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INVERSE EQUITABLE RINGS DOMINATION IN GRAPHS

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Abstract. A dominating set X is an equitable rings dominating set of G if for every vertex $v \in V(G) \setminus X$, there exists a vertex u in X such that $uv \in E(G)$, $|\deg(u) - \deg(v)| \leq 1$, and v is adjacent to at least two vertices in $V(G) \setminus X$. The equitable rings domination number G is the minimum cardinality of an equitable rings dominating set of G. Let $Y = V(G) \setminus X$, where X is a minimum equitable rings dominating set of G. A subset Z of Y is said to be an inverse equitable rings dominating set with respect to X if Z is an equitable rings dominating set of G. The inverse equitable rings domination number of G is the minimum cardinality of an inverse equitable rings dominating set of G, and is denoted by $\gamma_{eri}^{-1}(G)$. It is known that there does not exist an equitable rings dominating set in path P_k, and so its inverse does not exist. In this study, we present the inverse equitable rings dominating set in graphs and we provide a special result for the existence of inverse equitable rings dominating set in the join of two paths.

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1. Introduction

The study of domination in graphs has evolved over the years, introducing various concepts that explore how certain sets of vertices influence the entire structure. It all started in 1958 when Berge introduced the idea of domination [5], which was later refined in 1962 by Ore, who formally defined the terms "dominating set" and "domination number" [20]. A dominating set in a graph, in general, is a set of vertices D such that each vertex is either in D or is adjacent to a vertex in D [24]. Formally, a subset X of $V(G)$ is a dominating set of G if for every $v \in V(G) \setminus X$, there exists $u \in X$ such that $uv \in E(G)$. The minimum cardinality among dominating sets is called the domination number and is denoted by $\gamma(G)$.

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As the concept gained interest, researchers explored different ways to modify domination to fit various constraints and applications. This led to several variations, each introducing new conditions on how domination works in graphs. One important aspect of these studies is the identification of minimum dominating sets, γ-sets, which serve as the foundation for many extensions of domination theory. Over time, this concept was extended in various ways, one of which was equitable domination, first introduced by Anita, Arumugam, and Chellali in 2011 [4], cited in Caay's paper [6,7,9–11], a dominating set X of G is an equitable dominating set if for every $v \in V(G) \setminus X$, there exists $u \in X$ such that $uv \in E(G)$ and $|\deg(u) - \deg(v)| \leq 1$. The minimum cardinality of an equitable dominating set is called the equitable domination number, denoted by $\gamma_e(G)$, and an equitable dominating set whose cardinality is $\gamma_e(G)$ is called a γ_e-set of G.

In 2022, Abed and Al-Harere introduced another variation known as rings domination [1], which imposes a stricter adjacency condition. A dominating set X of G is a rings dominating set if every vertex $v \in V(G) \setminus X$ is adjacent to at least two vertices in $V(G) \setminus X$. The minimum cardinality of a rings dominating set is called the rings domination number, denoted by $\gamma_{ri}(G)$, and a rings dominating set whose cardinality is $\gamma_{ri}(G)$ is called a γ_{ri}-set of G. These developments set the stage for Caay's study, which combined equitable domination and rings domination . A dominating set X of G is an equitable rings dominating set if for every $v \in V(G) \setminus X$, there exists $u \in X$ such that $uv \in E(G)$, while also satisfying $|\deg(u) - \deg(v)| \leq 1$, and v is adjacent to at least two vertices in $V(G) \setminus X$. The minimum cardinality of such a set is called the equitable rings domination number, denoted by $\gamma_{eri}(G)$, and an equitable rings dominating set whose cardinality is $\gamma_{eri}(G)$ is called a γ_{eri}-set of G [6], [19].

As domination theory evolved, researchers also investigated variations of domination that consider ways a set can influence a graph, leading to *inverse domination*, first introduced by Kulli and Sigarkanti and they defined the *inverse domination number* $\gamma'(G)$ as the order of a smallest inverse dominating set of G [16], shifting the focus toward the resilience of domination in graphs. To establish a relationship between vertices within a dominating set, Mallinath and Kulkarni (2021) introduced *inverse equitable domination* [18]. Given an equitable dominating set X, a subset $D \subseteq V(G) \setminus X$ is called an *inverse equitable dominating set* with respect to X, if D is also an equitable dominating set of G. The *inverse equitable domination number*, denoted by ${\gamma_e}^{-1}(G)$, is the minimum cardinality of an inverse equitable dominating set. Their study determined the values of ${\gamma_e}^{-1}(G)$ for various standard graphs and established that it is always bounded above by the standard domination number, ${\gamma_e}^{-1}(G) \leq \gamma(G)$. The study of inverse domination continued in 2022, when Abed and Al-Harere introduced *inverse rings domination*, and was further introduced by Lundag and Caay [17]. Let X be a minimum rings dominating set of G and let $Y = V(G) \setminus X$. A rings dominating set $Z \subseteq Y$ is an *inverse rings dominating set* if every vertex $v \in V(G) \setminus Z$ is adjacent to at least two vertices in Y. The minimum cardinality of an inverse rings dominating set Z of G is called the *inverse rings domination number* of G, and is denoted by ${\gamma_{ri}}^{-1}(G)$. In this case, we call a minimum inverse rings dominating set Z of G to be a ${\gamma_{ri}}^{-1}$-set of G [1].

In 2024, Caay introduced the concept of *equitable rings domination* and established that there does not exist a $\gamma_{\text{eri}}(G)$-set in a path graph P_n [6]. Caay's work also examined this equitable rings domination in graphs formed by binary operations, including the *join of graphs* [6]. Since paths fail to satisfy the conditions for a $\gamma_{\text{eri}}(G)$-set, it is natural to explore whether the *join operation* could influence the existence of its inverse. Motivated by this, this study considers the join of graphs, examining how this operation influences inverse equitable rings domination and if it offers the necessary conditions for its existence. These results provide a foundation for further research, leading to the development of inverse equitable rings domination, which integrates ideas from inverse, equitable, and rings domination.

2. Preliminaries

This study focuses only on simple connected graphs, which are graphs without loops or multiple edges. A graph $G = (V(G), E(G))$ is defined on a set of vertices V, where the elements of $V(G)$ are called the *vertices* of G, and those of $E(G)$ are the *edges*. The reader may refer to [12,14] for some graph theoretic notions.

We define the neighborhood of $v \in (G)$, denoted by $N_G(v)$, as the set

$$N_G(v) := \{u \in V(G) \;:\; uv \in E(G)\}.$$

Given a subset $X \subseteq V(G)$, the set

$$N_G(X) = N(X) = \bigcup_{v\in X} N_G(v) \quad \text{and the set} \quad N_G[X] = N[X] = X \cup N(X)$$

are the *open neighborhood* and the *closed neighborhood* of X, respectively.

Given a vertex $v \in V(G)$, we define the *degree of* v, denoted by $\deg(v)$, to be the number of edges incident to v. The *maximum degree of* G, denoted by $\Delta(G)$, is the degree of the vertex in G having the maximum degree, and the *minimum degree of* G, denoted by $\delta(G)$, is the degree of the vertex in G having the minimum degree.

Moreover, a vertex v of G is called a *pendant vertex* if and only if $deg(v) = 1$. Otherwise, it is called a *non-pendant vertex*.

The *join* $G + H$ of the two graphs G and H is the graph with vertex set

$$V(G + H) = V(G) \cup V(H),$$

and the edge set

$$E(G + H) = E(G) \cup (H) \cup \{uv \;:\; u \in V(G), v \in V(H)\}.$$

Given two graphs G and H, we denote $deg_G(u)$ as the degree of u in G and we denote $deg_H(v)$ as the degree of v in H. Unless no join operation is done yet, we use the convention $deg(u)$ and $deg(v)$.

Remark 2.1. From the definition of join of two graphs, the degree of u in $G+H$ is given by $deg_{G+H}(u) = deg_G(u) + m$ and the degree of v in $G+H$ is given by $deg_{G+H}(v) = deg_H(v) + n$.

We now introduce the concept of domination in graphs.

Definition 2.2. [5] A subset $X \in V(G)$ is a dominating set of G if for every $v \in V(G) \setminus X$, there exists $u \in X$ such that $uv \in E(G)$. That is, $N[X] = V(G)$. The minimum cardinality of a dominating set X of G is called the domination number of G, and is denoted by $\gamma(G)$. In this case, we call a dominating set X of G to be a γ-set of G.

Remark 2.3. Let X be a dominating set of G. If there exists $v \in V(G) \setminus X$ such that $uv \in E(G)$, we say that u dominates v or v is dominated by u.

Definition 2.4. [16] Let X be a minimum dominating set of G and let $Y = V(G) \setminus X$. A subset $Z \subseteq Y$ is an inverse dominating set of G if for every $v \in V(G) \setminus Z$, there exists $u \in Z$ such that $uv \in E(G)$. The minimum cardinality of an inverse dominating set Z of G is called the *inverse domination number* of G, and is denoted by $\gamma^{-1}(G)$. In this case, we call a minimum inverse dominating set Z of G to be a γ^{-1}-set of G.

Theorem 2.5. [16] *Let P_n be a path graph of order n. Then*

$$\gamma^{-1}(P_n) = \begin{cases} \lceil \frac{n}{3} \rceil + 1 & \text{if } n = 3k \text{ for some } k \in \mathbb{Z} \\ \lceil \frac{n}{3} \rceil & \text{if } n \neq 3k. \end{cases}$$

Definition 2.6. [4] A dominating set X of G is an equitable dominating set of G if for every $v \in V(G) \setminus X$, there exists $u \in X$ such that $uv \in V(G)$ and $|\deg(u) - \deg(v)| \leq 1$. The minimum cardinality of an equitable dominating set X of G is called the equitable domination number of G, and is denoted by $\gamma_e(G)$. In this case, we call a minimum equitable dominating set X of G to be a γ_e-set of G.

Remark 2.7. Let X be an equitable dominating set of G. If there exists $v \in V(G) \setminus X$ such that $uv \in E(G)$, we say that u equitable dominates v or v is equitable dominated by u.

Definition 2.8. [18] Let X be a minimum equitable dominating set of G and let $Y = V(G) \setminus X$. A subset $X \subseteq Y$ is an inverse equitable dominating set if for every $v \in V(G) \setminus Z$, there exists $u \in Z$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leq 1$. The minimum cardinality of an inverse equitable dominating set Z of G is called the *inverse equitable domination number* of G, and is denoted by $\gamma_e(G)^{-1}$. In this case, we call a minimum inverse equitable dominating set Z of G to be a ${\gamma_e}^{-1}$-set of G.

Theorem 2.9. [25] *Let P_n be a path graph of order n. Then $\gamma_e(P_n) = \lceil \frac{n}{3} \rceil$.*

Remark 2.10. Let P_n be a path graph of order n. Since $\Delta(P_n) = 2$ and $\delta(P_n) = 1$, then $\gamma(P_n) = \gamma_e(P_n)$. It follows that $\gamma^{-1}(P_n) = \gamma_e^{-1}(P_n)$.

Remark 2.11. [16] Let P_n be a path graph of order n. If $n = 3k$ for some $k \in \mathbb{Z}$, then there exists a unique minimum dominating set $X = \{u_{3h-1} : 1 \leq h \leq \frac{n}{3}\}$ of P_n. Moreover, $\{u_i : u_{i+1} \in X\} \cup \{u_n\}$ and $\{u_1\} \cup \{u_{j+1} : u_j \in X\}$ are minimum inverse dominating sets of P_n.

Definition 2.12. [1] A dominating set X of G is a rings dominating set of G if every vertex $v \in V(G) \setminus X$ is adjacent to atleast two vertices $V(G) \setminus X$. The minimum cardinality of a rings dominating set X of G is called the rings domination number of G, and is denoted by $\gamma_{ri}(G)$. A rings dominating set X of a graph G whose cardinality is equal to the $\gamma_{ri}(G)$ is called the $\gamma_{\mathbf{ri}}$-set of G.

Remark 2.13. Let X be a rings dominating set of G. If there exists $v \in V(G) \setminus X$ such that $uv \in E(G)$, we say that u rings dominates v or v is rings dominated by u.

Definition 2.14. [1] Let X be a minimum rings dominating set of G and let $Y = V(G) \setminus X$. A rings dominating set $Z \subseteq Y$ is an *inverse rings dominating set* if every vertex $v \in V(G) \setminus Z$ is adjacent to at least two vertices in Y. The minimum cardinality of an inverse rings dominating set Z of G is called the inverse rings domination number of G, and is denoted by ${\gamma_{ri}}^{-1}(G)$. In this case, we call a minimum inverse rings dominating set Z of G to be a ${\gamma_{ri}}^{-1}$-set of G.

Remark 2.15. In this paper, if a vertex u rings dominate a vertex v, then we say u is a rings dominator of a vertex v, or v is a rings dominatee of a vertex u.

Remark 2.16. [1] For a γ_{ri}-set D of any graph G of order n, we have:

(1) The order of G is $n \geq 4$.
(2) For each vertex $v \in V(G) \setminus D$, $deg(v) \geq 3$.
(3) $1 \leq |D| \leq n - 3$.
(4) $3 \leq |V(G) \setminus D| \leq n - 1$.
(5) $1 \leq \gamma_{ri}(G) \leq |D| \leq n - 3$.

Definition 2.17. A dominating set $X \subseteq V(G)$ is said to be an equitable rings dominating set of G if for every $v \in V(G) \setminus X$, there exists $u \in X$ with $uv \in E(G)$ such that $|\deg(u) - \deg(v)| \leq 1$, and v is adjacent to at least two vertices in $V(G) \setminus X$. The minimum cardinality of an equitable rings dominating set of G is called equitable rings domination number of G, and is denoted by $\gamma_{eri}(G)$. An equitable rings dominating set X of G with $|X| = \gamma_{eri}(G)$ is said to be γ_{eri}-set of G.

Remark 2.18. The following remarks are essential in working the concept of this study.

(1) Given an equitable rings dominating set X of G, if $|X| = \gamma_{eri}(G)$, then we say that X is a $\gamma_{\mathbf{eri}}$-set of G.

(2) Given an equitable rings dominating set X of G, if there exists $v \in V(G) \setminus X$, then for every $u \in X$ with $uv \in E(G)$, we say that u equitable rings dominates v, or v is equitable rings dominated by u. In general, a set X is said to equitable rings dominate G.

The following lemmas directly follow from Definition 2.17.

Lemma 2.19. [6] *Let G be any graph of order n with $\Delta(G) - \delta(G) \leq 1$. Then $\gamma_{eri}(G) = 1$ if and only if $\Delta(G) = n - 1$.*

Lemma 2.20. [6] *Let G be any graph. If $u \in X \subseteq V(G)$ with $deg(u) = \Delta(G)$, then X is a γ_{eri}-set of G.*

We now introduce the main concept of the study.

Definition 2.21. Let X be a minimum equitable rings dominating set of G and let $Y = V(G) \setminus X$. A subset $Z \subseteq Y$ is said to be an inverse equitable rings dominating set of G if for every $v \in V(G) \setminus Z$, there exists $u \in Z$ with $uv \in E(G)$ such that $|deg(u) - deg(v)| \leq 1$, and v is adjacent to at least two vertices in $V(G) \setminus Z$. The minimum cardinality of an inverse equitable rings dominating set Z of G is called the *inverse equitable rings domination number* of G, and is denoted by ${\gamma_{eri}}^{-1}(G)$. In this case, we call a minimum inverse equitable rings dominating set Z of G to be a ${\gamma_{eri}}^{-1}$-set of G.

Remark 2.22. If Z is an inverse equitable rings dominating set of G such that $|Z| = \gamma_{eri}(G)$, then we say Z is a ${\gamma_{eri}}^{-1}$-set of G.

Remark 2.23. If there does not exist an inverse equitable rings dominating set of G, we say that ${\gamma_{eri}}^{-1}(G) = 0$.

Example 2.24. Consider the complete bipartite graph $G = K_{3,3}$.

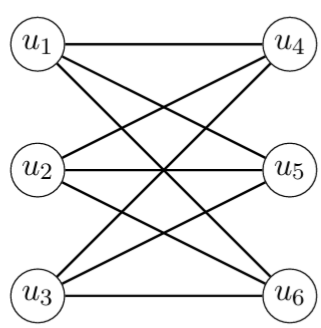

FIGURE 1. A Complete Bipartite Graph $K_{3,3}$

In Figure 1, we have two vertex partitions: $N_1 = \{u_1, u_2, u_3\}$ and $N_2 = \{u_4, u_5, u_6\}$. Observe that the vertices in N_1 can be dominated by just one vertex from N_2 and vice versa. This means that the minimum number of vertices we need to dominate the entire graph is 2. Note that $deg(u_i) = 3$

for $i \in \{1, 2, 3, 4, 5, 6\}$. Suppose we take $X = \{u_1, u_4\}$ to be a minimal dominating set of the graph. Notice that each vertex in $V(G) \setminus X$ is also connected to two vertices in $V(G) \setminus X$. Hence, X is a rings dominating set. Now, we can pick a different pair of vertices to be an inverse rings dominating set, say $\{u_2, u_4\}$. This pair also satisfies the condition for equitable rings domination. Thus, if we choose one pair to be a γ_{eri}-set, we are still left with two pairs that can be a ${\gamma_{eri}}^{-1}$-set. For example, if we take $D = \{u_1, u_4\}$ to be a minimum γ_{eri}-set. we can choose any different pair of vertices from $V(K_{3,3}) \setminus D$, one from each vertex partition, to be a ${\gamma_{eri}}^{-1}$-set. Therefore, $\gamma_{eri}(K_{3,3}) = {\gamma_{eri}}^{-1}(K_{3,3}) = 2$.

3. Inverse Equitable Rings Domination in Graphs

The following lemmas follow directly from Definition 2.21.

Lemma 3.1. *If there exist γ_{eri}-set and ${\gamma_{eri}}^{-1}$-set of a graph G, then $\gamma_{eri}(G) \leq {\gamma_{eri}}^{-1}(G)$.*

Proof. Let X be a γ_{eri}-set of G, and Z be a ${\gamma_{eri}}^{-1}$-set of G with respect to S. Then, $|X| = \gamma_{eri}(G)$ and $|Z| = {\gamma_{eri}}^{-1}(G)$ by Definition 2.17 and Definition 2.21, respectively. Note that Z is also an equitable rings dominating set of G. Since X is an equitable rings dominating set of G whose cardinality is minimum, it follows that $|X| \leq |Z|$, which implies that $\gamma_{eri}(G) \leq {\gamma_{eri}}^{-1}(G)$. □

Lemma 3.2. *Let G be any graph of order n such that $\Delta(G) - \delta(G) \leq 1$ and $\gamma_{eri}(G) = 1$. Let $X = \{u\}$ be a γ_{eri}-set of G. Then there exists $v \in V(G)$ different from u such that $deg(v) = deg(u)$ if and only if ${\gamma_{eri}}^{-1}(G) = 1$.*

Proof. Since $\Delta(G) - \delta(G) \leq 1$ and $\gamma_{eri}(G) = 1$, it follows that $\Delta(G) = n - 1$ by Lemma 2.19. Given that $X = \{u\}$ is a γ_{eri}-set of G, it follows that $deg(u) = n - 1$. Assume there exists $v \in V(G)$ such that $deg(v) = deg(u)$. Then $deg(v) = n - 1$. Take $Z = \{v\}$. Since $deg(v) = n - 1 = \Delta(G)$, it follows by Lemma 2.20 that Z is a γ_{eri}-set of G. Therefore, Z is an inverse equitable rings dominating set of G with respect to X, i.e. ${\gamma_{eri}}^{-1}(G) = 1$. Conversely, assume that ${\gamma_{eri}}^{-1}(G) = 1$. Let $Z \subseteq V(G) \setminus X$ be a ${\gamma_{eri}}^{-1}$-set of G. Since $X = \{u\}$ is a γ_{eri}-set of G, then $Z = \{v\}$ for some vertex v in G different from u. Since Z is a dominating set of G, then v must be adjacent to every vertex in G. Therefore, $deg(v) = n - 1$. □

Theorem 3.3. *An inverse equitable rings dominating set T of G must not contain a pendant.*

Proof. Let T be an inverse equitable rings dominating set of G such that $T \subseteq V(G) \setminus X$ for some γ_{eri}-set X of G. Suppose T contains a pendant. Let $x \in T$ such that $\deg_T(x) = 1$. This means that $x \in V(G) \setminus X$. This is a contradiction to the definition of an equitable rings dominating set X of G that every element $x \in V(G) \setminus S$ must be adjacent to at least two vertices in $x \in V(G) \setminus X$. Therefore, T must not contain a pendant. □

Proposition 3.4. *There does not exist a $\gamma_{eri}{}^{-1}$-set in a path graph P_n.*

Proposition 3.5. *There does not exist a $\gamma_{eri}{}^{-1}$-set in a cycle graph C_n.*

Propositions 3.4 and 3.5 are easy to prove. In fact, the proof follows from Propositions 4.3 and 4.4 of [6]

Proposition 3.6. *Let $n \geq 4$. Then $\gamma_{eri}(K_n) = 1 = \gamma_{eri}{}^{-1}(K_n)$.*

Proof. In a complete graph K_n for $n \geq 4$, $\Delta(K_n) = n-1 = \delta(K_n)$. By Lemma 2.19, it follows that $\gamma_{eri}(K_n) = 1$. Take $X = \{u\}$ where u is an arbitrary vertex in K_n. Since $\Delta(K_n) = n-1 = \delta(K_n)$, then every vertex in K_n has degree $n-1$. Therefore, by Lemma 3.2, $\gamma_{eri}{}^{-1}(K_n) = 1$. □

Proposition 3.7. *Let $n \geq 4$. If X is a γ_{ri}-set of K_n , there are $\displaystyle\sum_{i=1}^{n-4} \binom{n-1}{i}$ number of inverse rings dominating sets of K_n with respect to X.*

Proposition 3.8. *Let $G = K_{P_1,P_2,\cdots,P_k}$ be a complete k-partite graph, $k \geq 2$ such that $||P_i| - |P_j|| \leq 1$ for all $i \neq j$. Then $\gamma_{eri}(G) = \gamma_{eri}^{-1}(G)$ if and only if there exist partitions P_i and P_j, $j \neq j$ such that $|P_1| = |P_j| = \min\{|P_k| \mid P_k$ is a vertex partition of $G\}$*

Corollary 3.9. *There does not exist a $\gamma_{eri}{}^{-1}$-set of $G = K_{n_1,n_2,\cdots,n_k}$, $k \geq 3$, if there exists a vertex partition P_i such that $||P_i| - |P_j|| \geq 2$, for all $i \neq j$.*

Corollary 3.10. *Let $G = K_{P_1,P_2}$ be a complete bipartite graph such that $||P_1| - |P_2|| \leq 1$, and $|P_i| \geq 3$, $i = 1, 2$. Then $\gamma_{eri}(G) = \gamma_{eri}{}^{-1}(G)$. Moreover, if X is a γ_{eri}-set of G, then there are $|P_1| - 1 \times |P_2| - 1$ number of $\gamma_{eri}{}^{-1}$-sets with respect to X.*

Corollary 3.11. *There does not exist a $\gamma_{eri}{}^{-1}$-set of K_{n_1,n_2}, if $|n_1 - n_2| \geq 2$.*

4. The Inverse Equitable Rings Domination under a Binary Operation

In this, section, we then modify how we write the degree of a vertex to avoid confusion. Given two graphs G and H, we denote $deg_G(u)$ to refer the degree of u in the graph G alone and we denote $deg_H(v)$ to refer the degree of v in the graph H alone. Unless no binary operation is done yet, we will use the convention $deg(u)$ and $deg(v)$.

4.1. Join of Path-to-Path Graphs.

Since path graphs have no γ_{eri}-set [6] and thus have no $\gamma_{eri}{}^{-1}$-set by Proposition 3.4, we consider the paths under the binary operation: join of graphs. We explore if there exists a γ_{eri}-set and $\gamma_{eri}{}^{-1}$-set in the join of path-to-path graphs. Let P_n and P_m be path graphs of order n and m respectively. We denote X as a minimum equitable rings dominating set of the graph $P_n + P_m$, and Z as an inverse equitable rings dominating set of the graph. We then determine the $\gamma_{eri}(P_n + P_m)$ and $\gamma_{eri}{}^{-1}(P_n + P_m)$.

Lemma 4.1. *Let $V(P_n) = \{u_1, u_2, \ldots, u_n\}$ and $V(P_m) = \{v_1, v_2, \ldots, v_m\}$. The mapping $f : V(P_n + P_m) \longrightarrow V(P_n + P_m)$ defined by*

$$f(a) = \begin{cases} u_{n+1-i} & \text{if } a = u_i \in V(P_n) \\ v_{m+1-k} & \text{if } a = v_k \in V(P_m) \end{cases}$$

for $i = 1, 2, \ldots n$ and for $k = 1, 2, \ldots m$ is a graph automorphism on $P_n + P_m$.

Proof. For vertices u_i and u_j in $V(P_n)$, u_i is adjacent to u_j if and only if $|i - j| = 1$. Similarly, for vertices v_k and v_l in $V(P_m)$, v_k is adjacent to v_l if and only if $|k - l| = 1$. Moreover, by the definition of join of two graphs, each vertex in P_n is adjacent to every vertex in P_m.

i. Given that $1 \leq i \leq n$, it follows that $-n \leq -i \leq -1$ and thus, $1 \leq n + 1 - i \leq n$, implying that $f(u_i) = u_{n+1-i} \in V(P_n)$. Similarly, $1 \leq k \leq m$ implies that $-m \leq -k \leq -1$ and $1 \leq m + 1 - k \leq m$, implying that $f(v_k) = v_{m+1-k} \in V(P_m)$.
ii. Assume that u_i is adjacent is to u_j. Then $|i - j| = 1$. By the definition of the mapping f, $f(u_i) = u_{n+1-i}$ and $f(u_j) = u_{n+1-j}$. Since $|(n + 1 - i) - (n + 1 - j)| = | - i + j| = |i - j| = 1$, then $f(u_i)$ is adjacent to $f(u_j)$.
iii. Assume that v_k is adjacent is to v_l. Then $|k - l| = 1$. By the definition of the mapping f, $f(v_k) = v_{m+1-k}$ and $f(v_l) = v_{n+1-l}$. Since $|(m + 1 - k) - (m + 1 - l)| = | - k + l| = |k - l| = 1$, then $f(u_k)$ is adjacent to $f(u_l)$.
iv. For each $i = 1, 2, \ldots n$, u_i is adjacent to v_k for all $k = 1, 2, \ldots m$. By the definition of the mapping f, $f(u_i) = u_{n+1-i} \in V(P_n)$ and $f(v_k) = v_{m+1-k} \in V(P_m)$. Thus, $f(u_i)$ is adjacent to $f(v_k)$.

Therefore, f is a graph automorphism on $P_n + P_m$. □

Remark 4.2. . If X is a γ_{eri}-set of $P_n + P_m$, then $Z = \{f(a) \,|\, a \in X\}$ is also a γ_{eri}-set of $P_n + P_m$. Furthermore, if $Z \subseteq V(P_n + P_m) \setminus X$, then Z is a ${\gamma_{eri}}^{-1}$-set of $P_n + P_m$.

In this part, we let $V(P_n) = \{u_1, u_2, \ldots, u_n\}$ and $V(P_m) = \{v_1, v_2, \ldots, v_m\}$. We denote X as our desired equitable rings dominating set of $P_n + P_m$, and Z as our desired inverse equitable rings dominating set of $P_n + P_m$ with respect to X.

Theorem 4.3. *There exists a ${\gamma_{eri}}^{-1}$-set in the join $P_n + P_m$ of path graphs P_n and P_m if $n = m \geq 2$. Moreover*

$$\gamma_{eri}(P_n + P_m) = {\gamma_{eri}}^{-1}(P_n + P_m) = \begin{cases} 1 & \text{if } n = 2 \text{ or } n = 3 \\ 2 & \text{if } n \geq 4 \end{cases}.$$

Proof. Let $n = m \geq 2$. Consider the following cases.

Case 1. Let $n = 2$. Then $m = 2$ and we have $P_2 + P_2$, which is a graph of order 4.

	Vertex		Degree
P_n	Pendant	u_1, u_2	$1+m=3$
	Non-pendant	-	-
P_m	Pendant	v_1, v_2	$1+n=3$
	Non-pendant	-	-

TABLE 1. Degree of vertices in $P_2 + P_2$

From Table 1, $\Delta(P_2+P_2) = 3 = \delta(P_2+P_2)$. Since $\Delta(P_2+P_2) - \delta(P_2+P_2) = 0$ and $\Delta(P_2+P_2) = 3 = 4-1$ (recall that the order of the graph is 4), then $\gamma_{eri}(P_2+P_2) = 1$ by Lemma 2.19. From Table 1, every vertex in P_2+P_2 has degree 3. By Lemma 3.2, it follows that ${\gamma_{eri}}^{-1}(P_2+P_2) = 1$.

Case 2. Let $n = 3$. Then $m = 3$ and we have the graph $P_3 + P_3$ with $V(P_n) = \{u_1, u_2, u_3\}$ and $V(P_m) = \{v_1, v_2, v_3\}$, which is a graph of order 6.

	Vertex		Degree
P_n	Pendant	u_1, u_3	$1+m=4$
	Non-pendant	u_2	$2+m=5$
P_m	Pendant	v_1, v_3	$1+n=4$
	Non-pendant	v_2	$2+n=5$

TABLE 2. Degree of vertices in $P_3 + P_3$

From Table 2, $\Delta(P_3+P_3) = 5$ and $\delta(P_3+P_3) = 4$. Since $\Delta(P_3+P_3) - \delta(P_3+P_3) = 1$ and $\Delta(P_3+P_3) = 5 = 6-1$ (recall that the order of the graph is 6), then $\gamma_{eri}(P_3+P_3) = 1$ by Lemma 2.19. From Table 2, the vertices u_2 and v_2 have degree of 5. Without loss of generality, take $X = \{u_2\}$ be a γ_{eri}-set of P_3+P_3. Since $\Delta(P_3+P_3) - \delta(P_3+P_3) = 1$ and $\gamma_{eri}(P_3+P_3) = 1$, and there exists another vertex v_2 different from u_2 such that $deg(u_2) = deg(v_2)$, then ${\gamma_{eri}}^{-1}(P_2+P_2) = 1$ by Lemma 3.2.

Case 3. Let $n \geq 4$. Since $n = m$, the graph $P_n + P_m$ has an order $2n$ with $\Delta(P_n+P_m) = 2+n$ and $\delta(P_n+P_m) = 1+n$.

	Vertex		Degree	Degree in terms of n
P_n	Pendant	u_1, u_n	$1+m$	$1+n$
	Non-pendant	$u_2, u_3, \ldots, u_{n-1}$	$2+m$	$2+n$
P_m	Pendant	v_1, v_n	$1+n$	$1+n$
	Non-pendant	$v_2, v_3, \ldots, v_{m-1}$	$2+n$	$2+n$

TABLE 3. Degree of vertices in $P_n + P_m$ where $n = m$

Refer to the figure below for a clearer illustration.

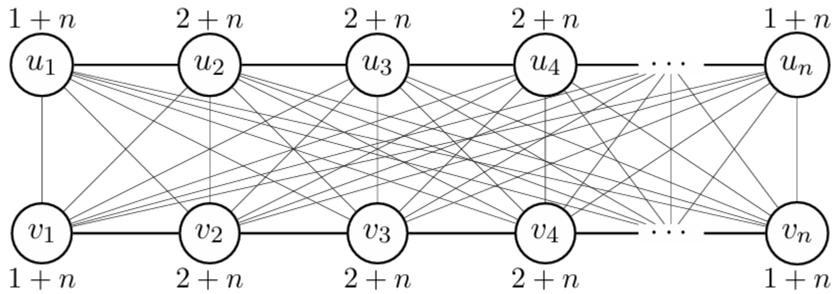

FIGURE 2. Graph $P_n + P_m$ where $n = m$ with the degree of its vertices

If a vertex in $P_n + P_m$ is adjacent to every other vertex in the graph, its order must be $2n - 1$ (here, the order of the graph is $2n$). But $\Delta(P_n + P_m) = 2 + n \neq 2n - 1$ for $n \geq 4$, which implies that no vertex in $P_n + P_m$ is adjacent to all other vertices in the graph. Consequently, $\gamma_{eri}(P_n + P_m) \neq 1$. Take $X = \{a, b\}$, where $a \in V(P_n)$ and $b \in V(P_m)$.

(a) By the definition of join of two graphs, each vertex in P_n is adjacent to every vertex in P_m. Thus, vertex a is adjacent to every vertex in $V(P_m) \setminus X$ and vertex b is adjacent to every vertex in $V(P_n) \setminus X$. Hence, X is a dominating set of $P_n + P_m$.

(b) $\Delta(P_n + P_m) = 2 + n$ and $\delta(P_2 + P_2) = 1 + n$ guarantees that $|deg(u) - deg(v)| \leq 1$ for any adjacent vertices u and v in $P_n + P_m$, satisfying the condition for equitable domination. Hence, X is an equitable dominating set of $P_n + P_m$.

(c) Recall that $X = \{a, b\}$, where $a \in V(P_n)$ and $b \in V(P_m)$. Consequently, $V(P_n + P_m) \setminus X$ contains $n + m - 2 = (n-1) + (m-1)$ vertices in the graph, particularly $n - 1$ vertices in P_n and $m - 1 = n - 1$ vertices in P_m.

In other words, $|V(P_n) \setminus X| = n - 1 = |V(P_m) \setminus X|$. By the definition of join of two graphs, each vertex in P_n is adjacent to every vertex in P_m. It follows that every vertex in $V(P_n) \setminus X$ is adjacent to at least $n - 1 \geq 4 - 1 = 3$ vertices in $V(P_m) \setminus X$, satisfying the condition for rings domination. Hence, X is a rings dominating set of $P_n + P_m$.

Therefore, X is a γ_{eri}-set of $P_n + P_m$ and $\gamma_{eri}(P_n + P_m) = 2$.

Take $Z = \{c, d\}$, where $c \in V(P_n)$ different from a, and $d \in V(P_m)$ different from b. It follows that $Z \subseteq V(P_n + P_m) \setminus X$. We only need to show that Z satisfies the conditions for equitable rings domination.

(a) By the definition of join of two graphs, each vertex in P_n is adjacent to every vertex in P_m. Thus, vertex c is adjacent to every vertex in $V(P_m) \setminus X$ and vertex d is adjacent to every vertex in $V(P_n) \setminus X$. Hence, Z is a dominating set of $P_n + P_m$.

(b) $\Delta(P_n + P_m) = 2 + n$ and $\delta(P_2 + P_2) = 1 + n$ guarantees that $|deg(u) - deg(v)| \leq 1$ for any adjacent vertices u and v in $P_n + P_m$, satisfying the condition for equitable domination. Hence, Z is an equitable dominating set of $P_n + P_m$.

(c) Recall that $Z = \{c, d\}$, where $a \neq c \in V(P_n)$ and $b \neq d \in V(P_m)$. Consequently, $V(P_n + P_m) \setminus Z$ contains $n + m - 2 = (n-1) + (m-1)$ vertices in the graph, particularly $n - 1$ vertices in P_n and $m - 1 = n - 1$ vertices in P_m. In other words, $|V(P_n) \setminus Z| = n - 1 = |V(P_m) \setminus Z|$. By the definition of join of two graphs, each vertex in P_n is adjacent to every vertex in P_m. It follows that every vertex in $V(P_n) \setminus Z$ is adjacent to at least $n - 1 \geq 4 - 1 = 3$ vertices in $V(P_m) \setminus Z$, satisfying the condition for rings domination. Hence, Z is a rings dominating set of $P_n + P_m$.

Therefore, Z is a ${\gamma_{eri}}^{-1}$-set of $P_n + P_m$ and ${\gamma_{eri}}^{-1}(P_n + P_m) = 2$ for $n \geq 4$.

□

Theorem 4.4. *There exists a γ_{eri}^{-1}-set in the join $P_n + P_m$ of path graphs P_n and P_m if $|n - m| = 1$ and $n \geq 2$. Moreover*

$$\gamma_{eri}(P_n + P_m) = {\gamma_{eri}}^{-1}(P_n + P_m) = \begin{cases} 1 & \text{if } n = 2 \\ 2 & \text{if } n \geq 3 \end{cases}.$$

Refer to the figure below for a clearer illustration.

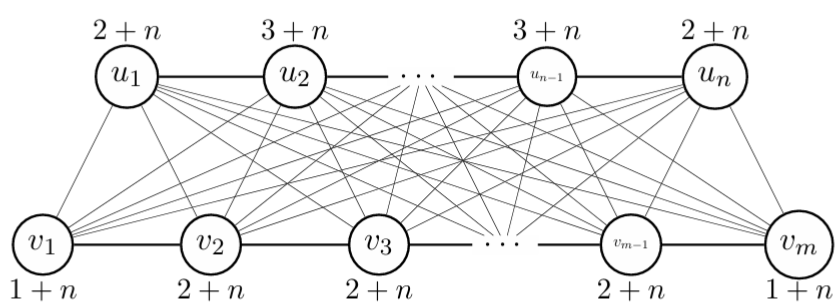

FIGURE 3. $P_n + P_m$ where $m = n + 1$ with the degree of its vertices

The following table summarizes the degree of the vertices in the graph.

	Vertex		Degree	Degree in terms of n
P_n	Pendant	u_1, u_n	$1+m$	$2+n$
	Non-pendant	$u_2, u_3, \ldots, u_{n-1}$	$2+m$	$3+n$
P_m	Pendant	v_1, v_n	$1+n$	$1+n$
	Non-pendant	$v_2, v_3, \ldots, v_{m-1}$	$2+n$	$2+n$

TABLE 4. Degree of vertices in $P_n + P_m$ where $n = m + 1$

Theorem 4.5. *There exists a γ_{eri}^{-1}-set in the join $P_n + P_m$ of path graphs P_n and P_m if $|n-m| = 2$ and $n \geq 2$. Morever,*

$$\gamma_{eri}(P_n + P_m) = \begin{cases} 2 & \text{if } n = 2 \\ 3 & \text{if } n = 3 \text{ or } n = 4 \\ \left\lceil \frac{n}{3} \right\rceil + 2 & \text{if } n \geq 5 \end{cases}$$

and

$$\gamma_{eri}{}^{-1}(P_n + P_m) = \begin{cases} \gamma_{eri}(P_n + P_m) & \text{if } n \neq 4 \\ 4 & \text{if } n = 4 \end{cases}$$

Refer to the figure below for a clearer illustration.

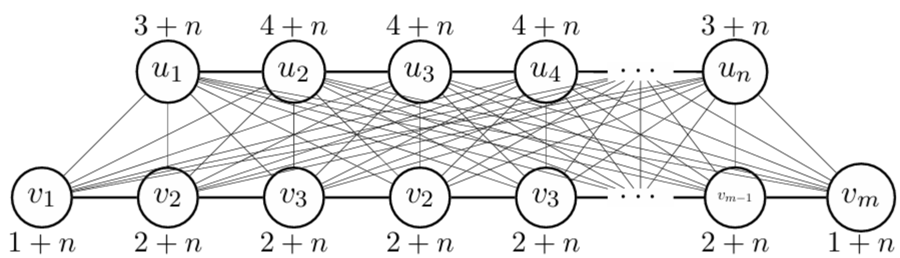

FIGURE 4. $P_n + P_m$ where $m = n + 2$ and $n \geq 5$ with the degree of its vertices

The following table summarizes the degree of the vertices in the graph.

	Vertex		Degree	Degree in terms of n
P_n	Pendant	u_1, u_n	$1+m$	$3+n$
	Non-pendant	$u_2, u_3, \ldots, u_{n-1}$	$2+m$	$4+n$
P_m	Pendant	v_1, v_n	$1+n$	$1+n$
	Non-pendant	$v_2, v_3, \ldots, v_{m-1}$	$2+n$	$2+n$

TABLE 5. Degree of vertices in $P_n + P_m$ where $n = m + 2$

Corollary 4.6. *There exists a $\gamma_{eri}{}^{-1}$-set in the join $P_n + P_m$ of path graphs P_n and P_m if ($|n-m| \geq 3$, $n = 3$, and $m = 3k+1$ for some $k \in \mathbb{Z}$) or ($|n-m| \geq 3$ and $n \geq 4$). Moreover, $\gamma_{eri}(P_n + P_m) = \gamma_e(P_n) + \gamma_e(P_m)$ and $\gamma_{eri}{}^{-1}-1(P_n + P_m) = \gamma_e^{-1}(P_n) + \gamma_e^{-1}(P_m)$.*

Refer to the figure below for a clearer illustration:

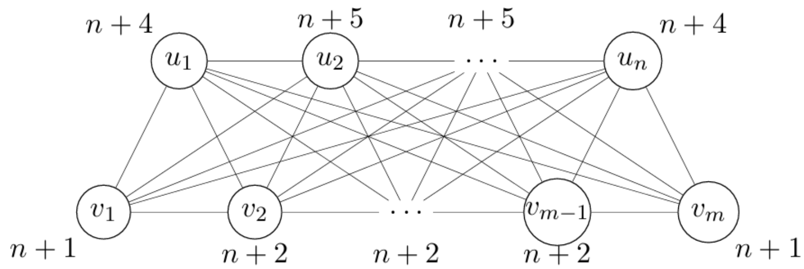

FIGURE 5. A Join of Graph $P_n + P_m$ where $|n-m| \geq 3$

The following degree of each vertex in the graph is summarized in the table below.

	Vertex	Degree	Minimum Degree in terms of n
P_n	u_1, u_n	$m+1$	$n+4$
	$u_2, u_3, \dots, u_{n-1}$	$m+2$	$n+5$
P_m	v_1, v_m	$n+1$	$n+1$
	$v_2, v_3, \dots, v_{m-1}$	$n+2$	$n+2$

TABLE 6. Degree of vertices in $P_n + P_m$ where $|n-m| \geq 3$

Hence, for vertices of P_n in $P_n + P_m$, we have the following degree difference:

i. $|deg(u_1) - deg(u_1)| = |deg(u_n) - deg(u_n)| = 0$

ii. $|deg(u_1) - deg(u_2)| = |deg(u_n) - deg(u_{n-1})| = |(m+1) - (m+2)| = 1$

iii. $|deg(u_i) - deg(u_i)| = 0$

For vertices of P_m in $P_n + P_m$, we have the following degree difference:

i. $|deg(v_1) - deg(v_1)| = |deg(v_m) - deg(v_m)| = 0$

ii. $|deg(v_1) - deg(v_2)| = |deg(v_m) - deg(v_{m-1})| = |(m+1) - (m+2)| = 1$

iii. $|deg(v_i) - deg(v_i)| = 0$

For any pair of vertices from P_n and P_m, we have the following degree difference:

i. $|deg(u_1) - deg(v_1)| = |deg(u_1) - deg(v_n)| = |deg(u_n) - deg(v_1)| = |deg(u_n) - deg(v_m)| = |(m+1) - (n+1)| \geq |(n+4) - (n+1)| = 3$

ii. $|deg(u_1) - deg(v_i)| = |deg(u_n) - deg(v_i) = |(m+1) - (n+2)| \geq |(n+4) - (n+2)| = 2$

iii. $|deg(u_i) - deg(v_1)| = |deg(u_i) - deg(v_m)| = |(m+2) - (n+1) \geq |(n+5) - (n+1)| = 4$

iv. $|deg(u_i) - deg(v_j)| = |(m+2) - (n+2)| \geq |(n+5) - (n+2)| = 3$

Theorem 4.7. *There does not exist a ${\gamma_{eri}}^{-1}$-set in $P_n + P_m$, where $n \leq m$, whenever any of the following is true:*

(1) $n = 1$. *Moreover, ${\gamma_{eri}}^{-1}(P_n + P_m) = 0$;*

(2) *$|n-m| \geq 3$ and $n=2$. Moreover, ${\gamma_{eri}}^{-1}(P_n+P_m)=0$.*

(3) *$|n-m| \geq 3$, $n=3$, and $m=3k$ or $m=3k+2$ for some $k \in \mathbb{Z}$. Moreover, ${\gamma_{eri}}^{-1}(P_n+P_m)=0$.*

Condition	Graph	$\gamma_{eri}(P_n+P_m)$	${\gamma_{eri}}^{-1}(P_n+P_m)$
$n=1$	P_1+P_m	0	0
$n=2$ and $m=3k\geq 5$	P_2+P_{3k}	0	0
$n=2$ and $m=3k+1\geq 5$	P_2+P_{3k+1}	$\lceil \frac{m}{3} \rceil + 1$	0
$n=2$ and $m=3k+2\geq 5$	P_2+P_{3k+2}	0	0
$n=3$ and $m=3k\geq 6$	P_3+P_{3k}	$\lceil \frac{m}{3} \rceil + 1$	0
$n=3$ and $m=3k+2\geq 6$	P_3+P_{3k+2}	$\lceil \frac{m}{3} \rceil + 1$	0

Table 7. Summary of P_n+P_m graphs with no ${\gamma_{eri}}^{-1}$-set

5. Concluding Remarks

The notion of inverse equitable rings domination has been successfully introduced in this paper with some key conditions for the existence of inverse equitable rings dominating sets in graphs, and determined their inverse equitable rings domination numbers up to a binary operation: join of path-to-path graphs. One definite extension of this research is the inverse equitable rings domination number of the join of path-to-cycle, join of cycle-to-cycle, corona of graphs, and cartesian product of graphs that the authors had already started working on. Meanwhile, the authors recommend other researchers to study results and existence of this notion to other binary operations not covered in this study. Moreover, exploring the applications of this concept in other areas of mathematics could be an interesting research direction.

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Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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