1(t)) - E(x_2(t))\|$, that is,

$$\leq \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\,\|x_1 - x_2\|^2 + \sqrt{1 - 2\eta\theta + \eta^2\phi^2}\,\|x_1 - x_2\|^2 + \sqrt{1 - 2\xi\omega + \xi^2 v^2}\,\|x_1 - x_2\|^2 = L,$$

with $L = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2} + \sqrt{1 - 2\eta\theta + \eta^2\phi^2} + \sqrt{1 - 2\xi\omega + \xi^2 v^2} = l_1 + l_2 + l_3$.

Let $L < 1$, then $l_1 < 1,\ l_2 < 1\ l_3 < 1$. For $l_1 < 1$, we have that $0 < \rho < \frac{2\alpha}{\beta^2}$. For $l_2 < 1$, we have that $0 < \eta < \frac{2\theta}{\phi^2}$ and for $l_3 < 1$, we have that $0 < \xi < \frac{2\omega}{v^2}$. From the conditions $0 < \rho < \frac{2\alpha}{\beta^2}$, $0 < \eta < \frac{2\theta}{\phi^2}$, $0 < \xi < \frac{2\omega}{v^2}$, it follows that $L < 1$ and the mapping $E(x(t))$ is the contraction mapping and consequently it has a fixed point $E(x(t)) = x \in K$, a closed and convex set satisfying the variational inequality (a).

Uniqueness: Let $x_1(t) \neq x_2(t)$ be the solution of variational inequality (a), then

$$\langle\frac{dx_1(t)}{dt} + Ax_1(t) + Fx_1(t), v - x_1(t)\rangle \geq 0, \forall\, v \in K,\ t \geq t_0 \qquad (6)$$

and

$$\langle\frac{dx_2(t)}{dt} + Ax_2(t) + Fx_2(t), v - x_2(t)\rangle \geq 0, \forall\, v \in K,\ t \geq t_0 \qquad (7)$$

Take $v = x_2(t)$ in Equ. 7 and $v = x_1(t)$ in Equ. 6 to have

$$\langle\frac{dx_1(t)}{dt} + Ax_1(t) + Fx_1(t), x_2(t) - x_1(t)\rangle \geq 0, \forall\, x_1(t), x_2(t) \in K \qquad (8)$$

and

$$\langle\frac{dx_2(t)}{dt} + Ax_2(t) + Fx_2(t), x_1(t) - x_2(t)\rangle \geq 0, \forall\, x_1(t), x_2(t) \in K. \qquad (9)$$

In turn this implies that

$$\langle\frac{-dx_2(t)}{dt} - Ax_2(t) - Fx_2(t), x_2(t) - x_1(t)\rangle \geq 0, \forall\, x_1(t), x_2(t) \in K \qquad (10)$$

Adding Equ. 8 and Equ. 10, we have

$$\langle\frac{dx_1(t)}{dt} - \frac{dx_2(t)}{dt} + Ax_1(t) - Ax_2(t) + Fx_1(t) - Fx_2(t), x_2(t) - x_1(t)\rangle \geq 0, \forall\, x_1(t), x_2(t) \in K. \qquad (11)$$

This implies that

$$\langle\frac{d(x_1(t)-x_2(t))}{dt}+A(x_1(t)-x_2(t))+F(x_1(t)-x_2(t)),x_2(t)-x_1(t)\rangle\geq 0,\forall\, x_1(t),x_2(t)\in K. \quad (12)$$

Furthermore,

$$\langle\frac{d(x_1(t)-x_2(t))}{dt}+A(x_1(t)-x_2(t))+F(x_1(t)-x_2(t)),x_1(t)-x_2(t)\rangle\leq 0,\forall\, x_1(t),x_2(t)\in K. \quad (13)$$

Thus,

$$\langle\frac{dx_1}{dt}-\frac{dx_2}{dt},x_1-x_2\rangle+\langle Ax_1(t)-Ax_2(t),x_1(t)-x_2(t)\rangle+\langle Fx_1(t)-Fx_2(t),x_1(t)-x_2(t)\rangle\leq 0. \quad (14)$$

Since, $\frac{d}{dt}$ is strongly monotone with constant $\alpha>0$, A is strongly monotone with constant $\theta>0$ and F is strongly monotone with constant $\omega>0$ such that

$$\alpha\left\|x_1(t)-x_2(t)\right\|^2+\theta\left\|x_1(t)-x_2(t)\right\|^2+\omega\left\|x_1(t)-x_2(t)\right\|^2$$

$$\leq\langle\frac{d(x_1-x_2}{dt},x_1-x_2\rangle+\langle A(x_1(t)-x_2(t),x_1(t)-x_2(t)\rangle+\langle F(x_1(t)-x_2(t)),x_1(t)-x_2(t)\rangle\leq 0.$$

From this, we have that

$$\alpha\left\|x_1(t)-x_2(t)\right\|^2+\theta\left\|x_1(t)-x_2(t)\right\|^2+\omega\left\|x_1(t)-x_2(t)\right\|^2\leq 0.$$

This implies that

$$(\alpha+\theta+\omega)\left\|x_1(t)-x_2(t)\right\|^2\leq 0$$

and

$$\left\|x_1(t)-x_2(t)\right\|^2\leq 0.$$

Thus, finally we have that $x_1(t)=x_2(t)$, which implies the uniqueness of the solution.

4. Stability of unilaterally constrained dynamical systems

4.1. **Stability Analysis of Linear Evolution Variational Inequalities (LEVI).**

Theorem 2 show that solutions corresponding to (a) and (b) are stable for given $A\in\mathbf{R}^{n,n}$ belonging to set $\mho_K$.

Theorem 2. *Let $K\subset\mathbf{R}^n$ be a set satisfying hypothesis:*

(H1): Set K is closed,

(H2): Set K is convex,

(H3): $0\in K$.

Then, solutions to (a) and (b) in (P) are stable for given $A\in\mho_K$.

Proof. To show that solutions of (a) and (b) are stable for given $A \in \mho_K$, it is sufficient to show that there exists a positive semi-definite matrix G such that

$$\lambda_i \left(GA + A^t G \right) \geq 0, \quad \forall i.$$

For existence of G such that $\lambda_i(GG^t) \geq 0$ or $\sigma_i(G) \geq 0 \; \forall i$, SVD, the famous linear algebra tool applied on A yields

$$A = U \begin{pmatrix} \sigma_1 & 0 \\ 0 & B \end{pmatrix} V^t,$$

where U, V are unitary matrices. Let $\sigma_1 = \left\| Av \right\|_2, \left\| v \right\|_2 = 1$ for $v \in \mathbf{R}^{n,1}$. Take u_1, the very first column of U as $u_1 = \frac{Av}{\sigma_1}$, then

$$\frac{\left\| Av \right\|_2}{\sigma_1} = \frac{\left\| Av \right\|_2}{\left\| A \right\|_2} = 1.$$

The quantity $\sigma_1 = \left\| A \right\|_2$ denotes largest singular value of $A \in \mathbf{R}^{n,n}$. The structure of $U_2 \in \mathbf{R}^{m \times n-1}$, $V_2 \in \mathbf{R}^{n \times n-1}$ such that $U = (u_1 : U_2)$ and $V = (v_1 : V_2)$. Thus, the structure of given A in view of U, V appear as

$$(u_1 : U_2)\, A\, (v_1 : V_2) = \begin{pmatrix} \sigma_1 u_1^t u_1 & u_1^t A V_2 \\ \sigma_1 U_2^t u_1 & U_2^t A V_2 \end{pmatrix} = \begin{pmatrix} \sigma_1 & a \\ 0 & B \end{pmatrix}.$$

In above, $u_1^t u_1 = 1$, $U_2^t u_1 = 0$, $a = V_2^t A^t u_1$, and $B = U_2^t A V_2$. Let $a = 0$, then

$$\sigma_1^2 = \left\| A \right\|_2^2 = max \left(\frac{\left\| U^t A V x \right\|_2^2}{\left\| x \right\|_2^2} \right) = max \left(\frac{\left\| \begin{pmatrix} \sigma_1 & a^t \\ 0 & B \end{pmatrix} x \right\|_2^2}{\left\| x \right\|_2^2} \right).$$

Let $x \longmapsto a$, yields

$$U^t A V = \begin{pmatrix} \sigma_1 & 0 \\ 0 & B \end{pmatrix} V^t.$$

In turn,

$$A = U \begin{pmatrix} \sigma_1 & 0 \\ 0 & B \end{pmatrix} V^t.$$

Let $S = \begin{pmatrix} I & 0 \\ 0 & \alpha \end{pmatrix}$, $\alpha > 0$ such that S^{-1} exists, then $G := S^{-1} A = \begin{pmatrix} A_{11} & A_{12} \\ \frac{1}{\alpha} A_{21} & \frac{1}{\alpha} A_{22} \end{pmatrix}$. The largest singular value of G is dependent on choice of α. The Schur complement lemma yields inequality for positive-definite matrix I and G as

$$\begin{pmatrix} I & G \\ G^t & I \end{pmatrix} > 0 \Leftrightarrow \left(I - G I^{-1} G^t \right) \geq 0.$$

The above inequality results $\lambda_i\left(I - GG^t\right) \geq 0,\ \forall i$. In turn this yields

$$0 \leq \lambda_i\left(I - GG^t\right) \geq 0,\ \forall i.$$

In other words,

$$\sigma_i\left(G\right) \geq 0,\ \forall i.$$

Finally, we conclude that for given $A \in \mathbf{R}^{n,n}$, a positive semi-definite matrix there exists a matrix G such that $\lambda_i\left(GA + A^tG\right) \geq 0,\ \forall i$.

Theorem 3 is about the computation of a positive definite matrix G for a given matrix $A \in \mho_K^+$ and to discuss the asymptotic stability of solutions of (a) and (b) in (P).

Theorem 3. *Let $K \subset \mathbf{R}^n$ be a non-empty closed and convex set and $0 \in K$. Then, solutions to (a) and (b) in (P) are asymptotically stable for given $A \in \mho_K^+$.*

Proof. To show that solutions to (a) and (b) in (P) are asymptotically stable for given $A \in \mho_K^+$, it is sufficient to show that there exists a positive-definite matrix G such that

$$\lambda_i\left(GA + A^tG\right) > 0,\ \forall i.$$

For given $A \in \mathbf{R}^{n,n}$, let $\lambda \in \Lambda(A)$ denotes the it's spectrum. Assume that $Re(\lambda_i) > 0,\ \forall i$. Furthermore,

$$\Lambda(A) = \{\alpha_j, (\lambda_j + \iota\mu_j)_j\},\ \alpha_j, \mu_j > 0,\ \forall i.$$

The given $A \in \mathbf{R}^{n,n}$ possesses the same spectrum as of a block diagonal matrix whose blocks have the forms:

(F_1) : The family of matrices $(\alpha_j I_m + J_m)$ with J_m being $m \times m$ nil-potent Jordan matrix.

(F_2) : With $M = \begin{pmatrix} \lambda_j & \mu_j \\ -\mu_j & \lambda_j \end{pmatrix}$, construct the block matrix $\begin{pmatrix} M & I_m & 0_m \\ 0_m & M & I_m \\ 0_m & 0_m & M \end{pmatrix}$. For (F_1) : Let $\epsilon > 0$, a small positive parameter so that $(\alpha_j I_m + J_m)$ possesses the same spectrum as $G = (\alpha_j I_m + \varepsilon J_m)$ possesses. Furthermore,

$$G + G^t = (2\alpha_j I_m + \epsilon(J_m + J_m^t))$$

with $(G + G^t) > 0$, being a positive definite matrix.

For (F_2): The given $A \in \mathbf{R}^{n,n}$ have the same spectrum as $G = \begin{pmatrix} M & \epsilon I_m & 0_m \\ 0_m & M & \epsilon I_m \\ 0_m & 0_m & M \end{pmatrix}$ and $G + G^t =$

$\begin{pmatrix} 2\lambda_j I_m & 0_m & 0_m \\ 0_m & 2\lambda_j I_m & 0_m \\ 0_m & 0_m & 2\lambda_j I_m \end{pmatrix} + \epsilon K$. Thus it can be seen that $(G + G^t) > 0$ for $\epsilon > 0$, the admissible perturbation level.

4.2. **Instability of Linear Evolution Variational Inequalities.**

For given $A \in \mathbf{R}^{n,n}$, we aim to determine G, a positive definite matrix such that

$$\big((G + G^t)A + \alpha G\big) < 0, \;\; \alpha > 0.$$

The existence of $G > 0$ is given in Theorem 2. We aim to determine an admissible perturbation matrix $(\epsilon P(t)), \;\epsilon > 0$, a small positive parameter, the perturbation level and $P(t)$ a matrix valued functions for all t. The matrix Frobenius norm of $P(t)$ satisfies

$$\Big\|P(t)\Big\|_F = \sqrt{\Sigma_{i,j} p_{i,j}^2} \leq 1, \;\; \forall t.$$

Furthermore, $diag(P(t)) = 0$. The computation of perturbation matrix $\epsilon P(t)$ shifts the negative spectrum of the matrix valued function $\big((G + G^t)A + \alpha G\big)$ such that

$$\lambda_i \left(((G + G^t)A + \alpha G) + \epsilon P(t)\right) > 0, \;\; \forall i.$$

The eigenvalue problem under consideration is

$$x^*(t) \left(((G + G^t)A + \alpha G) + \epsilon P(t)\right) x(t) = \lambda(t)x(t), \tag{15}$$

where $x(t)$ being an eigenvector corresponding to the smallest negative eigenvalue. Furthermore, assume that $\Big\|x(t)\Big\|_2 \leq 1, \;\forall t$. Differentiating Equ. 15 with respect to t yields

$$\lambda(t)x^*(t)\frac{d}{dt}(x(t)) + \epsilon x^*(t)\frac{d}{dt}(P(t)x(t)) = \frac{d}{dt}(\lambda(t)) + \lambda(t)x^*(t)\frac{d}{dt}(x(t)). \tag{16}$$

Upon making use of Equ. 15, we have that

$$\lambda(t)x^*(t) = x^*(t) \left(((G + G^t)A + \alpha G) + \epsilon P(t)\right).$$

Thus, Equ. 16 becomes

$$\frac{d}{dt}(\lambda(t)) = \epsilon x^*(t)\frac{d}{dt}(P(t))x(t). \tag{17}$$

Let $x^*(t)\frac{d}{dt}(x(t)) = 0$ in Equ.16 and $\frac{d}{dt}(P(t)) = \phi(t)$, then this yields an optimization problem to maximize the matrix values function $P(t)$.

4.2.1. *Optimization Problem.* For $\phi(t) = \frac{d}{dt}(P(t))$ construct and then maximize the quantity $(x_1^*(t)\phi(t)x_1(t))$ subject to constraints on $P(t)$, $\frac{d}{dt}(P(t))$ and $diag(\frac{d}{dt}(P(t)))$. The maximization problem in order to maximize the growth of smallest negative eigenvalues is:

$$(A) \longleftarrow \begin{cases} max\ \left(x_1^*(t)\frac{d}{dt}(P(t))x_1(t)\right) \\ Subject\ \ to \\ \langle \frac{d}{dt}(P(t)), P(t)\rangle = 0 \\ diag(\frac{d}{dt}(P(t))) = 0. \end{cases}$$

The solution to maximization problem (A) is given by Lemma 1.

Lemma 1. *Let $P(t),\ \forall t$ be a non-zero matrix valued function with $\left\|P(t)\right\|_F = \sqrt{\Sigma_{i,j}p_{i,j}^2} \leq 1$. Let $x_1(t),\ x_1^*(t)$ be right and left eigenvectors associated with smallest negative eigenvalue $\lambda_1(t)$ of the matrix valued function $\big((G+G^t)A + \alpha G\big) + \epsilon P(t)$. The solution to problem (A) is the differential equation*

$$\frac{d}{dt}(P(t)) = Proj_{P(t)}(x_1(t)x_1^*(t)) - \langle Proj_{P(t)}(x_1(t)x_1^*(t)), P(t)\rangle P(t),$$

where $Proj_{P(t)}(x_1(t)x_1^*(t))$ is the projection of matrices $(x_1(t)x_1^*(t))$ onto manifold induced by family of matrices $P(t)$.

Proof. The proof involve the computation of an orthogonal projection onto the manifold of $P(t)$. We refer [13] for a complete details on proof.

Remark 1. *The solution of (A), that is,*

$$\frac{d}{dt}(P(t)) = Proj_{P(t)}(x_1(t)x_1^*(t)) - \langle Proj_{P(t)}(x_1(t)x_1^*(t)), P(t)\rangle P(t)$$

ensures that

$$\frac{d}{dt}(\lambda_1(t)) > 0,\ \forall t.$$

The very next negative eigenvalue $\lambda_2(t)$ to be a part of the positive spectrum of matrix valued function $\big(((G+G^t)A + \alpha G) + \epsilon P(t)\big)$ while keeping $\lambda_1(t)$ remains positive. Furthermore, the maximization problem addressed to maximize both $\lambda_1(t)$ and $\lambda_2(t)$ such that $\lambda_1(t) > 0$ and $\lambda_2(t) > 0$ is given as follows:

$$(B) \longleftarrow \begin{cases} max\ \left(x_1^*(t)\frac{d}{dt}(P(t))x_1(t)\right) \\ Subject\ \ to \\ x_2^*(t)\frac{d}{dt}\left(P(t)\right)x_2(t) = x_1^*(t)\frac{d}{dt}\left(P(t)\right)x_1(t) \\ \langle \frac{d}{dt}(P(t)), P(t)\rangle = 0 \\ diag(\frac{d}{dt}(P(t))) = 0. \end{cases}$$

Remark 2. *The vectors $x_1^*(t),\ x_1(t),\ x_2^*(t),\ x_2(t)$ in (B) are the left and right eigenvectors to $\lambda_1(t)$ and $\lambda_2(t)$, respectively.*

For solution of optimization problem (B), we refer [13]. Finally, the eigenvalue problem

$$\left(((G+G^t)A+\alpha G)+\epsilon P(t)\right)x(t)=\lambda(t)x(t), \tag{18}$$

yields the positive spectrum of matrix valued function $\left((G+G^t)A+\alpha G\right)$ for an admissible perturbation level $\epsilon>0,\ P(t)$.

Remark 3. *The addition of $-\beta I$ for $\beta>0$ and I being an identity matrix to the matrix valued function $\left((G+G^t)A+\alpha G\right)$ yields a purely negative spectrum.*

5. Conclusion

In this article, the new results on existence, uniqueness, stability and instability analysis of Linear Evaluation Variational Inequalities LEVI are presented. For the existence and uniqueness, the results are provided on the solutions in a closed and convex set in the real Hilbert space satisfying the given LEVI. The construction of a positive semi-definite matrix is provided in order to discuss the stability and instability analysis of LEVI. Furthermore, the results on the asymptotic stability of LEVI are presented. Finally, an optimization problem is developed to study the behaviour of the spectrum of matrices corresponding to LEVI.

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Authors' Contributions

M.U.R and R.L. wrote the main manuscript text. M.U.R., and R.L. reviewed the manuscript. All authors have read and agreed to the published version of the manuscript.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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