Asia Pac. J. Math. 2025 **12**:10

EXPLORING QUADRATIC APPROXIMATIONS IN NONLINEAR SYSTEMS

RACHEL B. APDO[1], ROLLY N. APDO[2,*], ROLANDO N. PALUGA[1]

[1]Department of Mathematics, Caraga State University, Philippines

[2]Department of Secondary Education, Caraga State University, Philippines

[*]Corresponding author: rnapdo@carsu.edu.ph

Received Oct. 14, 2024

Abstract. Nonlinear ordinary differential equations (ODEs) play a critical role in modeling complex dynamical systems across various scientific fields. However, the difficulty in finding exact solutions often necessitates the use of approximation methods. This paper explores quadratic approximations, focusing on the inclusion of only the second-order terms from the Taylor series expansion. By using only the second-order terms, we aim to provide new insights about the solution of the system. We demonstrate the existence of nonzero real solutions for systems of ODEs involving quadratic terms and present comparisons between these quadratic approximations and exact solutions. The findings demonstrate that quadratic approximations effectively capture the dynamics of nonlinear systems, providing deeper insights into their behavior and may offer a reliable method for analyzing such systems.

2020 Mathematics Subject Classification. 34A34.

Key words and phrases. approximation; nonlinear ordinary differential equations; mathematical modeling.

1. Introduction

Nonlinear ordinary differential equations (ODEs) are important for understanding complex dynamical phenomena across various scientific disciplines. However, finding exact solutions for nonlinear ODEs presents a significant challenge, which is why we often use approximation methods to understand how these systems behave.

One such method involves the concept of linearization. Linearization is the process of converting a nonlinear system into a simple linear system. Some well-known methods for doing this include perturbation techniques ([1], [2]), substitution methods [3], changing variables ([4], [5]), and using the Jacobian matrix ([6], [7]). While these methods are useful, they mostly focus on linear terms, which might not fully capture the behavior of systems that are far from equilibrium. Among these techniques, the Jacobian matrix method is particularly well-known. However, it mainly considers the

DOI: 10.28924/APJM/12-10

first-order terms of a Taylor series expansion and ignores higher-order terms like quadratic terms. This has led researchers to explore more methods by considering the quadratic terms of the Taylor series expansion in the approximation. In the study of Bogoliubov et al., they have presented asymptotic methods for dealing with nonlinear oscillations, relying on quadratic terms for approximation. The study highlights how quadratic terms can help model systems where linear approaches fail to capture nonlinear behaviors [8]. Another research by Coullet and Spiegel derives amplitude equations using quadratic terms to describe the behavior of systems near critical points where multiple instabilities compete. The analysis focuses on the nonlinear interactions that arise in these systems, with the quadratic terms playing a crucial role in determining the dynamics [9]. The said studies emphasized the importance of quadratic approximations in understanding complex dynamical behaviors focusing on quadratic terms including the linear terms.

This paper focuses exclusively on utilizing quadratic terms, without the inclusion of linear terms, from the Taylor series expansion to solve ODEs. Through this approach, we aim to uncover new insights and deepen the understanding of the unique dynamics that arise in systems governed solely by quadratic nonlinearities.

The structure of this paper is organized as follows: Section 2 explores the existence of nonzero solutions for differential equations that involve quadratic terms. In Section 3, we approximate a system of nonlinear differential equations using the Taylor series expansion, specifically retaining only the quadratic terms, and compare the solutions of the original nonlinear system with those of the quadratic-only system. Finally, Section 4 concludes the paper with a summary of our findings and a discussion of their broader implications.

2. Existence of Nonzero Solutions to the Quadratic Approximations

This section aims to demonstrate the presence of nonzero real solutions in a system of differential equations involving quadratic terms. We establish the same because many real-world problems modeled by differential equations require nonzero solutions to be meaningful. For example, in epidemiology, nonzero solutions in a disease model could represent the persistence of an infection in a population, which is crucial for understanding and controlling disease spread.

The theorem establishes the existence of nonzero real solutions for a system of differential equations involving quadratic terms with n variables. The corollaries offer additional evidence of nonzero real solutions, specifically in the cases where $n = 2$ or $n = 3$. Examples are provided for each case to illustrate the concept further.

Theorem 2.1. *The system of differential equations involving quadratic terms*

$$\begin{cases} \dfrac{dx_1}{dt} &= \sum_{j=1}^{n} a_{1j}x_j^2 \\ \dfrac{dx_2}{dt} &= \sum_{j=1}^{n} a_{2j}x_j^2 \\ &\vdots \\ \dfrac{dx_n}{dt} &= \sum_{j=1}^{n} a_{nj}x_j^2 \end{cases}$$

has a nonzero real solution whenever the system of equations

$$\begin{cases} \sum_{j=1}^{n} a_{1j}y_j^2 + y_1 &= 0 \\ \sum_{j=1}^{n} a_{2j}y_j^2 + y_2 &= 0 \\ &\vdots \\ \sum_{j=1}^{n} a_{nj}y_j^2 + y_n &= 0 \end{cases}$$

has a nonzero real solution.

Proof. Consider the system of equations:

$$\begin{cases} \sum_{j=1}^{n} a_{1j}y_j^2 + y_1 &= 0 \\ \sum_{j=1}^{n} a_{2j}y_j^2 + y_2 &= 0 \\ &\vdots \\ \sum_{j=1}^{n} a_{nj}y_j^2 + y_n &= 0 \end{cases} \tag{2.1}$$

Let $(y_1, y_2, \ldots, y_n) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a nonzero real solution of the system (2.1). That is,

$$\begin{cases} \sum_{j=1}^{n} a_{1j}\lambda_j^2 + \lambda_1 &= 0 \\ \sum_{j=1}^{n} a_{2j}\lambda_j^2 + \lambda_2 &= 0 \\ &\vdots \\ \sum_{j=1}^{n} a_{nj}\lambda_j^2 + \lambda_n &= 0 \end{cases}$$

Equivalently,

$$\begin{cases} \sum_{j=1}^{n} a_{1j}\lambda_j^2 &= -\lambda_1 \\ \sum_{j=1}^{n} a_{2j}\lambda_j^2 &= -\lambda_2 \\ &\vdots \\ \sum_{j=1}^{n} a_{nj}\lambda_j^2 &= -\lambda_n \end{cases}$$

Let $(x_1, x_2, \ldots, x_n) = \left(\dfrac{\lambda_1}{t}, \dfrac{\lambda_2}{t}, \ldots, \dfrac{\lambda_n}{t}\right)$ where $t > 0$. Then

$$\begin{cases} \dfrac{dx_1}{dt} &= -\dfrac{\lambda_1}{t^2} \\ \dfrac{dx_2}{dt} &= -\dfrac{\lambda_2}{t^2} \\ &\vdots \\ \dfrac{dx_n}{dt} &= -\dfrac{\lambda_n}{t^2} \end{cases} \tag{2.2}$$

Moreover,

$$\sum_{j=1}^{n} a_{1j} x_j^2 = \sum_{j=1}^{n} a_{1j} \left(\frac{\lambda_j}{t}\right)^2 = \frac{1}{t^2} \sum_{j=1}^{n} a_{1j} (\lambda_j)^2 = -\frac{\lambda_1}{t^2},$$

$$\sum_{j=1}^{n} a_{2j} x_j^2 = \sum_{j=1}^{n} a_{2j} \left(\frac{\lambda_j}{t}\right)^2 = \frac{1}{t^2} \sum_{j=1}^{n} a_{2j} (\lambda_j)^2 = -\frac{\lambda_2}{t^2}$$

and

$$\sum_{j=1}^{n} a_{nj} x_j^2 = \sum_{j=1}^{n} a_{nj} \left(\frac{\lambda_j}{t}\right)^2 = \frac{1}{t^2} \sum_{j=1}^{n} a_{nj} (\lambda_j)^2 = -\frac{\lambda_n}{t^2}$$

Hence,

$$\begin{cases} \sum_{j=1}^{n} a_{1j} x_j^2 = -\frac{\lambda_1}{t^2} \\ \sum_{j=1}^{n} a_{2j} x_j^2 = -\frac{\lambda_2}{t^2} \\ \sum_{j=1}^{n} a_{3j} x_j^2 = -\frac{\lambda_3}{t^2} \\ \vdots \\ \sum_{j=1}^{n} a_{nj} x_j^2 = -\frac{\lambda_n}{t^2} \end{cases}$$

Using the system (2.2), we get

$$\begin{cases} \frac{dx_1}{dt} &= \sum_{j=1}^{n} a_{1j} x_j^2 \\ \frac{dx_2}{dt} &= \sum_{j=1}^{n} a_{2j} x_j^2 \\ &\vdots \\ \frac{dx_n}{dt} &= \sum_{j=1}^{n} a_{nj} x_j^2 \end{cases}$$

Thus, $(x_1, x_2, \dots, x_n) = \left(\frac{\lambda_1}{t}, \frac{\lambda_2}{t}, \dots, \frac{\lambda_n}{t}\right)$ is a nonzero real solution to the system of differential equations involving quadratic terms since $(y_1, y_2, \dots, y_n) = (\lambda_1, \lambda_2, \dots, \lambda_n)$ is nonzero. □

To illustrate the solution of a system of differential equations involving quadratic terms, we will examine the solutions for the cases of $n = 2$ and $n = 3$.

Corollary 2.1. *The system of differential equations involving quadratic terms*

$$\begin{cases} \frac{dx_1}{dt} &= a_{11} x_1^2 + a_{12} x_2^2 \\ \frac{dx_2}{dt} &= a_{21} x_1^2 + a_{22} x_2^2 \end{cases}$$

has a nonzero real solution. Moreover, a nonzero real solution is given for each of the four cases:
Case 1: If $a_{11} \neq 0$ and $a_{22} \neq 0$, then the nonzero real solution is

$$\begin{cases} x_1 = \frac{1}{t}\left(\omega_1 + \frac{4a_{22}^2 - 3a_{11}a_{12} - 3a_{22}^2}{9|A|^2 \omega_1} - \frac{2a_{22}}{|A|}\right) \\ x_2 = \frac{1}{t}\left(\omega_2 + \frac{4a_{11}^2 - 3a_{21}a_{22} - 3a_{11}^2}{9|A|^2 \omega_2} - \frac{2a_{11}}{|A|}\right) \end{cases}$$

where

$$\omega_1 = \left\{ \frac{(81a_{12}^2a_{22}^2|A|^4 + 96a_{12}a_{22}^4|A|^3 - 108a_{12}a_{22}^2(a_{11}a_{12} + a_{22}^2)|A|^3}{18|A|^4} \right.$$
$$+ \frac{12(a_{11}a_{12} + a_{22}^2)^3|A|^2 - 12a_{22}^2(a_{11}a_{12} + a_{22}^2)^2|A|^2)^{\frac{1}{2}}}{18|A|^4}$$
$$\left. + \frac{2a_{22}(a_{11}a_{12} + a_{22}^2) - 3a_{12}a_{22}|A|}{6|A|^3} - \frac{8a_{22}^3}{27|A|^3} \right\}^{\frac{1}{3}}$$

and

$$\omega_2 = \left\{ \frac{(81a_{11}^2a_{21}^2|A|^4 + 96a_{11}^4a_{21}|A|^3 - 108a_{11}^2a_{21}(a_{21}a_{22} + a_{11}^2)|A|^3)^{\frac{1}{2}}}{18|A|^4} \right.$$
$$+ \frac{12(a_{21}a_{22} + a_{11}^2)^3|A|^2 - 12a_{11}^2(a_{21}a_{22} + a_{11}^2)^2|A|^2}{18|A|^4}$$
$$\left. + \frac{2a_{11}(a_{21}a_{22} + a_{11}^2) - 3a_{11}a_{21}|A|}{6|A|^3} - \frac{8a_{11}^3}{27|A|^3} \right\}^{\frac{1}{3}}$$

Case 2: If $a_{11} = 0$ and $a_{22} = 0$, then the nonzero real solution is

$$\begin{cases} x_1 = -\dfrac{1}{t\sqrt[3]{a_{12}a_{21}^2}} \\ x_2 = -\dfrac{1}{t\sqrt[3]{a_{12}^2a_{21}}} \end{cases}$$

Case 3: If $a_{11} = 0$ and $a_{22} \neq 0$, then the nonzero real solution is

$$\begin{cases} x_1 &= -\dfrac{a_{12}}{t}\left(\omega_3 - \dfrac{a_{22}}{3a_{12}^2a_{21}\omega_3}\right)^2 \\ x_2 &= \dfrac{1}{t}\left(\omega_3 - \dfrac{a_{22}}{3a_{12}^2a_{21}\omega_3}\right) \end{cases}$$

where

$$\omega_3 = \left(\frac{1}{6a_{12}^2a_{21}}\sqrt{\frac{4a_{22}^3}{3a_{12}^2a_{21}} + 9} - \frac{1}{2a_{12}^2a_{21}} \right)^{\frac{1}{3}}$$

Case 4: If $a_{11} \neq 0$ and $a_{22} = 0$, then the nonzero real solution is

$$\begin{cases} x_1 &= \dfrac{1}{t}\left(\omega_4 - \dfrac{a_{11}}{3a_{12}a_{21}^2\omega_4}\right) \\ x_2 &= -\dfrac{a_{21}}{t}\left(\omega_4 - \dfrac{a_{11}}{3a_{12}a_{21}^2\omega_4}\right)^2 \end{cases}$$

where

$$\omega_4 = \left(\frac{1}{6a_{12}a_{21}^2}\sqrt{\frac{4a_{11}^3}{3a_{12}a_{21}^2} + 9} - \frac{1}{2a_{12}a_{21}^2} \right)^{\frac{1}{3}}$$

Proof. Consider the system of differential equations:

$$\begin{cases} \dfrac{dx_1}{dt} &= a_{11}x_1^2 + a_{12}x_2^2 \\ \dfrac{dx_2}{dt} &= a_{21}x_1^2 + a_{22}x_2^2 \end{cases} \tag{2.3}$$

By Theorem 2.1, the system (2.3) has a nonzero real solution whenever the system of equations

$$\begin{cases} a_{11}y_1^2 + a_{12}y_2^2 + y_1 &= 0 \\ a_{21}y_1^2 + a_{22}y_2^2 + y_2 &= 0 \end{cases} \tag{2.4}$$

has a nonzero real solution.

If there exists a nonzero real solution $(y_1, y_2) = (\lambda_1, \lambda_2)$ of the system (2.4), then $(x_1, x_2) = \left(\frac{\lambda_1}{t}, \frac{\lambda_2}{t}\right)$ where $t > 0$ is a nonzero real solution of the system (2.3).

Now, we form the following system of equations:

$$\begin{cases} a_{11}\lambda_1^2 + a_{12}\lambda_2^2 + \lambda_1 &= 0 \\ a_{21}\lambda_1^2 + a_{22}\lambda_2^2 + \lambda_2 &= 0 \end{cases}$$

We solve for λ_2 from the second equation of the system.

$$\lambda_2 = \frac{-1 \pm \sqrt{1 - 4a_{22}a_{21}\lambda_1^2}}{2a_{22}}$$

Substitute λ_2 to the first equation and we get

$$a_{11}\lambda_1^2 + a_{12}\left(\frac{-1 \pm \sqrt{1 - 4a_{22}a_{21}\lambda_1^2}}{2a_{22}}\right)^2 + \lambda_1 = 0$$

After some algebraic manipulations, we obtain

$$\left(\frac{a_{11}a_{22} - a_{12}a_{21}}{a_{22}}\right)^2 \lambda_1^4 + 2\left(\frac{a_{11}a_{22} - a_{12}a_{21}}{a_{22}}\right)\lambda_1^3 + \left(\frac{a_{11}a_{12} + a_{22}^2}{a_{22}^2}\right)\lambda_1^2 + \frac{a_{12}}{a_{22}}\lambda_1 = 0$$

On the other hand, if we first solve for λ_1 using the first equation in the system and then substitute it into the second equation, we can get

$$\left(\frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11}}\right)^2 \lambda_1^4 + 2\left(\frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11}}\right)\lambda_1^3 + \left(\frac{a_{21}a_{22} + a_{11}^2}{a_{11}^2}\right)\lambda_1^2 + \frac{a_{21}}{a_{11}}\lambda_1 = 0$$

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Then $|A| = a_{11}a_{22} - a_{12}a_{21}$. So we get

$$\frac{|A|^2}{a_{22}^2}\lambda_1^4 + \frac{2|A|}{a_{22}}\lambda_1^3 + \left(\frac{a_{11}a_{12} + a_{22}^2}{a_{22}^2}\right)\lambda_1^2 + \frac{a_{12}}{a_{22}}\lambda_1 = 0 \tag{2.5}$$

and

$$\frac{|A|^2}{a_{11}^2}\lambda_1^4 + \frac{2|A|}{a_{11}}\lambda_1^3 + \left(\frac{a_{21}a_{22} + a_{11}^2}{a_{11}^2}\right)\lambda_1^2 + \frac{a_{21}}{a_{11}}\lambda_1 = 0 \tag{2.6}$$

Consider the following cases:

Case 1: $a_{11} \neq 0$ and $a_{22} \neq 0$

Multiplying both sides of the equation (2.5) by a_{22}, we have

$$|A|^2 \lambda_1^4 + 2a_{22}|A|\lambda_1^3 + \left(a_{11}a_{12} + a_{22}^2\right)\lambda_1^2 + a_{12}a_{22}\lambda_1 = 0$$

Equivalently,

$$\lambda_1 = 0 \text{ or } |A|^2 \lambda_1^3 + 2a_{22}|A|\lambda_1^2 + \left(a_{11}a_{12} + a_{22}^2\right)\lambda_1 + a_{12}a_{22} = 0$$

Multiplying both sides of the equation (2.6) by a_{11}, we have

$$\lambda_2 = 0 \text{ or } |A|^2 \lambda_2^3 + 2a_{11}|A|\lambda_2^2 + \left(a_{21}a_{22} + a_{11}^2\right)\lambda_2 + a_{11}a_{21} = 0$$

We solve for λ_1 and λ_2 and substitute to $x_1 = \dfrac{\lambda_1}{t}$ and $x_2 = \dfrac{\lambda_2}{t}$.
The nonzero real solution of the system of differential equation is:

$$\begin{cases} x_1 = \dfrac{1}{t}\left(\omega_1 + \dfrac{4a_{22}^2 - 3a_{11}a_{12} - 3a_{22}^2}{9|A|^2\omega_1} - \dfrac{2a_{22}}{|A|}\right) \\ x_2 = \dfrac{1}{t}\left(\omega_2 + \dfrac{4a_{11}^2 - 3a_{21}a_{22} - 3a_{11}^2}{9|A|^2\omega_2} - \dfrac{2a_{11}}{|A|}\right) \end{cases}$$

where

$$\begin{aligned} \omega_1 = \Bigg\{ & \frac{(81a_{12}^2a_{22}^2|A|^4 + 96a_{12}a_{22}^4|A|^3 - 108a_{12}a_{22}^2(a_{11}a_{12} + a_{22}^2)|A|^3}{18|A|^4} \\ & + \frac{12(a_{11}a_{12} + a_{22}^2)^3|A|^2 - 12a_{22}^2(a_{11}a_{12} + a_{22}^2)^2|A|^2)^{\frac{1}{2}}}{18|A|^4} \\ & + \frac{2a_{22}(a_{11}a_{12} + a_{22}^2) - 3a_{12}a_{22}|A|}{6|A|^3} - \frac{8a_{22}^3}{27|A|^3} \Bigg\}^{\frac{1}{3}} \end{aligned}$$

and

$$\begin{aligned} \omega_2 = \Bigg\{ & \frac{(81a_{11}^2a_{21}^2|A|^4 + 96a_{11}^4a_{21}|A|^3 - 108a_{11}^2a_{21}(a_{21}a_{22} + a_{11}^2)|A|^3)^{\frac{1}{2}}}{18|A|^4} \\ & + \frac{12(a_{21}a_{22} + a_{11}^2)^3|A|^2 - 12a_{11}^2(a_{21}a_{22} + a_{11}^2)^2|A|^2}{18|A|^4} \\ & + \frac{2a_{11}(a_{21}a_{22} + a_{11}^2) - 3a_{11}a_{21}|A|}{6|A|^3} - \frac{8a_{11}^3}{27|A|^3} \Bigg\}^{\frac{1}{3}} \end{aligned}$$

Case 2: $a_{11} = 0$ and $a_{22} = 0$

The following system

$$\begin{cases} a_{11}\lambda_1^2 + a_{12}\lambda_2^2 + \lambda_1 & = & 0 \\ a_{21}\lambda_1^2 + a_{22}\lambda_2^2 + \lambda_2 & = & 0 \end{cases}$$

becomes

$$\begin{cases} a_{12}\lambda_2^2 + \lambda_1 &= 0 \\ a_{21}\lambda_1^2 + \lambda_2 &= 0 \end{cases}$$

We solve for λ_1 and λ_2 and substitute to $x_1 = \dfrac{\lambda_1}{t}$ and $x_2 = \dfrac{\lambda_2}{t}$.

The nonzero real solution of the system of differential equations is:

$$\begin{cases} x_1 = -\dfrac{1}{t\sqrt[3]{a_{12}a_{21}^2}} \\ x_2 = -\dfrac{1}{t\sqrt[3]{a_{12}^2a_{21}}} \end{cases}$$

Case 3: $a_{11} = 0$ and $a_{22} \neq 0$

The following system

$$\begin{cases} a_{11}\lambda_1^2 + a_{12}\lambda_2^2 + \lambda_1 &= 0 \\ a_{21}\lambda_1^2 + a_{22}\lambda_2^2 + \lambda_2 &= 0 \end{cases}$$

becomes

$$\begin{cases} a_{12}\lambda_2^2 + \lambda_1 &= 0 \\ a_{21}\lambda_1^2 + a_{22}\lambda_2^2 + \lambda_2 &= 0 \end{cases}$$

We solve for λ_1 from the first equation.

$$\lambda_1 = -a_{12}\lambda_2^2$$

Substitute this to the second equation.

$$a_{12}^2a_{21}\lambda_2^4 + a_{22}\lambda_2^2 + \lambda_2 = 0$$

or

$$\lambda_2 = 0 \text{ or } a_{12}^2a_{21}\lambda_2^3 + a_{22}\lambda_2 + 1 = 0$$

We solve for λ_1 and λ_2 and substitute to $x_1 = \dfrac{\lambda_1}{t}$ and $x_2 = \dfrac{\lambda_2}{t}$.

The nonzero real solution of the system of differential equations is:

$$\begin{cases} x_1 &= -\dfrac{a_{12}}{t}\left(\omega_3 - \dfrac{a_{22}}{3a_{12}^2a_{21}\omega_3}\right)^2 \\ x_2 &= \dfrac{1}{t}\left(\omega_3 - \dfrac{a_{22}}{3a_{12}^2a_{21}\omega_3}\right) \end{cases}$$

where

$$\omega_3 = \left(\frac{1}{6a_{12}^2a_{21}}\sqrt{\frac{4a_{22}^3}{3a_{12}^2a_{21}} + 9} - \frac{1}{2a_{12}^2a_{21}}\right)^{\frac{1}{3}}$$

Case 4: $a_{11} \neq 0$ and $a_{22} = 0$

The following system

$$\begin{cases} a_{11}\lambda_1^2 + a_{12}\lambda_2^2 + \lambda_1 &= 0 \\ a_{21}\lambda_1^2 + a_{22}\lambda_2^2 + \lambda_2 &= 0 \end{cases}$$

becomes

$$\begin{cases} a_{11}\lambda_1^2 + a_{12}\lambda_2^2 + \lambda_1 &= 0 \\ a_{21}\lambda_1^2 + \lambda_2 &= 0 \end{cases}$$

We solve for λ_2 from the second equation.

$$\lambda_2 = -a_{21}\lambda_1^2$$

Substitute this to the second equation.

$$a_{12}a_{21}^2\lambda_1^4 + a_{11}\lambda_1^2 + \lambda_1 = 0$$

or

$$\lambda_1 = 0 \text{ or } a_{12}a_{21}^2\lambda_1^3 + a_{11}\lambda_1 + 1 = 0$$

We solve for λ_1 and λ_2 and substitute to $x_1 = \dfrac{\lambda_1}{t}$ and $x_2 = \dfrac{\lambda_2}{t}$.

The nonzero real solution of the system of differential equations is:

$$\begin{cases} x_1 &= \dfrac{1}{t}\left(\omega_4 - \dfrac{a_{11}}{3a_{12}a_{21}^2\omega_4}\right) \\ x_2 &= -\dfrac{a_{21}}{t}\left(\omega_4 - \dfrac{a_{11}}{3a_{12}a_{21}^2\omega_4}\right)^2 \end{cases}$$

where

$$\omega_4 = \left(\frac{1}{6a_{12}a_{21}^2}\sqrt{\frac{4a_{11}^3}{3a_{12}a_{21}^2} + 9} - \frac{1}{2a_{12}a_{21}^2}\right)^{\frac{1}{3}}$$

This completes the proof. □

Remark 2.1. *The rest of the solutions of system of differential equations*

$$\begin{cases} \dfrac{dx_1}{dt} &= a_{11}x_1^2 + a_{12}x_2^2 \\ \dfrac{dx_2}{dt} &= a_{21}x_1^2 + a_{22}x_2^2 \end{cases}$$

are given for each case:

Case 1: If $a_{11} \neq 0$ and $a_{22} \neq 0$, then the zero and complex solutions are

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases},$$

$$\begin{cases} x_1 = \dfrac{1}{t}\left[\dfrac{\left(-1+\sqrt{3}i\right)\omega_1}{2} - \dfrac{\left(1+\sqrt{3}i\right)\left(4a_{22}^2 - 3a_{11}a_{12} - 3a_{22}^2\right)}{18|A|^2\omega_1} - \dfrac{2a_{22}}{|A|}\right] \\ x_2 = \dfrac{1}{t}\left[\dfrac{\left(-1-\sqrt{3}i\right)\omega_2}{2} - \dfrac{\left(1-\sqrt{3}i\right)\left(4a_{11}^2 - 3a_{21}a_{22} - 3a_{11}^2\right)}{18|A|^2\omega_2} - \dfrac{2a_{11}}{|A|}\right] \end{cases}$$

and

$$\begin{cases} x_1 = \dfrac{1}{t}\left[\dfrac{\left(-1-\sqrt{3}i\right)\omega_1}{2} - \dfrac{\left(1-\sqrt{3}i\right)\left(4a_{22}^2-3a_{11}a_{12}-3a_{22}^2\right)}{18|A|^2\omega_1} - \dfrac{2a_{22}}{|A|}\right] \\ x_2 = \dfrac{1}{t}\left[\dfrac{\left(-1+\sqrt{3}i\right)\omega_2}{2} - \dfrac{\left(1+\sqrt{3}i\right)\left(4a_{11}^2-3a_{21}a_{22}-3a_{11}^2\right)}{18|A|^2\omega_2} - \dfrac{2a_{11}}{|A|}\right] \end{cases}$$

where

$$\omega_1 = \left\{\frac{(81a_{12}^2a_{22}^2|A|^4 + 96a_{12}a_{22}^4|A|^3 - 108a_{12}a_{22}^2(a_{11}a_{12}+a_{22}^2)|A|^3}{18|A|^4}\right.$$
$$+\frac{12(a_{11}a_{12}+a_{22}^2)^3|A|^2 - 12a_{22}^2(a_{11}a_{12}+a_{22}^2)^2|A|^2)^{\frac{1}{2}}}{18|A|^4}$$
$$\left.+\frac{2a_{22}(a_{11}a_{12}+a_{22}^2)-3a_{12}a_{22}|A|}{6|A|^3} - \frac{8a_{22}^3}{27|A|^3}\right\}^{\frac{1}{3}}$$

and

$$\omega_2 = \left\{\frac{(81a_{11}^2a_{21}^2|A|^4 + 96a_{11}^4a_{21}|A|^3 - 108a_{11}^2a_{21}(a_{21}a_{22}+a_{11}^2)|A|^3)^{\frac{1}{2}}}{18|A|^4}\right.$$
$$+\frac{12(a_{21}a_{22}+a_{11}^2)^3|A|^2 - 12a_{11}^2(a_{21}a_{22}+a_{11}^2)^2|A|^2}{18|A|^4}$$
$$\left.+\frac{2a_{11}(a_{21}a_{22}+a_{11}^2)-3a_{11}a_{21}|A|}{6|A|^3} - \frac{8a_{11}^3}{27|A|^3}\right\}^{\frac{1}{3}}$$

Case 2: If $a_{11}=0$ and $a_{22}=0$, then the zero and complex solutions are

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}, \begin{cases} x_1 = \dfrac{1+\sqrt{3}i}{2t\sqrt[3]{a_{12}a_{21}^2}} \\ x_2 = \dfrac{1-\sqrt{3}i}{2t\sqrt[3]{a_{12}^2a_{21}}} \end{cases} \text{ and } \begin{cases} x_1 = \dfrac{1-\sqrt{3}i}{2t\sqrt[3]{a_{12}a_{21}^2}} \\ x_2 = \dfrac{1+\sqrt{3}i}{2t\sqrt[3]{a_{12}^2a_{21}}} \end{cases}$$

Case 3: If $a_{11}=0$ and $a_{22}\neq 0$, then the zero and complex solutions are

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}, \begin{cases} x_1 = -\dfrac{a_{12}}{t}\left(\dfrac{\left(-1+\sqrt{3}i\right)\omega_3}{2} + \dfrac{\left(1+\sqrt{3}i\right)a_{22}}{6a_{12}^2a_{21}\omega_3}\right)^2 \\ x_2 = \dfrac{\left(-1+\sqrt{3}i\right)\omega_3}{2} + \dfrac{\left(1+\sqrt{3}i\right)a_{22}}{6a_{12}^2a_{21}\omega_3} \end{cases}$$

and

$$\begin{cases} x_1 = -\dfrac{a_{12}}{t}\left(\dfrac{\left(-1-\sqrt{3}i\right)\omega_3}{2} + \dfrac{\left(1-\sqrt{3}i\right)a_{22}}{6a_{12}^2a_{21}\omega_3}\right)^2 \\ x_2 = \dfrac{\left(-1-\sqrt{3}i\right)\omega_3}{2} + \dfrac{\left(1-\sqrt{3}i\right)a_{22}}{6a_{12}^2a_{21}\omega_3} \end{cases}$$

where

$$\omega_3 = \left(\frac{1}{6a_{12}^2a_{21}}\sqrt{\frac{4a_{22}^3}{3a_{12}^2a_{21}}+9} - \frac{1}{2a_{12}^2a_{21}}\right)^{\frac{1}{3}}$$

Case 4: If $a_{11} \neq 0$ and $a_{22} = 0$, then the zero and complex solutions are

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}, \begin{cases} x_1 &= \dfrac{\left(-1+\sqrt{3}i\right)\omega_4}{2} + \dfrac{\left(1+\sqrt{3}i\right)a_{11}}{6a_{12}a_{21}^2\omega_4} \\ x_2 &= -\dfrac{a_{21}}{t}\left(\dfrac{\left(-1+\sqrt{3}i\right)\omega_4}{2} + \dfrac{\left(1+\sqrt{3}i\right)a_{11}}{6a_{12}a_{21}^2\omega_4}\right)^2 \end{cases}$$

and

$$\begin{cases} x_1 &= \dfrac{\left(-1-\sqrt{3}i\right)\omega_4}{2} + \dfrac{\left(1-\sqrt{3}i\right)a_{11}}{6a_{12}a_{21}^2\omega_4} \\ x_2 &= -\dfrac{a_{21}}{t}\left(\dfrac{\left(-1-\sqrt{3}i\right)\omega_4}{2} + \dfrac{\left(1-\sqrt{3}i\right)a_{11}}{6a_{12}a_{21}^2\omega_4}\right)^2 \end{cases}$$

where

$$\omega_4 = \left(\frac{1}{6a_{12}a_{21}^2}\sqrt{\frac{4a_{11}^3}{3a_{12}a_{21}^2}+9} - \frac{1}{2a_{12}a_{21}^2}\right)^{\frac{1}{3}}$$

Example 2.1. *Find the nonzero real solutions of the following system of differential equations*

$$\begin{cases} \dfrac{dx}{dt} &= -(x-1)^2 + \dfrac{3}{2}(y-1)^2 \\ \dfrac{dy}{dt} &= -\dfrac{1}{3}(x-1)^2 + (y-1)^2 \end{cases} \tag{2.7}$$

Solution:

Let $x_1 = x - 1$ and $x_2 = y - 1$. Then $\dfrac{dx_1}{dt} = \dfrac{dx}{dt}$ and $\dfrac{dx_2}{dt} = \dfrac{dy}{dt}$.

Now,

$$\begin{cases} \dfrac{dx_1}{dt} &= -x_1^2 + \dfrac{3}{2}x_2^2 \\ \dfrac{dx_2}{dt} &= -\dfrac{1}{3}x_1^2 + x_2^2 \end{cases}$$

We have $a_{11} = -1, a_{12} = \dfrac{3}{2}, a_{21} = -\dfrac{1}{3}$ and $a_{22} = 1$. Since $a_{11} \neq 0$ and $a_{22} \neq 0$, by Corollary 2.1, the nonzero real solution is of the form:

$$\begin{cases} x_1 = \dfrac{1}{t}\left(\omega_1 + \dfrac{4a_{22}^2 - 3a_{11}a_{12} - 3a_{22}^2}{9|A|^2\omega_1} - \dfrac{2a_{22}}{|A|}\right) \\ x_2 = \dfrac{1}{t}\left(\omega_2 + \dfrac{4a_{11}^2 - 3a_{21}a_{22} - 3a_{11}^2}{9|A|^2\omega_2} - \dfrac{2a_{11}}{|A|}\right) \end{cases}$$

where

$$\begin{aligned} \omega_1 = \Bigg\{ & \frac{(81a_{12}^2a_{22}^2|A|^4 + 96a_{12}a_{22}^4|A|^3 - 108a_{12}a_{22}^2(a_{11}a_{12}+a_{22}^2)|A|^3}{18|A|^4} \\ & + \frac{12(a_{11}a_{12}+a_{22}^2)^3|A|^2 - 12a_{22}^2(a_{11}a_{12}+a_{22}^2)^2|A|^2)^{\frac{1}{2}}}{18|A|^4} \\ & + \frac{2a_{22}(a_{11}a_{12}+a_{22}^2) - 3a_{12}a_{22}|A|}{6|A|^3} - \frac{8a_{22}^3}{27|A|^3} \Bigg\}^{\frac{1}{3}} \end{aligned}$$

and

$$\omega_2 = \left\{ \frac{\left(81a_{11}^2a_{21}^2|A|^4 + 96a_{11}^4a_{21}|A|^3 - 108a_{11}^2a_{21}(a_{21}a_{22} + a_{11}^2)|A|^3\right)^{\frac{1}{2}}}{18|A|^4} + \frac{12(a_{21}a_{22} + a_{11}^2)^3|A|^2 - 12a_{11}^2(a_{21}a_{22} + a_{11}^2)^2|A|^2}{18|A|^4} + \frac{2a_{11}(a_{21}a_{22} + a_{11}^2) - 3a_{11}a_{21}|A|}{6|A|^3} - \frac{8a_{11}^3}{27|A|^3} \right\}^{\frac{1}{3}}$$

Substituting the values, we get the nonzero real solutions rounded to five decimal places:

$$\begin{cases} x_1 = \dfrac{4.13263}{t} \\ x_2 = -\dfrac{4.58710}{t} \end{cases}, \begin{cases} x_1 = \dfrac{1.14043}{t} \\ x_2 = -\dfrac{4.24298}{t} \end{cases}, \begin{cases} x_1 = -\dfrac{1.27307}{t} \\ x_2 = \dfrac{0.66478}{t} \end{cases},$$

$$\begin{cases} x_1 = -\dfrac{1.27307}{t} \\ x_2 = -\dfrac{0.80301}{t} \end{cases} \text{ and } \begin{cases} x_1 = -\dfrac{1.27307}{t} \\ x_2 = -\dfrac{3.86176}{t} \end{cases}$$

Since $x_1 = x - 1$ and $x_2 = y - 1$, we obtain the nonzero real solutions of the system (3.5):

$$\begin{cases} x = \dfrac{4.13263 + t}{t} \\ y = \dfrac{t - 4.58710}{t} \end{cases}, \begin{cases} x = \dfrac{1.14043 + t}{t} \\ y = \dfrac{t - 4.24298}{t} \end{cases}, \begin{cases} x = \dfrac{t - 1.27307}{t} \\ y = \dfrac{0.66478 + t}{t} \end{cases},$$

$$\begin{cases} x = \dfrac{t - 1.27307}{t} \\ y = \dfrac{t - 0.80301}{t} \end{cases} \text{ and } \begin{cases} x = \dfrac{t - 1.27307}{t} \\ y = \dfrac{t - 3.86176}{t} \end{cases}$$

Corollary 2.2. *The system of differential equations involving quadratic terms*

$$\begin{cases} \dfrac{dx_1}{dt} = \sum_{j=1}^{3} a_{1j}x_j^2 \\ \dfrac{dx_2}{dt} = \sum_{j=1}^{3} a_{2j}x_j^2 \\ \dfrac{dx_3}{dt} = \sum_{j=1}^{3} a_{3j}x_j^2 \end{cases}$$

has a nonzero real solution. Moreover, the nonzero real solution is

$$(x_1, x_2, x_3) = \left(\frac{\lambda_1}{t}, \frac{\lambda_2}{t}, \frac{\lambda_3}{t}\right)$$

where λ_1, λ_2 and λ_3 are nonzero real roots of $P_1(y_1)$, $P_2(y_2)$ and $P_3(y_3)$, respectively such that

$$P_1(y_1) = \pi_{1,0} + \pi_{1,1}y_1 + \pi_{1,2}y_1^2 + \pi_{1,3}y_1^3 + \pi_{1,4}y_1^4 + \pi_{1,5}y_1^5 + \pi_{1,6}y_1^6 + \pi_{1,7}y_1^7,$$

$$P_2(y_2) = \pi_{2,0} + \pi_{2,1}y_2 + \pi_{2,2}y_2^2 + \pi_{2,3}y_2^3 + \pi_{2,4}y_2^4 + \pi_{2,5}y_2^5 + \pi_{2,6}y_2^6 + \pi_{2,7}y_2^7$$

and

$$P_3(y_3) = \pi_{3,0} + \pi_{3,1}y_3 + \pi_{3,2}y_3^2 + \pi_{3,3}y_3^3 + \pi_{3,4}y_3^4 + \pi_{3,5}y_3^5 + \pi_{3,6}y_3^6 + \pi_{3,7}y_3^7$$

where

$$\pi_{1,0} = \delta_{1,0}^2 \left(a_{12}a_{13}\Delta(1,1)\delta_{1,1} - a_{12}a_{13}\delta_{1,3} + a_{32}\delta_{1,0}\delta_{1,7}\right)$$

$$\pi_{1,1} = \delta_{1,0} \big(a_{12}a_{13}\delta_{1,0}\delta_{1,1}|A| + a_{12}a_{13}\delta_{1,0}\delta_{1,2} + \Delta(1,1)^2\delta_{1,4} + \delta_{1,3}^2 - 2\Delta(1,1)\delta_{1,1}\delta_{1,3} - \delta_{1,0}^2\delta_{1,6} \big)$$

$$\pi_{1,2} = 2\delta_{1,0} \big(\Delta(1,1)|A|\delta_{1,4} - \delta_{1,3}(\delta_{1,1}|A| + \delta_{1,2}) + \Delta(1,1)\delta_{1,1}(\delta_{1,2} + \Delta(1,1)^2\delta_{1,0}) - \Delta(1,1)^2\delta_{1,0}\delta_{1,3} - 2\delta_{1,0}^3\delta_{1,5}\big)$$

$$\pi_{1,3} = \delta_{1,0}(\delta_{1,4}|A|^2 + \delta_{1,2}(2|A|\delta_{1,1} + \delta_{1,3} + 2\Delta(1,1)^2\delta_{1,0}) + 6\Delta(1,1)^2\delta_{1,0}\delta_{1,1}|A| - 4\Delta(1,1)\delta_{1,0}\delta_{1,3}|A| + \Delta(1,1)^4\delta_{1,0}^2 + 4\Delta(3,2)\Delta(2,3)\delta_{1,0}^3)$$

$$\pi_{1,4} = 2\delta_{1,0}^2|A|^2 \big((3\Delta(1,1)\delta_{1,1}\delta_{1,3})|A| + 2\Delta(1,1)\delta_{1,2} + 2\Delta(1,1)^3\delta_{1,0}\big)$$

$$\pi_{1,5} = 2\delta_{1,0}^2|A|^2 \left(\delta_{1,1}|A| + \delta_{1,2} + 3\Delta(1,1)^2\delta_{1,0}\right)$$

$$\pi_{1,6} = 4\Delta(1,1)\delta_{1,0}^3|A|^3$$

$$\pi_{1,7} = \delta_{1,0}^3|A|^4$$

$$\pi_{2,0} = \delta_{2,0}^2 \left(a_{21}a_{23}\Delta(2,2)\delta_{2,1} - a_{21}a_{23}\delta_{2,3} + a_{31}\delta_{2,0}\delta_{2,7}\right)$$

$$\pi_{2,1} = \delta_{2,0} \big(a_{21}a_{23}\delta_{2,0}\delta_{2,1}|A| + a_{21}a_{23}\delta_{2,0}\delta_{2,2} + \Delta(2,2)^2\delta_{2,4} + \delta_{2,3}^2 - 2\Delta(2,2)\delta_{2,1}\delta_{2,3} - \delta_{2,0}^2\delta_{2,6} \big)$$

$$\pi_{2,2} = 2\delta_{2,0} \big(\Delta(2,2)|A|\delta_{2,4} - \delta_{2,3}(\delta_{2,1}|A| + \delta_{2,2}) + \Delta(2,2)\delta_{2,1}(\delta_{2,2} + \Delta(2,2)^2\delta_{2,0}) - \Delta(2,2)^2\delta_{2,0}\delta_{2,3} - 2\delta_{2,0}^3\delta_{2,5}\big)$$

$$\pi_{2,3} = \delta_{2,0}(\delta_{2,4}|A|^2 + \delta_{2,2}(2|A|\delta_{2,1} + \delta_{2,3} + 2\Delta(2,2)^2\delta_{2,0}) + 6\Delta(2,2)^2\delta_{2,0}\delta_{2,1}|A| - 4\Delta(2,2)\delta_{2,0}\delta_{2,3}|A| + \Delta(2,2)^4\delta_{2,0}^2 + 4\Delta(3,1)\Delta(1,3)\delta_{2,0}^3)$$

$$\pi_{2,4} = 2\delta_{2,0}^2|A|^2 \big((3\Delta(2,2)\delta_{2,1}\delta_{2,3})|A| + 2\Delta(2,2)\delta_{2,2} + 2\Delta(2,2)^3\delta_{2,0}\big)$$

$$\pi_{2,5} = 2\delta_{2,0}^2|A|^2 \left(\delta_{2,1}|A| + \delta_{2,2} + 3\Delta(2,2)^2\delta_{2,0}\right)$$

$$\pi_{2,6} = 4\Delta(2,2)\delta_{2,0}^3|A|^3$$

$$\pi_{2,7} = \delta_{2,0}^3|A|^4$$

$$\pi_{3,0} = \delta_{3,0}^2 \left(a_{32}a_{31}\Delta(3,3)\delta_{3,1} - a_{32}a_{31}\delta_{3,3} + a_{12}\delta_{3,0}\delta_{3,7}\right)$$

$$\pi_{3,1} = \delta_{3,0} \big(a_{32}a_{31}\delta_{3,0}\delta_{3,1}|A| + a_{32}a_{31}\delta_{3,0}\delta_{3,2} + \Delta(3,3)^2\delta_{3,4} + \delta_{3,3}^2 - 2\Delta(3,3)\delta_{3,1}\delta_{3,3} - \delta_{3,0}^2\delta_{3,6}\big)$$

$$\pi_{3,2} = 2\delta_{3,0}\left(\Delta(3,3)|A|\delta_{3,4} - \delta_{3,3}(\delta_{3,1}|A| + \delta_{3,2})\right.$$
$$\left. + \Delta(3,3)\delta_{3,1}(\delta_{3,2} + \Delta(3,3)^2\delta_{3,0}) - \Delta(3,3)^2\delta_{3,0}\delta_{3,3} - 2\delta_{3,0}^3\delta_{3,5}\right)$$

$$\pi_{3,3} = \delta_{3,0}(\delta_{3,4}|A|^2 + \delta_{3,2}(2|A|\delta_{3,1} + \delta_{3,3} + 2\Delta(3,3)^2\delta_{3,0})$$
$$+ 6\Delta(3,3)^2\delta_{3,0}\delta_{3,1}|A| - 4\Delta(3,3)\delta_{3,0}\delta_{3,3}|A| + \Delta(3,3)^4\delta_{3,0}^2$$
$$+ 4\Delta(1,2)\Delta(2,1)\delta_{3,0}^3)$$

$$\pi_{3,4} = 2\delta_{3,0}^2|A|^2\left((3\Delta(3,3)\delta_{3,1}\delta_{3,3})|A| + 2\Delta(3,3)\delta_{3,2} + 2\Delta(3,3)^3\delta_{3,0}\right)$$
$$\pi_{3,5} = 2\delta_{3,0}^2|A|^2\left(\delta_{3,1}|A| + \delta_{3,2} + 3\Delta(3,3)^2\delta_{3,0}\right)$$
$$\pi_{3,6} = 4\Delta(3,3)\delta_{3,0}^3|A|^3$$
$$\pi_{3,7} = \delta_{3,0}^3|A|^4$$

and

$$\delta_{1,0} = \Delta(2,1)\Delta(3,1)$$
$$\delta_{1,1} = a_{13}\Delta(2,1)^2 + a_{12}\Delta(3,1)^2$$
$$\delta_{1,2} = \Delta(3,2)\Delta(2,1)^3 - \Delta(2,3)\Delta(3,1)^3$$
$$\delta_{1,3} = a_{23}\Delta(2,1)^3 - a_{32}\Delta(3,1)^3$$
$$\delta_{1,4} = a_{13}^2\Delta(2,1)^4 + a_{12}^2\Delta(3,1)^4 + 3a_{12}a_{13}\Delta(2,1)^2\Delta(3,1)^2$$
$$\delta_{1,5} = a_{23}\Delta(2,3) + a_{32}\Delta(3,2)$$
$$\delta_{1,6} = a_{13}^2\Delta(2,1)\Delta(2,3) - a_{12}^2\Delta(3,1)\Delta(3,2) - 4a_{23}a_{32}\Delta(2,1)\Delta(3,1)$$
$$\delta_{1,7} = a_{13}^2\Delta(2,1) - a_{12}^2\Delta(3,1)$$
$$\delta_{2,0} = \Delta(1,2)\Delta(3,2)$$
$$\delta_{2,1} = a_{23}\Delta(1,2)^2 + a_{21}\Delta(3,2)^2$$
$$\delta_{2,2} = \Delta(3,1)\Delta(1,2)^3 - \Delta(1,3)\Delta(3,2)^3$$
$$\delta_{2,3} = a_{13}\Delta(1,2)^3 - a_{31}\Delta(3,2)^3$$
$$\delta_{2,4} = a_{23}^2\Delta(1,2)^4 + a_{21}^2\Delta(3,2)^4 + 3a_{21}a_{23}\Delta(1,2)^2\Delta(3,2)^2$$
$$\delta_{2,5} = a_{13}\Delta(1,3) + a_{31}\Delta(3,1)$$
$$\delta_{2,6} = a_{23}^2\Delta(1,2)\Delta(1,3) - a_{21}^2\Delta(3,2)\Delta(3,1) - 4a_{13}a_{31}\Delta(1,2)\Delta(3,2)$$
$$\delta_{2,7} = a_{23}^2\Delta(1,2) - a_{21}^2\Delta(3,2)$$
$$\delta_{3,0} = \Delta(2,3)\Delta(1,3)$$
$$\delta_{3,1} = a_{13}\Delta(2,3)^2 + a_{32}\Delta(1,3)^2$$
$$\delta_{3,2} = \Delta(1,2)\Delta(2,3)^3 - \Delta(2,1)\Delta(1,3)^3$$
$$\delta_{3,3} = a_{21}\Delta(2,3)^3 - a_{12}\Delta(1,3)^3$$
$$\delta_{3,4} = a_{31}^2\Delta(2,3)^4 + a_{32}^2\Delta(1,3)^4 + 3a_{32}a_{31}\Delta(2,3)^2\Delta(1,3)^2$$
$$\delta_{3,5} = a_{21}\Delta(2,1) + a_{12}\Delta(1,2)$$
$$\delta_{3,6} = a_{31}^2\Delta(2,3)\Delta(2,1) - a_{32}^2\Delta(1,3)\Delta(1,2) - 4a_{21}a_{12}\Delta(2,3)\Delta(1,3)$$

$\delta_{3,7} = a_{31}^2\Delta(2,3) - a_{32}^2\Delta(1,3)$

Proof. Consider the system of differential equations:

$$\begin{cases} \frac{dx_1}{dt} &= \sum_{j=1}^{3} a_{1j}x_j^2 \\ \frac{dx_2}{dt} &= \sum_{j=1}^{3} a_{2j}x_j^2 \\ \frac{dx_3}{dt} &= \sum_{j=1}^{3} a_{3j}x_j^2 \end{cases} \tag{2.8}$$

By Theorem 2.1, the system (2.8) has a nonzero real solution whenever the system of equations

$$\begin{cases} \sum_{j=1}^{3} a_{1j}y_j^2 + y_1 &= 0 \\ \sum_{j=1}^{3} a_{2j}y_j^2 + y_2 &= 0 \\ \sum_{j=1}^{3} a_{3j}y_j^2 + y_3 &= 0 \end{cases} \tag{2.9}$$

has a nonzero real solution.

If there exists a nonzero real solution $(y_1, y_2, y_3) = (\lambda_1, \lambda_2, \lambda_3)$ of the system (2.9), then $(x_1, x_2, x_3) = \left(\frac{\lambda_1}{t}, \frac{\lambda_2}{t}, \frac{\lambda_3}{t}\right)$ where $t > 0$ is a nonzero real solution of the system (2.8).
Now, we form the following system of equations:

$$\begin{cases} \sum_{j=1}^{3} a_{1j}\lambda_j^2 + \lambda_1 &= 0 \qquad (1) \\ \sum_{j=1}^{3} a_{2j}\lambda_j^2 + \lambda_2 &= 0 \qquad (2) \\ \sum_{j=1}^{3} a_{3j}\lambda_j^2 + \lambda_3 &= 0 \qquad (3) \end{cases}$$

Evaluate $a_{23}(1) - a_{13}(2)$ and solve for λ_2 yields:

$$\lambda_2 = \frac{a_{13} \pm \sqrt{a_{13}^2 - 4\left(a_{12}a_{23} - a_{13}a_{22}\right)\left[\left(a_{11}a_{23} - a_{13}a_{21}\right)\lambda_1^2 + a_{23}\lambda_1\right]}}{2\left(a_{12}a_{23} - a_{13}a_{22}\right)}$$

Evaluate $a_{32}(1) - a_{12}(3)$ and solve for λ_3 yields:

$$\lambda_3 = \frac{a_{12} \pm \sqrt{a_{12}^2 - 4\left(a_{13}a_{32} - a_{12}a_{33}\right)\left[\left(a_{11}a_{32} - a_{12}a_{31}\right)\lambda_1^2 + a_{32}\lambda_1\right]}}{2\left(a_{13}a_{32} - a_{12}a_{33}\right)}$$

We substitute λ_2 and λ_3 to (1). After a lengthy algebraic manipulation, we get $\lambda_1 = 0$ or λ_1 is a nonzero real root of $P_1(y_1)$ such that

$$P_1(y_1) = \pi_{1,0} + \pi_{1,1}y_1 + \pi_{1,2}y_1^2 + \pi_{1,3}y_1^3 + \pi_{1,4}y_1^4 + \pi_{1,5}y_1^5 + \pi_{1,6}y_1^6 + \pi_{1,7}y_1^7$$

where

$\pi_{1,0} = \delta_{1,0}^2\left(a_{12}a_{13}\Delta(1,1)\delta_{1,1} - a_{12}a_{13}\delta_{1,3} + a_{32}\delta_{1,0}\delta_{1,7}\right)$

$$\begin{aligned}\pi_{1,1} = \delta_{1,0}\big(&a_{12}a_{13}\delta_{1,0}\delta_{1,1}|A| + a_{12}a_{13}\delta_{1,0}\delta_{1,2} + \Delta(1,1)^2\delta_{1,4} + \delta_{1,3}^2 \\ &- 2\Delta(1,1)\delta_{1,1}\delta_{1,3} - \delta_{1,0}^2\delta_{1,6}\big)\end{aligned}$$

$$\pi_{1,2} = 2\delta_{1,0}\,(\Delta(1,1)|A|\delta_{1,4} - \delta_{1,3}(\delta_{1,1}|A| + \delta_{1,2})$$
$$+ \Delta(1,1)\delta_{1,1}(\delta_{1,2} + \Delta(1,1)^2\delta_{1,0}) - \Delta(1,1)^2\delta_{1,0}\delta_{1,3} - 2\delta_{1,0}^3\delta_{1,5}\big)$$

$$\pi_{1,3} = \delta_{1,0}(\delta_{1,4}|A|^2 + \delta_{1,2}(2|A|\delta_{1,1} + \delta_{1,3} + 2\Delta(1,1)^2\delta_{1,0})$$
$$+ 6\Delta(1,1)^2\delta_{1,0}\delta_{1,1}|A| - 4\Delta(1,1)\delta_{1,0}\delta_{1,3}|A| + \Delta(1,1)^4\delta_{1,0}^2$$
$$+ 4\Delta(3,2)\Delta(2,3)\delta_{1,0}^3)$$

$$\pi_{1,4} = 2\delta_{1,0}^2|A|^2\left((3\Delta(1,1)\delta_{1,1}\delta_{1,3})|A| + 2\Delta(1,1)\delta_{1,2} + 2\Delta(1,1)^3\delta_{1,0}\right)$$
$$\pi_{1,5} = 2\delta_{1,0}^2|A|^2\left(\delta_{1,1}|A| + \delta_{1,2} + 3\Delta(1,1)^2\delta_{1,0}\right)$$
$$\pi_{1,6} = 4\Delta(1,1)\delta_{1,0}^3|A|^3$$
$$\pi_{1,7} = \delta_{1,0}^3|A|^4$$

and

$$\delta_{1,0} = \Delta(2,1)\Delta(3,1)$$
$$\delta_{1,1} = a_{13}\Delta(2,1)^2 + a_{12}\Delta(3,1)^2$$
$$\delta_{1,2} = \Delta(3,2)\Delta(2,1)^3 - \Delta(2,3)\Delta(3,1)^3$$
$$\delta_{1,3} = a_{23}\Delta(2,1)^3 - a_{32}\Delta(3,1)^3$$
$$\delta_{1,4} = a_{13}^2\Delta(2,1)^4 + a_{12}^2\Delta(3,1)^4 + 3a_{12}a_{13}\Delta(2,1)^2\Delta(3,1)^2$$
$$\delta_{1,5} = a_{23}\Delta(2,3) + a_{32}\Delta(3,2)$$
$$\delta_{1,6} = a_{13}^2\Delta(2,1)\Delta(2,3) - a_{12}^2\Delta(3,1)\Delta(3,2) - 4a_{23}a_{32}\Delta(2,1)\Delta(3,1)$$
$$\delta_{1,7} = a_{13}^2\Delta(2,1) - a_{12}^2\Delta(3,1)$$

We do the same line of reasoning in solving λ_2 and λ_3.

After a lengthy algebraic manipulation for λ_2, we get $\lambda_2 = 0$ or λ_2 is a nonzero real root of $P_2(y_2)$ such that

$$P_2(y_2) = \pi_{2,0} + \pi_{2,1}y_2 + \pi_{2,2}y_2^2 + \pi_{2,3}y_2^3 + \pi_{2,4}y_2^4 + \pi_{2,5}y_2^5 + \pi_{2,6}y_2^6 + \pi_{2,7}y_2^7$$

where

$$\pi_{2,0} = \delta_{2,0}^2\,(a_{21}a_{23}\Delta(2,2)\delta_{2,1} - a_{21}a_{23}\delta_{2,3} + a_{31}\delta_{2,0}\delta_{2,7})$$

$$\pi_{2,1} = \delta_{2,0}\left(a_{21}a_{23}\delta_{2,0}\delta_{2,1}|A| + a_{21}a_{23}\delta_{2,0}\delta_{2,2} + \Delta(2,2)^2\delta_{2,4} + \delta_{2,3}^2\right.$$
$$\left.- 2\Delta(2,2)\delta_{2,1}\delta_{2,3} - \delta_{2,0}^2\delta_{2,6}\right)$$

$$\pi_{2,2} = 2\delta_{2,0}\,(\Delta(2,2)|A|\delta_{2,4} - \delta_{2,3}(\delta_{2,1}|A| + \delta_{2,2})$$
$$+\Delta(2,2)\delta_{2,1}(\delta_{2,2} + \Delta(2,2)^2\delta_{2,0}) - \Delta(2,2)^2\delta_{2,0}\delta_{2,3} - 2\delta_{2,0}^3\delta_{2,5}\big)$$

$$\pi_{2,3} = \delta_{2,0}(\delta_{2,4}|A|^2 + \delta_{2,2}(2|A|\delta_{2,1} + \delta_{2,3} + 2\Delta(2,2)^2\delta_{2,0})$$
$$+ 6\Delta(2,2)^2\delta_{2,0}\delta_{2,1}|A| - 4\Delta(2,2)\delta_{2,0}\delta_{2,3}|A|$$
$$+ \Delta(2,2)^4\delta_{2,0}^2 + 4\Delta(3,1)\Delta(1,3)\delta_{2,0}^3)$$

$$\pi_{2,4} = 2\delta_{2,0}^2|A|^2\left((3\Delta(2,2)\delta_{2,1}\delta_{2,3})|A| + 2\Delta(2,2)\delta_{2,2} + 2\Delta(2,2)^3\delta_{2,0}\right)$$
$$\pi_{2,5} = 2\delta_{2,0}^2|A|^2\left(\delta_{2,1}|A| + \delta_{2,2} + 3\Delta(2,2)^2\delta_{2,0}\right)$$
$$\pi_{2,6} = 4\Delta(2,2)\delta_{2,0}^3|A|^3$$
$$\pi_{2,7} = \delta_{2,0}^3|A|^4$$

and

$$\delta_{2,0} = \Delta(1,2)\Delta(3,2)$$
$$\delta_{2,1} = a_{23}\Delta(1,2)^2 + a_{21}\Delta(3,2)^2$$
$$\delta_{2,2} = \Delta(3,1)\Delta(1,2)^3 - \Delta(1,3)\Delta(3,2)^3$$
$$\delta_{2,3} = a_{13}\Delta(1,2)^3 - a_{31}\Delta(3,2)^3$$
$$\delta_{2,4} = a_{23}^2\Delta(1,2)^4 + a_{21}^2\Delta(3,2)^4 + 3a_{21}a_{23}\Delta(1,2)^2\Delta(3,2)^2$$
$$\delta_{2,5} = a_{13}\Delta(1,3) + a_{31}\Delta(3,1)$$
$$\delta_{2,6} = a_{23}^2\Delta(1,2)\Delta(1,3) - a_{21}^2\Delta(3,2)\Delta(3,1) - 4a_{13}a_{31}\Delta(1,2)\Delta(3,2)$$
$$\delta_{2,7} = a_{23}^2\Delta(1,2) - a_{21}^2\Delta(3,2)$$

After a lengthy algebraic manipulation for λ_3, we get $\lambda_3 = 0$ or λ_3 is a nonzero real root of $P_3(y_3)$ such that

$$P_3(y_3) = \pi_{3,0} + \pi_{3,1}y_3 + \pi_{3,2}y_3^2 + \pi_{3,3}y_3^3 + \pi_{3,4}y_3^4 + \pi_{3,5}y_3^5 + \pi_{3,6}y_3^6 + \pi_{3,7}y_3^7$$

where

$$\pi_{3,0} = \delta_{3,0}^2\left(a_{32}a_{31}\Delta(3,3)\delta_{3,1} - a_{32}a_{31}\delta_{3,3} + a_{12}\delta_{3,0}\delta_{3,7}\right)$$

$$\pi_{3,1} = \delta_{3,0}\big(a_{32}a_{31}\delta_{3,0}\delta_{3,1}|A| + a_{32}a_{31}\delta_{3,0}\delta_{3,2} + \Delta(3,3)^2\delta_{3,4} + \delta_{3,3}^2$$
$$-2\Delta(3,3)\delta_{3,1}\delta_{3,3} - \delta_{3,0}^2\delta_{3,6}\big)$$

$$\pi_{3,2} = 2\delta_{3,0}\left(\Delta(3,3)|A|\delta_{3,4} - \delta_{3,3}(\delta_{3,1}|A| + \delta_{3,2})\right.$$
$$\left.+\Delta(3,3)\delta_{3,1}(\delta_{3,2} + \Delta(3,3)^2\delta_{3,0}) - \Delta(3,3)^2\delta_{3,0}\delta_{3,3} - 2\delta_{3,0}^3\delta_{3,5}\right)$$

$$\pi_{3,3} = \delta_{3,0}(\delta_{3,4}|A|^2 + \delta_{3,2}(2|A|\delta_{3,1} + \delta_{3,3} + 2\Delta(3,3)^2\delta_{3,0})$$
$$+ 6\Delta(3,3)^2\delta_{3,0}\delta_{3,1}|A| - 4\Delta(3,3)\delta_{3,0}\delta_{3,3}|A| + \Delta(3,3)^4\delta_{3,0}^2$$
$$+ 4\Delta(1,2)\Delta(2,1)\delta_{3,0}^3)$$

$$\pi_{3,4} = 2\delta_{3,0}^2|A|^2\left((3\Delta(3,3)\delta_{3,1}\delta_{3,3})|A| + 2\Delta(3,3)\delta_{3,2} + 2\Delta(3,3)^3\delta_{3,0}\right)$$
$$\pi_{3,5} = 2\delta_{3,0}^2|A|^2\left(\delta_{3,1}|A| + \delta_{3,2} + 3\Delta(3,3)^2\delta_{3,0}\right)$$

$\pi_{3,6} = 4\Delta(3,3)\delta_{3,0}^3|A|^3$

$\pi_{3,7} = \delta_{3,0}^3|A|^4$

and

$\delta_{3,0} = \Delta(2,3)\Delta(1,3)$

$\delta_{3,1} = a_{13}\Delta(2,3)^2 + a_{32}\Delta(1,3)^2$

$\delta_{3,2} = \Delta(1,2)\Delta(2,3)^3 - \Delta(2,1)\Delta(1,3)^3$

$\delta_{3,3} = a_{21}\Delta(2,3)^3 - a_{12}\Delta(1,3)^3$

$\delta_{3,4} = a_{31}^2\Delta(2,3)^4 + a_{32}^2\Delta(1,3)^4 + 3a_{32}a_{31}\Delta(2,3)^2\Delta(1,3)^2$

$\delta_{3,5} = a_{21}\Delta(2,1) + a_{12}\Delta(1,2)$

$\delta_{3,6} = a_{31}^2\Delta(2,3)\Delta(2,1) - a_{32}^2\Delta(1,3)\Delta(1,2) - 4a_{21}a_{12}\Delta(2,3)\Delta(1,3)$

$\delta_{3,7} = a_{31}^2\Delta(2,3) - a_{32}^2\Delta(1,3)$

Observe that $P_1(y_1), P_2(y_2)$ and $P_3(y_3)$ are odd degree polynomials with nonzero constant terms. This means that there exists a nonzero real root λ_1 for $P(y_1)$, λ_2 for $P(y_2)$ and λ_3 for $P(y_3)$. Hence, there exists a nonzero real solution $(x_1, x_2, x_3) = \left(\frac{\lambda_1}{t}, \frac{\lambda_2}{t}, \frac{\lambda_3}{t}\right)$ of the system of differential equations. $\square$

Example 2.2. *Find the nonzero real solutions of the following system of differential equations*

$$\begin{cases} \dfrac{dx}{dt} &= -\dfrac{1}{2}(x-1)^2 - \dfrac{9}{8}(y-1)^2 \\ \dfrac{dy}{dt} &= \dfrac{3}{4}(y-1)^2 + \dfrac{1}{3}(z-1)^2 \\ \dfrac{dz}{dt} &= -\dfrac{1}{2}(x-1)^2 - \dfrac{9}{8}(y-1)^2 - 2(z-1)^2 \end{cases} \tag{2.10}$$

Solution:

Let $x_1 = x-1$, $x_2 = y-1$ and $x_3 = z-1$. Then $\frac{dx_1}{dt} = \frac{dx}{dt}$, $\frac{dx_2}{dt} = \frac{dy}{dt}$ and $\frac{dx_3}{dt} = \frac{dz}{dt}$.

Now,

$$\begin{cases} \dfrac{dx_1}{dt} &= -\dfrac{1}{2}x_1^2 - \dfrac{9}{8}x_2^2 \\ \dfrac{dx_2}{dt} &= \dfrac{3}{4}x_2^2 + \dfrac{1}{3}x_3^2 \\ \dfrac{dx_3}{dt} &= -\dfrac{1}{2}x_1^2 - \dfrac{9}{8}x_2^2 - 2x_3^2 \end{cases} \tag{2.11}$$

We have $a_{11} = -\frac{1}{2}$, $a_{12} = -\frac{9}{8}$, $a_{13} = 0$, $a_{21} = 0$, $a_{22} = \frac{3}{4}$, $a_{23} = \frac{1}{3}$, $a_{31} = -\frac{1}{2}$, $a_{32} = -\frac{9}{8}$ and $a_{33} = -2$.
By Corollary 3.5.1, the nonzero real solution of the system is of the form $(x_1, x_2, x_3) = \left(\frac{\lambda_1}{t}, \frac{\lambda_2}{t}, \frac{\lambda_3}{t}\right)$ where $(y_1, y_2, y_3) = (\lambda_1, \lambda_2, \lambda_3)$ is a nonzero real solution of the system

$$\begin{cases} -\dfrac{1}{2}y_1^2 - \dfrac{9}{8}y_2^2 + y_1 &= 0 \\ \dfrac{3}{4}y_2^2 + \dfrac{1}{3}y_3^2 + y_2 &= 0 \\ -\dfrac{1}{2}y_1^2 - \dfrac{9}{8}y_2^2 - 2y_3^2 + y_3 &= 0 \end{cases} \tag{2.12}$$

We get the nonzero real solution of system (2.12) rounded to five decimal places:

$$\begin{cases} \lambda_1 = 0.00837 \\ \lambda_2 = -0.08608 \\ \lambda_3 = -0.49149 \end{cases}$$

The nonzero real solution of the system (2.11) is

$$\begin{cases} x_1 = \frac{0.00837}{t} \\ x_2 = -\frac{0.08608}{t} \\ x_3 = -\frac{0.49149}{t} \end{cases}$$

Since $x_1 = x - 1$, $x_2 = y - 1$ and $x_3 = z - 1$, the nonzero real solution of the system (2.10) is

$$\begin{cases} x = \frac{t + 0.00837}{t} \\ y = \frac{t - 0.08608}{t} \\ z = \frac{t - 0.49149}{t} \end{cases}$$

3. Quadratic Approximation of a System of Nonlinear Ordinary Differential Equations

Consider the system of nonlinear ordinary differential equations of the form

$$\begin{cases} \frac{dz_1}{dt} & = & f_1\left(z_1, z_2, \ldots z_n\right) \\ \frac{dz_2}{dt} & = & f_2\left(z_1, z_2, \ldots z_n\right) \\ & \vdots & \\ \frac{dz_n}{dt} & = & f_n\left(z_1, z_2, \ldots z_n\right) \end{cases} \tag{3.1}$$

Using only the quadratic terms of the Taylor Series Expansion, the approximation of the system is as follows:

$$\begin{cases} \frac{dz_1}{dt} & = & \sum_{j=1}^{n} a_{1j}\left(z_j - a_j\right)^2 \\ \frac{dz_2}{dt} & = & \sum_{j=1}^{n} a_{2j}\left(z_j - a_j\right)^2 \\ & \vdots & \\ \frac{dz_n}{dt} & = & \sum_{j=1}^{n} a_{nj}\left(z_j - a_j\right)^2 \end{cases} \tag{3.2}$$

where $a = (a_1, a_2, \ldots, a_n)$ is an equilibrium point.

Let $x_j = z_j - a_j$ where $1 \leq j \leq n$. Then the quadratic estimate for the system of differential equations becomes

$$\begin{cases} \dfrac{dx_1}{dt} & = & \sum_{j=1}^{n} a_{1j}x_j^2 \\ \dfrac{dx_2}{dt} & = & \sum_{j=1}^{n} a_{2j}x_j^2 \\ & \vdots & \\ \dfrac{dx_n}{dt} & = & \sum_{j=1}^{n} a_{nj}x_j^2 \end{cases} \tag{3.3}$$

The initial corollaries and theorem from the previous section show that it is possible to obtain a nonzero real solution for a system of differential equations involving quadratic terms. It is worth noting that the proof of the existence of a nonzero real solution for a system of differential equations with quadratic terms is typically non-trivial and requires advanced mathematical techniques. However, once it is established, it can provide valuable insights into the system's behavior.

We consider the following system of nonlinear differential equations for which an exact solution is known. We would like to approximate this system using a system of differential equations with quadratic terms by using a Taylor series expansion. Specifically, we will retain the quadratic terms of the expansion and discard higher-order terms. Then we will compare the solutions of the system of nonlinear differential equations and system of differential equations with quadratic.

Example 3.1. *Find the approximate solutions of the following system of nonlinear differential equations*

$$\begin{cases} \dfrac{dx}{dt} & = & \dfrac{1}{2}xy^3 - \dfrac{1}{2x} \\ \dfrac{dy}{dt} & = & \dfrac{1}{3y^2} - \dfrac{1}{3}x^2y \end{cases} \tag{3.4}$$

Solution:

Observe that the solution for this system is

$$\begin{cases} x & = & \sqrt{\dfrac{t+1}{t}} \\ y & = & \sqrt[3]{\dfrac{t-1}{t}} \end{cases}$$

where $t \geq 1$.

We will solve for the equilibrium points of the system by setting $\dfrac{dx}{dt} = 0$ and $\dfrac{dy}{dt} = 0$.

$$\begin{cases} \dfrac{1}{2}xy^3 - \dfrac{1}{2x} & = & 0 \\ \dfrac{1}{3y^2} - \dfrac{1}{3}x^2y & = & 0 \end{cases}$$

We obtain $(x_0, y_0) = (1, 1)$ as one of the equilibrium points.

We will find the approximate system of differential equations of the system (3.4) at the point $(1,1)$ by collecting the quadratic terms of the Taylor expansion and discard higher-order terms.
Let $F_1(x,y) = \frac{1}{2}xy^3 - \frac{1}{2x}$ and $F_2(x,y) = \frac{1}{3y^2} - \frac{1}{3}x^2y$. Then the estimate system of differential equations is given by

$$\begin{cases} \frac{dx}{dt} &= \frac{1}{2}\left(\frac{\partial^2 F_1}{\partial x^2}(1,1)(x-1)^2 + \frac{\partial^2 F_1}{\partial y^2}(1,1)(y-1)^2\right) \\ \frac{dy}{dt} &= \frac{1}{2}\left(\frac{\partial^2 F_2}{\partial x^2}(1,1)(x-1)^2 + \frac{\partial^2 F_2}{\partial y^2}(1,1)(y-1)^2\right) \end{cases}$$

We evaluate the right-hand side of both equations. So we have

$$\begin{cases} \frac{dx}{dt} &= -(x-1)^2 + \frac{3}{2}(y-1)^2 \\ \frac{dy}{dt} &= -\frac{1}{3}(x-1)^2 + (y-1)^2 \end{cases} \tag{3.5}$$

Note that the solution of this system of differential equations are

$$\begin{cases} x = \dfrac{t+4.13263}{t} \\ y = \dfrac{t-4.58710}{t} \end{cases}, \begin{cases} x = \dfrac{t+1.14043}{t} \\ y = \dfrac{t-4.24298}{t} \end{cases}, \begin{cases} x = \dfrac{t-1.27307}{t} \\ y = \dfrac{t+0.66478}{t} \end{cases},$$

$$\begin{cases} x = \dfrac{t-1.27307}{t} \\ y = \dfrac{t-0.80301}{t} \end{cases} \text{ and } \begin{cases} x = \dfrac{t-1.27307}{t} \\ y = \dfrac{t-3.86176}{t} \end{cases}$$

where $t > 0$.

The following figures show that the solutions of the system of nonlinear differential equations and the estimate system of differential equations gets closer and closer as $t \to \infty$.

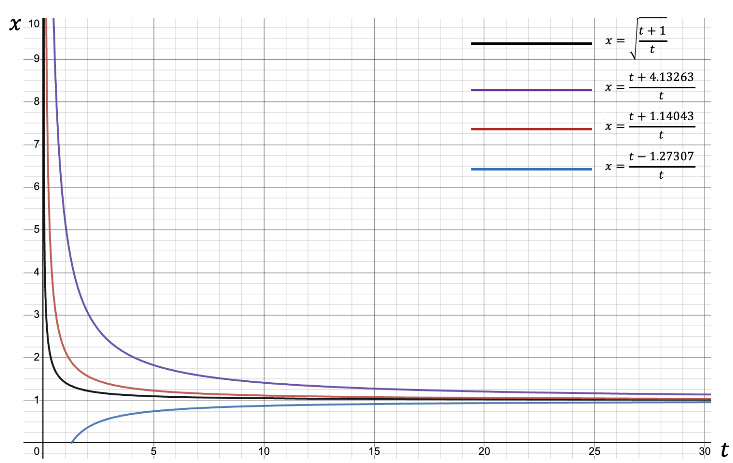

FIGURE 1. Solutions x of the system and the estimates over time

The figure 1 illustrates the behavior of solutions x over time t, with the black curve representing the exact solution and the other colored curves (purple, red, and blue) showing quadratic approximations.

The exact solution decreases steadily, approaching zero as t increases. All the colored approximations (purple, red, and blue) are relatively close to the exact solution, with the red and blue approximations closely following the exact behavior, showing only minor deviations as t increases. The purple approximation slightly overestimates the solution, but it still remains close to the exact curve for a significant portion of the time interval.

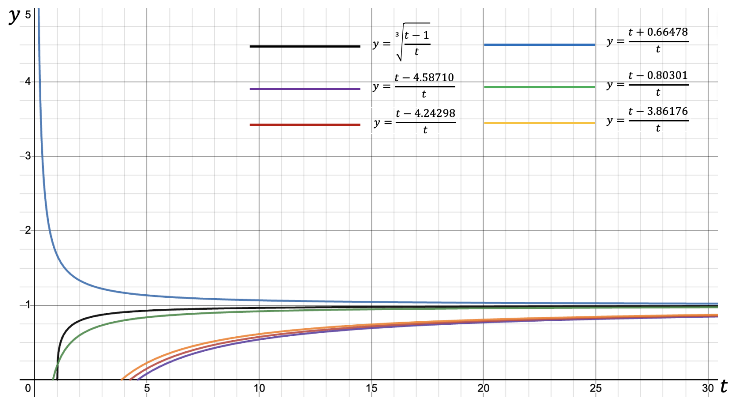

FIGURE 2. Solutions y of the system and the estimates over time

The figure 2 shows the behavior of solutions y over time t, with the black curve representing the exact solution and the other colored curves (blue, green, purple, red, and orange) depicting various quadratic approximations. All the colored approximations are generally close to the exact solution, with the blue and green curves providing the closest estimates. The other approximations (purple, red, and orange) also follow the overall trend of the exact solution, converging towards it as t increases, though with slight deviations. Despite minor differences, the approximations remain relatively accurate in capturing the behavior of the exact solution throughout the time interval.

Example 3.2. *Find the approximate solution of the following system of nonlinear differential equations*

$$\begin{cases} \dfrac{dx}{dt} &= -\dfrac{1}{4}x^2 + \dfrac{1}{2}xy - \dfrac{1}{4}y^2 \\ \dfrac{dy}{dt} &= \dfrac{1}{4}x^2 - \dfrac{1}{2}yz + \dfrac{1}{4}z^2 \\ \dfrac{dz}{dt} &= -\dfrac{1}{4}x^2 - \dfrac{1}{4}y^2 - z^2 - \dfrac{1}{2}xy + xz + yz \end{cases} \tag{3.6}$$

Solution:

Observe that the solution for this system is

$$\begin{cases} x &= \sqrt{\dfrac{t+1}{t}} \\ y &= \sqrt{\dfrac{t-1}{t}} \\ z &= \sqrt{\dfrac{t+1}{t}} \end{cases}$$

where $t \geq 1$.

We will solve for the equilibrium points of the system by setting $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 0$ and $\frac{dz}{dt} = 0$.

$$\begin{cases} -\frac{1}{4}x^2 + \frac{1}{2}xy - \frac{1}{4}y^2 & = & 0 \\ \frac{1}{4}x^2 - \frac{1}{2}yz + \frac{1}{4}z^2 & = & 0 \\ -\frac{1}{4}x^2 - \frac{1}{4}y^2 - z^2 - \frac{1}{2}xy + xz + yz & = & 0 \end{cases}$$

We obtain $(x_0, y_0, z_0) = (1, 1, 1)$ as one of the equilibrium points.

We will find the estimate system of differential equations of the system (3.6) at the point $(1, 1, 1)$ by collecting the quadratic terms of the Taylor expansion and discard higher-order terms.

Let $F_1(x, y, z) = -\frac{1}{4}x^2 + \frac{1}{2}xy - \frac{1}{4}y^2$, $F_2(x, y, z) = \frac{1}{4}x^2 - \frac{1}{2}yz + \frac{1}{4}z^2$ and $F_3(x, y, z) = -\frac{1}{4}x^2 - \frac{1}{4}y^2 - z^2 - \frac{1}{2}xy + xz + yz$. Then the estimate system of differential equations is given by

$$\begin{cases} \frac{dx}{dt} & = & \frac{1}{2}\left(\frac{\partial^2 F_1}{\partial x^2}(1,1,1)(x-1)^2 + \frac{\partial^2 F_1}{\partial y^2}(1,1,1)(y-1)^2 + \frac{\partial^2 F_1}{\partial z^2}(1,1,1)(z-1)^2\right) \\ \frac{dy}{dt} & = & \frac{1}{2}\left(\frac{\partial^2 F_2}{\partial x^2}(1,1,1)(x-1)^2 + \frac{\partial^2 F_2}{\partial y^2}(1,1,1)(y-1)^2 + \frac{\partial^2 F_2}{\partial z^2}(1,1,1)(z-1)^2\right) \\ \frac{dz}{dt} & = & \frac{1}{2}\left(\frac{\partial^2 F_3}{\partial x^2}(1,1,1)(x-1)^2 + \frac{\partial^2 F_3}{\partial y^2}(1,1,1)(y-1)^2 + \frac{\partial^2 F_3}{\partial z^2}(1,1,1)(z-1)^2\right) \end{cases}$$

We evaluate the right-hand side of the equations. So we have

$$\begin{cases} \frac{dx}{dt} & = & -\frac{1}{2}(x-1)^2 - \frac{9}{8}(y-1)^2 \\ \frac{dy}{dt} & = & \frac{3}{4}(y-1)^2 + \frac{1}{3}(z-1)^2 \\ \frac{dz}{dt} & = & -\frac{1}{2}(x-1)^2 - \frac{9}{8}(y-1)^2 - 2(z-1)^2 \end{cases} \tag{3.7}$$

Note that the solution of system (3.7) is

$$\begin{cases} x = \frac{t + 0.00837}{t} \\ y = \frac{t - 0.08608}{t} \\ z = \frac{t - 0.49149}{t} \end{cases}$$

where $t > 0$.

The following figures show that the solutions of the system of nonlinear differential equations and the estimate system of differential equations with quadratic terms gets closer and closer as $t \to \infty$.

Figures 3, 4, and 5 all illustrate the solutions of the system over time, with black curves representing the exact solutions and green curves showing the quadratic approximations. In each case, the approximations are generally very close to the exact solutions, with only minimal deviations observed. In Figure 3, the approximation for x follows the exact solution closely, with only a small difference as t increases. Similarly, in Figure 4, the approximation for y accurately tracks the exact curve, showing

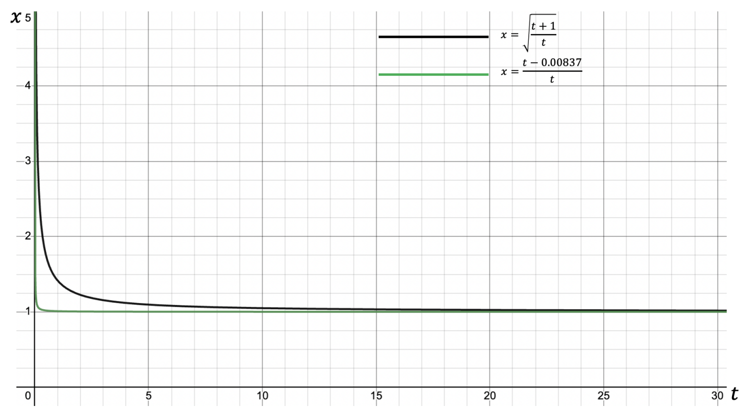

FIGURE 3. Solutions x of the system and the estimates over time

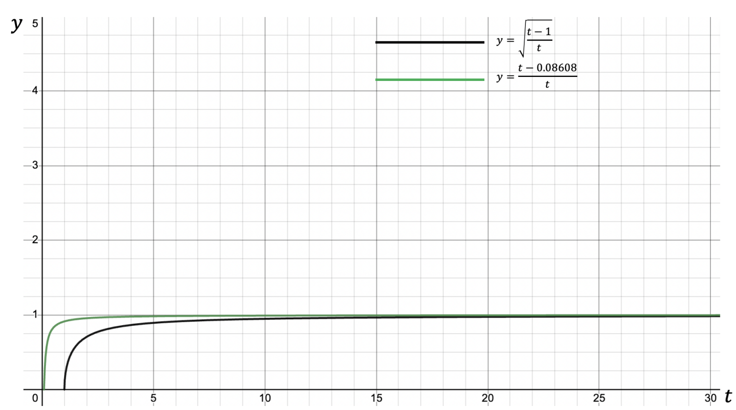

FIGURE 4. Solutions y of the system and the estimates over time

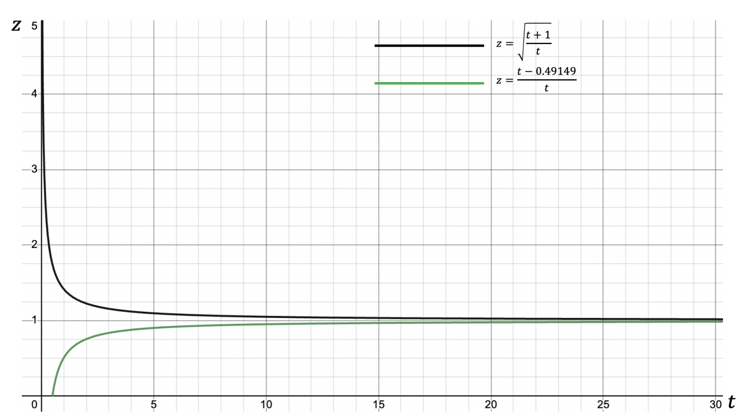

FIGURE 5. Solutions z of the system and the estimates over time

alignment throughout. In Figure 5, the approximation for z initially underestimates the exact solution but converges toward it as time progresses. Overall, the green approximations in all three figures remain close to the exact solutions.

4. Conclusion

This study shows that quadratic approximations provide a valuable alternative for analyzing the behavior of nonlinear systems governed by ordinary differential equations. By focusing exclusively on quadratic terms, these approximations may offer enhanced accuracy, particularly in capturing nonlinear dynamics. The existence of nonzero real solutions further demonstrates the practical utility of these approximations. Through a series of examples, we have shown that quadratic approximations can closely align with exact solutions, thus expanding the range of tools available for studying nonlinear systems. This approach opens new avenues for better understanding and predicting the behavior of complex systems in various scientific disciplines.

Authors' Contributions. All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] J. Kevorkian, J.D. Cole, Perturbation Methods in Applied Mathematics, Springer New York, 2013. https://doi.org/10.1007/978-1-4757-4213-8.

[2] A.H. Nayfeh, Perturbation Methods, Wiley, 2008.

[3] C.M. Bender, S.A. Orszag, Advanced Mathematical Methods for Scientists and Engineers I, Springer New York, 1999. https://doi.org/10.1007/978-1-4757-3069-2.

[4] M.W. Hirsch, S. Smale, R.L. Devaney, Differential Equations, Dynamical Systems, and an Introduction to Chaos, Academic Press, 2004.

[5] V.I. Arnold, R. Cooke, Ordinary Differential Equations, Springer, 1992.

[6] J. Guckenheimer, P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Springer New York, 2013. https://doi.org/10.1007/978-1-4612-1140-2.

[7] L. Perko, Differential Equations and Dynamical Systems, Springer New York, 2013. https://doi.org/10.1007/978-1-4613-0003-8.

[8] N.N. Bogoliubov, Y.A. Mitropolsky, J. Gillis, Asymptotic Methods in the Theory of Non-Linear Oscillations, Phys. Today 16 (1963), 61–61. https://doi.org/10.1063/1.3050754.

[9] P.H. Coullet, E.A. Spiegel, Amplitude Equations for Systems with Competing Instabilities, SIAM J. Appl. Math. 43 (1983), 776–821. https://doi.org/10.1137/0143052.