

ON THE NONEXISTENCE OF CERTAIN PERFECT PERMUTATION CODES UNDER CROWN POSET METRICS

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ABSTRACT. In this study, we investigated the non-existence of perfect permutation codes under the crown poset metric for different types of permutation codes. These include the permutation code generated by permutation $(n + 1 \ n + 2 \ \cdots \ 2n)$, permutation code generated by two permutations $(1 \ 2 \ 3 \ \cdots \ n)$ and $(n + 1 \ n + 2 \ n + 3 \ \cdots \ 2n)$, permutation code generated by permutation $(1 \ 3 \ 5 \ \cdots \ 2n - 1)(2 \ 4 \ 6 \ \cdots \ 2n)$, permutation code generated by permutation $(1 \ n + 1 \ 2 \ n + 2 \ \cdots \ n \ 2n)$, and permutation code generated by permutation $(1 \ 2n \ 2 \ 2n - 1 \ \cdots \ n \ n + 1)$.

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1. INTRODUCTION

Slepian [4] introduced the concept of permutation modulation and demonstrated that permutation codes exhibit a strong performance on an additive white Gaussian noise (AWGN) channel. Moreover, the maximum-likelihood decoders for these permutation codes were relatively straightforward. However, the implementation of these codes requires large lookup tables, rendering them quite complex. Wolf [13] introduced the concept of utilizing permutation codes for the development of high-rate runlength-limited codes that possess both error detection and correction capabilities. Wolf demonstrated that these codes asymptotically achieve the capacity of a noiseless, runlength-limited constrained channel. However, they did not provide efficient encoding or decoding mechanisms. Accordingly, Datta et al. [22] presented an enumeration scheme designed to encode and decode permutation codes with low complexity. Although the enumeration scheme is applicable to any use of permutation codes, the authors focused on its application in the construction of runlength-limited codes. In this case, enumerative encoding and decoding algorithms based on enumeration of combinations [6] were employed. However, a limitation of this indirect enumeration technique is its requirement for successive reenumeration of symbols within codewords, as combinations, rather than permutations, are counted.

In this correspondence, Milenkovic et al. [18] proposed an algorithm based on the enumeration of permutations of a multiset, which obviates the need for reenumeration and thereby enhances efficiency. Ian F. Blake [11] introduced permutation codes for discrete channels by employing the concept presented in [5]. Permutation codes have garnered renewed interest due to their crucial applications in data transmission over powerlines [9], [17], [3], [24], [15], as well as their role in designing block ciphers [7], [25] and advancing multilevel flash memory technologies [1], [2], [8].

In 2015, Buzaglo et al. [23] established novel bounds for the size of permutation codes. They demonstrated the nonexistence of perfect single-error-correcting codes in S_n for cases where, $n > 4$ is a prime or $3 < n < 11$. In 2016, Kong et al. [14] established that perfect permutation codes does not exists by using the Ulam metric. They developed a technique for determining sphere size through the application of Young tableaux. This approach is crucial for proving the nonexistence of perfect permutation codes. In 2021, Wang et al. [29] addressed the unresolved problem initially proposed by Buzaglo and Etzion [23]. They introduced a polynomial expression to represent the size of a ball in S_n using the Kendall τ metric for a specified radius r and identified certain conditions that are sufficient to prove the absence of perfect permutation codes. Moreover, they demonstrated that no perfect t -error-correcting code exists in S_n under the Kendall τ metric for the specific values of $1 < t < 6$ and n or $\frac{5}{8}(\frac{n}{2}) \leq 2t + 1 \leq (\frac{n}{2})$. In 2023, Wang et al. [27] demonstrated the nonexistence of perfect error-correcting codes in S_n under the Hamming metric for an expanded range of values of t and n . They specifically proposed sufficient conditions for the nonexistence of perfect permutation codes. Furthermore, they established that no perfect error-correcting code exists in S_n under the Hamming metric for certain values of $t = 1, 2, 3, 4$ and n , or when $2t + 1 \leq n \leq \max\{4t^2e^{-2+1/t} - 2, 2t + 1\}$ for $t \geq 2$, or $\min\{\frac{e}{2}\sqrt{n+2}, \lfloor \frac{n-1}{2} \rfloor\} \leq t \leq \lfloor \frac{n-1}{2} \rfloor$ for $n \geq 7$, where, e represents Napier's constant. In 2024, Wang et al. [28] conducted a study on the nonexistence of perfect codes in S_n under the l_∞ metric. They established a sufficient condition for the nonexistence of perfect permutation codes within this metric framework. In addition, they utilized these conditions to demonstrate the absence of perfect t -error-correcting codes in S_n under the Hamming metric for certain values of t and n . Specifically, they proved that no perfect t -error-correcting code exists in S_n under the l_∞ metric for $1 \leq t \leq 3$. Additionally, they demonstrated that a perfect t -error-correcting code does not exist in S_n under the l_∞ metric for the values of t and n , where, $1 \leq t \leq 3$ and $2t + 1 \leq n$, or for t and n , where, $R_{2t+1}(n) = 0, 1, 2t$. Here, $0 \leq R_d(n) \leq d$ represents the remainder when $n \in \mathbb{N}$ is divided by the $d \in \mathbb{N}$.

In 1995, Brualdi et al. [20] introduced a metric concept for vector spaces based on partial order over a finite set, known as a poset. These metrics, referred to as poset metrics, extend both the traditional Hamming metric and Niederreiter Rosenbloom Tsfasman metric. General poset metric offers new insights into many classical coding theory invariants, such as minimum distance, packing, and covering radius, as well as fundamental results related to perfect and MDS codes and syndrome decoding. This

perspective enhances the understanding of these invariants and properties when viewed through the lens of the classical and widely used Hamming metric, in which perfect codes with a poset metric are studied using three different approaches: a) fixing a family of codes, b) fixing a family of posets, and c) operations on a given perfect code [16]. Utilizing the approach of fixing a family of posets, Ahn et al. [12] comprehensively characterized the parameters of single and double error-correcting perfect linear codes with a crown poset structure by solving a Ramanujan–Nagell-type Diophantine equation. In [10], Kim et al. provided a more concise proof of the same result by analyzing a generator matrix of a perfect linear code. Furthermore, they integrated their method with the Johnson bound in coding theory to demonstrate the non-existence of triple-error-correcting perfect binary codes with a crown-poset structure. Despite their practical significance and theoretical elegance, the concept of permutation codes has not been extensively investigated within the framework of poset metrics. Consequently, driven by the theoretical and practical importance of both poset metrics and permutation codes, this study examined the nonexistence of certain perfect permutation codes under the crown poset metric. In this study, we selected a certain permutation codes generated by one or two permutations owing to their well-defined algebraic structures. These codes naturally interact with the layered dependency model of the crown poset, enabling a rigorous examination of how group actions influence invariants of coding theory, such as the minimum distance, packing radius, and covering radius.

In this article, we have worked on nonexistence of certain perfect permutation codes under the crown poset metric. The paper is organized as follows: In Section 2, definition of poset code is recalled. In Section 3, permutation codes under the poset metric are defined. In Section 4, permutation codes under the crown poset metric are presented. Nonexistence of certain perfect permutation codes under crown poset metric is discussed in Section 5. Section 6 contains concluding remarks.

Abbreviations and Acronyms:

Notation	Meaning
$G = \langle\langle g \rangle\rangle$	G is generated by g , i.e., G contains g and all its powers.
$I = \langle \text{supp}(x) \rangle$	I is the smallest ideal containing the support of x .
$[2n] = \{1, 2, \dots, n, n+1, \dots, 2n\}$	The set of integers from 1 to $2n$.

TABLE 1. List of Notations and Their Meanings

2. PRELIMINARIES

In 1995, Brualdi et al. [20] introduced the concept of a poset metric, which is a metric defined on a vector space $\mathbb{F}_q^n$ over a field $\mathbb{F}_q$, with partial ordering ($\preceq$) imposed on a finite set P containing elements from 1 to n , specifically, $P = \{1, 2, \dots, n\}$. An ideal I is defined as a subposet of P with the characteristic that if $y \in I$ and $z \preceq y$, then $z \in I$. The poset weight of vector $y \in \mathbb{F}_q^n$ is determined by

the cardinality of the smallest ideal of P that encompasses the support of y . Formally, this is expressed as $\varpi_P(y) = |\langle \text{supp}(y) \rangle|$, where, $\text{supp}(y) = \{j | y_j \neq 0\}$. The poset distance between any two vectors y and z in $\mathbb{F}_q^n$ is defined as $d_P(y, z) = \varpi_P(y - z)$. This distance adheres to all properties of a metric and is referred to as the *poset metric*. A linear subspace C of $\mathbb{F}_q^n$ that possesses a poset metric with dimension k and minimum distance d_P , is termed as a *poset code* with parameters $[n, k, d_P]$.

3. PERMUTATION CODES UNDER POSET METRIC

In this article, $[n]$ denotes the set $\{1, 2, \dots, n\}$. The set of all permutation over $[n]$ is called symmetric group denoted as S_n . Let $P = ([n], \preceq)$ be a poset over $[n]$. Given a permutation $\pi \in S_n$ we define permutation poset weight of π by

$$wt_{PP}(\pi) = |\langle \text{supp}(\pi) \rangle|$$

where, $\text{supp}(\pi) = \{i \in [n] : \pi(i) \neq i\}$.

Definition 3.1. *Permutation poset distance between any two permutations in S_n is defined as $d_{PP}(\sigma, \pi) = wt_{PP}(\sigma, \pi)$ where, $wt_{PP}(\sigma, \pi) = |\langle \text{supp}(\sigma, \pi) \rangle|$, $\text{supp}(\sigma, \pi) = \{i \in [n] | \sigma(i) \neq \pi(i)\}$ and $\langle \text{supp}(\sigma, \pi) \rangle$ is the smallest ideal generated by $\text{supp}(\sigma, \pi)$.*

Theorem 3.2. [21] *If P is a poset of n elements, then permutation poset distance i.e., d_{PP} is a metric on S_n .*

We call the metric $d_{PP}(\cdot, \cdot)$ on S_n as *Permutation Poset metric*.

Definition 3.3. *Permutation codes having length n and minimum distance δ_{PP} are defined as a subsets of symmetric group S_n .*

Definition 3.4. *For a given a permutation code $C \subset S_n$, Minimum Distance is denoted as $\delta_{PP}(C)$ and defined as $\delta_{PP}(C) := \min\{d_{PP}(\pi, \sigma) : \pi, \sigma \in C, \pi \neq \sigma\}$.*

Definition 3.5. *For a given a permutation code $C \subset S_n$, ball of radius r is denoted as $B_{d_{PP}}(\pi, r)$ and defined as $B_{d_{PP}}(\pi, r) = \{\sigma \in S_n | d_{PP}(\sigma, \pi) \leq r\}$.*

Definition 3.6. *For a given a permutation code $C \subset S_n$, the Packing Radius is denoted as $R_{d_{PP}}(C)$ and defined as $R_{d_{PP}}(C) = \max\{r \in [n] : B_{d_{PP}}(\pi, r) \cap B_{d_{PP}}(\sigma, r) = \emptyset, \forall \pi, \sigma \in C, \pi \neq \sigma\}$.*

Definition 3.7. *For a given a permutation code $C \subset S_n$, Covering Radius is denoted as $CV_{d_{PP}}(C)$ and defined as $CV_{d_{PP}}(C) := \min\{r \in \mathbb{Z}^+ : S_n = \bigcup_{\pi \in C} B_{d_{PP}}(\pi, r)\}$.*

Definition 3.8. *Given a subset C of S_n , the covering radius of C is denoted as $CV_{d_{PP}}(C)$, is defined as $CV_{d_{PP}}(C) = \max_{h \in S_n} \min_{g \in C} d(g, h)$. In other words, it is the smallest radius r such that balls of radius r centered at the elements of C covers the entire space S_n .*

Definition 3.9. A permutation code is said to be perfect permutation code if its packing radius is equal to its covering radius.

Theorem 3.10. [21] For a given permutation code C having parameter (n, M, δ_{PP}) , the packing radius $R_{d_{PP}}(C)$ satisfies the following inequalities:

$$\lfloor \frac{\delta_{PP}(C) - 1}{2} \rfloor \leq R_{d_{PP}}(C) \leq \delta_{PP}(C) - 1 \quad (1)$$

Theorem 3.11. [21] For a given permutation code C having parameter (n, M, δ_{PP}) , the covering radius $CR_{d_{PP}}(C)$ satisfies the following inequality:

$$CR_{d_{PP}}(C) \leq n \quad (2)$$

4. PERMUTATION CODES UNDER CROWN POSET METRIC

Definition 4.1. Crown Poset is a poset $Cr = ([2n], \preceq_{cr})$ where the only relations are:

$$n \preceq_{cr} n + 1, \quad n \preceq_{cr} 2n \quad \text{and} \quad j \preceq_{cr} n + j, \quad j \preceq_{cr} n + j + 1, \quad \forall 1 \leq j \leq n - 1. \quad (3)$$

Let $[2n]$ denotes the set $\{1, 2, \dots, 2n\}$. The set of all permutation over $[2n]$ is called symmetric group denoted by S_{2n} . Let $Cr = ([2n], \preceq)$ be a crown poset over $[2n]$. Given a permutation $\pi \in S_{2n}$ we define permutation crown poset weight of π by

$$wt_{cr}(\pi) = |\langle \text{supp}(\pi) \rangle|$$

where, $\text{supp}(\pi) = \{i \in [2n] : \pi(i) \neq i\}$.

Definition 4.2. Permutation crown poset distance between any two permutation belong to S_{2n} is defined as $d_{cr}(\sigma, \pi) = wt_{cr}(\sigma, \pi)$ where, $wt_{cr}(\sigma, \pi) = |\langle \text{supp}(\sigma, \pi) \rangle|$, $\text{supp}(\sigma, \pi) = \{i \in [2n] | \sigma(i) \neq \pi(i)\}$ and $\langle \text{supp}(\sigma, \pi) \rangle$ is the smallest ideal generated by $\text{supp}(\sigma, \pi)$.

Remark 4.3. Let $Cr = ([2n], \preceq)$ be a crown poset over $[2n]$ then permutation crown poset distance i.e., d_{cr} is a metric on S_{2n} .

Definition 4.4. For a given permutation code $C \subset S_{2n}$, minimum distance is denoted as $\delta_{cr}(C)$ and defined as $\delta_{cr}(C) := \min\{d_{cr}(\pi, \sigma) : \pi, \sigma \in C, \pi \neq \sigma\}$.

Definition 4.5. Permutation codes having length $2n$, size M and minimum distance δ_{cr} under crown poset metric are defined as a subsets of symmetric group S_{2n} .

Definition 4.6. For a given a permutation code $C \subset S_{2n}$, ball of radius r is denoted as $B(\pi, r)$ and defined as $B(\pi, r) = \{\sigma \in S_{2n} | d_{cr}(\sigma, \pi) \leq r\}$.

Definition 4.7. For a given a permutation code $C \subset S_{2n}$, the Packing Radius is denoted as $R_{d_{cr}}(C)$ and defined as $R_{d_{cr}}(C) = \max\{r \in [2n] : B(\pi, r) \cap B(\sigma, r) = \emptyset, \forall \pi, \sigma \in C, \pi \neq \sigma\}$.

Definition 4.8. For a given a permutation code $C \subset S_{2n}$, Covering Radius is denoted as $CV_{d_{cr}}(C)$ and defined as $CV_{d_{cr}}(C) := \min\{r \in \mathbb{Z}^+ : S_{2n} = \bigcup_{\pi \in C} B(\pi, r)\}$.

Definition 4.9. Given a subset C of S_{2n} , the covering radius of C is denoted as $CV_{d_{cr}}(C)$, is defined as $CV_{d_{cr}}(C) = \max_{h \in S_{2n}} \min_{g \in C} d(g, h)$. In other words, it is the smallest radius r such that balls of radius r centered at the elements of C covers the entire space S_{2n} .

Definition 4.10. A permutation code is said to be perfect permutation code if its packing radius is equal to its covering radius.

Theorem 4.11. [21] For a given permutation code C having parameter $(2n, M, \delta_{cr})$, the packing radius $R_{d_{cr}}(C)$ satisfies the following inequalities:

$$\lfloor \frac{\delta_{cr}(C) - 1}{2} \rfloor \leq R_{d_{cr}}(C) \leq \delta_{cr}(C) - 1 \quad (4)$$

Theorem 4.12. [21] For a given permutation code C having parameter $(2n, M, \delta_{cr})$, the covering radius $CV_{d_{cr}}(C)$ satisfies the following inequality:

$$CV_{d_{cr}}(C) \leq 2n \quad (5)$$

5. NONEXISTENCE OF CERTAIN PERFECT PERMUTATION CODES UNDER CROWN POSET METRIC

Theorem 5.1. The minimum distance of the cyclic group generated by $(n + 1 \ n + 2 \ \cdots \ 2n)$ permutation, denoted as, C_1 is $2n$ under the crown poset metric.

Proof. The support of any two distinct permutations belong to C_1 contains $\{n + 1, n + 2, n + 3, \dots, 2n\}$ elements. Subsequently, its ideal contains $\{1, 2, \dots, 2n\}$. Therefore, $d_{cr}(\tau, \pi) = 2n$ for all $\tau, \pi \in C_1$ and $\tau \neq \pi$. Hence, minimum distance of the C_1 is $2n$ under the crown poset metric. $\square$

Theorem 5.2. The packing radius of C_1 is $n + 1$, $n + 1$ and $n + 2$ when $n \in \mathbb{N} \setminus \{1\}$ is odd, $n \equiv 0 \pmod{4}$ and $n \equiv 2 \pmod{4}$, respectively, under the crown poset metric.

Proof. As established in the Theorem 4.11 packing radius of C_1 bounded between $\lfloor \frac{2n-1}{2} \rfloor$ and $2n - 1$.

Case 1: If $n \equiv 1 \pmod{4}$, $n > 1$ and $n \equiv 3 \pmod{4}$

To establish that the packing radius of C_1 for $n \equiv 1 \pmod{4}$, $n > 1$ and $n \equiv 3 \pmod{4}$ is not $n + 2$ it is sufficient to present a counterexample.

Consider, I as the Identity permutation, $\sigma = (n + 1 \ n + 2 \ \cdots \ 2n) \in C_1$ and $\alpha = \left(\frac{n-1}{2} \ n + \frac{n+1}{2} \ n + \frac{n+3}{2} \ \cdots \ 2n\right) \in S_{2n}$ then,

$$\begin{aligned}
\text{supp}(I, \alpha) &= \left\{ \frac{n-1}{2}, n + \frac{n+1}{2}, n + \frac{n+3}{2}, \dots, 2n \right\} \\
\Rightarrow \langle \text{supp}(I, \alpha) \rangle &= \text{supp}(I, \alpha) \cup \left\{ \frac{n+1}{2}, \frac{n+3}{2}, \dots, n \right\}. \\
\Rightarrow d_{cr}(I, \alpha) &= n + 2.
\end{aligned} \tag{6}$$

Hence, $\alpha \in B(I, n + 2)$.

$$\begin{aligned}
\text{supp}(\sigma, \alpha) &= \left\{ \frac{n-1}{2}, n + 1, n + 2, \dots, n + \frac{n-1}{2}, 2n \right\} \\
\langle \text{supp}(\sigma, \alpha) \rangle &= \text{supp}(\sigma, \alpha) \cup \left\{ 1, 2, \dots, \frac{n-3}{2}, n-1, n \right\} \\
\Rightarrow d_{cr}(\sigma, \alpha) &= n + 2
\end{aligned} \tag{7}$$

Therefore, $\alpha \in B(\sigma, n + 2)$

By using (6) and (7), we can conclude that $\alpha \in B(\sigma, n + 2) \cap B(I, n + 2)$. Hence, By using definition of packing radius, we can conclude that packing radius of C_1 is less than $n + 2$ when $n \in \mathbb{N} \setminus \{1\}$ is odd. Now, we will prove that packing radius of C_1 is $n + 1$ where, $n \equiv 1 \pmod{4}$ and $n \equiv 3 \pmod{4}$. Consider, $\nu, \theta \in C_1$ then, $\text{supp}(\nu, \theta) = \{n + 1, n + 2, \dots, 2n\}$ and $d_{cr}(\nu, \theta) = 2n$. Suppose that $\beta \in B(\nu, n + 1) \cap B(\theta, n + 1)$ then $d_{cr}(\nu, \beta) \leq n + 1$ and $d_{cr}(\theta, \beta) \leq n + 1$.

If there are $\frac{n+1}{2}$ elements in $\text{supp}(\nu, \beta)$ from set $\{n + 1, n + 2, \dots, 2n\}$, then its ideal contains at least $n + 2$ elements. Consequently, $d_{cr}(\nu, \beta) \geq n + 2 > n + 1$, which results in a contradiction. Therefore, $\text{supp}(\nu, \beta)$ contains less than or equal to $\frac{n-1}{2}$ elements from $\{n + 1, n + 2, \dots, 2n\}$. If $\text{supp}(\nu, \beta)$ contains less than or equal to $\frac{n-1}{2}$ elements from $\{n + 1, n + 2, \dots, 2n\}$, then $\text{supp}(\theta, \beta)$ contains at least $\frac{n+1}{2}$ elements from set $\{n + 1, n + 2, \dots, 2n\}$. Therefore, $|\langle \text{supp}(\theta, \beta) \rangle| \geq n + 2$, which implies that $d_{cr}(\theta, \beta) \geq n + 2$, leading to a contradiction. Hence, we conclude that $B(\nu, n + 1) \cap B(\theta, n + 1) = \phi$. Therefore, the packing radius of C_1 is $n + 1$ when $n \equiv 1 \pmod{4}$, $n > 1$, and $n \equiv 3 \pmod{4}$.

Case 2: If $n \equiv 0 \pmod{4}$

To demonstrate that the packing radius of C_1 for $n \equiv 0 \pmod{4}$ is not $n + 2$ it is sufficient to provide counterexample.

Consider, I as the Identity permutation, $\pi = \left(n + 1 \ n + 1 + \frac{n}{2}\right) \left(n + 2 \ n + 2 + \frac{n}{2}\right) \cdots \left(n + \frac{n}{2} \ 2n\right) \in C_1$ and $\alpha = \left(n + 1 \ n + 1 + \frac{n}{2}\right) \cdots \left(n + \frac{n}{4} \ n + \frac{n}{4} + \frac{n}{2}\right) \in S_{2n}$ then,

$$\begin{aligned}
\text{supp}(I, \alpha) &= \left\{ n + 1, n + 2, \dots, n + \frac{n}{4}, n + 1 + \frac{n}{2}, \dots, n + \frac{n}{4} + \frac{n}{2} \right\} \\
\Rightarrow \langle \text{supp}(I, \alpha) \rangle &= \text{supp}(I, \alpha) \cup \left\{ 1, 2, \dots, \frac{n}{4}, \frac{n}{2} + 1, \frac{n}{2} + 2, \dots, \frac{n}{2} + \frac{n}{4}, \frac{n}{2}, n \right\}. \\
\Rightarrow d_{cr}(I, \alpha) &= n + 2.
\end{aligned} \tag{8}$$

Hence, $\alpha \in B(I, n + 2)$.

$$\begin{aligned}
\text{supp}(\pi, \alpha) &= \left\{ n + \frac{n}{4} + 1, n + \frac{n}{4} + 2, \dots, n + \frac{n}{2}, n + \frac{n}{4} + 1 + \frac{n}{2}, \dots, 2n \right\} \\
\langle \text{supp}(\pi, \alpha) \rangle &= \text{supp}(\pi, \alpha) \cup \left\{ \frac{n}{4} + 1, \dots, \frac{n}{2}, \frac{n}{2} + \frac{n}{4} + 1, \frac{n}{2} + \frac{n}{4} + 2, \dots, n, \frac{n}{4} - 1, \frac{n}{2} + \frac{n}{4} \right\} \\
\therefore d_{cr}(\pi, \alpha) &= n + 2 \\
\implies \alpha &\in B(\pi, n + 2)
\end{aligned} \tag{9}$$

By using (8) and (9), we can infer that $\alpha \in B(\pi, n + 2) \cap B(I, n + 2)$. Hence, by using definition of packing radius, we can deduce that packing radius of C_1 is less than $n + 2$ when $n \equiv 0 \pmod{4}$.

Now, we will prove that the packing radius of C_1 is $n + 1$ where, $n \equiv 0 \pmod{4}$. Let, $\nu, \mu \in C_1$ then, $\text{supp}(\nu, \mu) = \{n + 1, n + 2, \dots, 2n\}$ and $d_{cr}(\nu, \mu) = 2n$.

Suppose that $\gamma \in B(\nu, n + 1) \cap B(\mu, n + 1)$ then $d_{cr}(\nu, \gamma) \leq n + 1$ and $d_{cr}(\mu, \gamma) \leq n + 1$. If there are $\frac{n}{2} + 1$ elements in $\text{supp}(\nu, \gamma)$ from the set $\{n + 1, n + 2, \dots, 2n\}$, then its ideal contains at least $n + 3$ elements, therefore, it follows that $d_{cr}(\nu, \gamma) \geq n + 3 > n + 1$, which constitutes a contradiction. Therefore, $\text{supp}(\nu, \gamma)$ must contain fewer than or equal to $\frac{n}{2}$ elements from the set $\{n + 1, n + 2, \dots, 2n\}$. If $\text{supp}(\nu, \gamma)$ contains fewer than $\frac{n}{2}$ elements from set $\{n + 1, n + 2, \dots, 2n\}$, it follows that $\text{supp}(\mu, \gamma)$ must include at least $\frac{n}{2} + 1$ elements from this set. Consequently, $|\langle \text{supp}(\mu, \gamma) \rangle| \geq n + 3 > n + 1$. This leads to the conclusion that $d_{cr}(\mu, \gamma) > n + 1$, which is a contradiction. If $\text{supp}(\nu, \gamma)$ includes $\frac{n}{2}$ elements from set $\{n + 1, n + 2, \dots, 2n\}$, it follows that $\text{supp}(\mu, \gamma)$ must contain at least $\frac{n}{2} + 1$ elements from the same set. Consequently, $|\langle \text{supp}(\mu, \gamma) \rangle| \geq n + 3 > n + 1$. This leads to the conclusion that $d_{cr}(\mu, \gamma) > n + 1$, which is a contradiction. Hence, we deduced that the $B(\nu, n + 1) \cap B(\mu, n + 1) = \phi$. Therefore, Packing radius of C_1 is $n + 1$ when $n \equiv 0 \pmod{4}$.

Case 3: If $n \equiv 2 \pmod{4}$

To refute the claim that the packing radius of T for $n \equiv 2 \pmod{4}$ is not $n + 3$ it is adequate to offer a counterexample.

Consider, I as the Identity permutation, $\sigma = (n + 1 \ n + 2 \ \dots \ 2n) \in C_1$ and $\alpha = (n + 1 \ n + 2 \ \dots \ n + \frac{n}{2} + 1) \in S_{2n}$ then,

$$\begin{aligned}
\text{supp}(I, \alpha) &= \left\{ n + 1, n + 2, \dots, n + \frac{n}{2} + 1 \right\} \\
\implies \langle \text{supp}(I, \alpha) \rangle &= \text{supp}(I, \alpha) \cup \left\{ 1, 2, \dots, \frac{n}{2} + 1, n \right\} \\
\implies d_{cr}(I, \alpha) &= n + 3.
\end{aligned} \tag{10}$$

Hence, $\alpha \in B(I, n + 3)$.

$$\begin{aligned}
\text{supp}(\sigma, \alpha) &= \left\{ n + \frac{n}{2} + 1, n + \frac{n}{2} + 2, \dots, 2n \right\} \\
\implies \langle \text{supp}(\sigma, \alpha) \rangle &= \text{supp}(\sigma, \alpha) \cup \left\{ \frac{n}{2}, \frac{n}{2} + 1, \dots, n \right\} \\
\implies d_{cr}(\sigma, \alpha) &= n + 1
\end{aligned} \tag{11}$$

$\therefore \alpha \in B(\sigma, n + 3)$

By using (10) and (11), we can ascertain that $\alpha \in B(\sigma, n+3) \cap B(I, n+3)$. Hence, By using definition of packing radius, we can deduce that packing radius of C_1 is less than $n+3$ when $n \equiv 2 \pmod{4}$.

Now, we will establish that the packing radius of C_1 is $n+2$ where, $n \equiv 2 \pmod{4}$. Let, $\nu, \theta \in C_1$ then, $\text{supp}(\nu, \theta) = \{n+1, n+2, \dots, 2n\}$ and $d_{cr}(\nu, \theta) = 2n$. Suppose that $\beta \in B(\nu, n+2) \cap B(\theta, n+2)$ then $d_{cr}(\nu, \beta) \leq n+2$ and $d_{cr}(\theta, \beta) \leq n+2$. If there are $\frac{n}{2} + 1$ elements in $\text{supp}(\nu, \beta)$ from set $\{n+1, n+2, \dots, 2n\}$, then its ideal must contain at least $n+3$ elements. Consequently, $d_{cr}(\nu, \beta) \geq n+3 > n+2$, which results in a contradiction. Therefore, $\text{supp}(\nu, \beta)$ contains at most $\frac{n}{2}$ elements from set $\{n+1, n+2, \dots, 2n\}$. If $\text{supp}(\nu, \beta)$ contains fewer than $\frac{n}{2}$ elements from set $\{n+1, n+2, \dots, 2n\}$, it follows that $\text{supp}(\theta, \beta)$ must include at least $\frac{n}{2} + 1$ elements from this set. Consequently, $|\langle \text{supp}(\theta, \beta) \rangle| \geq n+3 > n+2$, This leads to the conclusion that $d_{cr}(\theta, \beta) \geq n+2$, which is a contradiction. If $\text{supp}(\nu, \beta)$ includes $\frac{n}{2}$ elements from set $\{n+1, n+2, \dots, 2n\}$, it follows that $\text{supp}(\theta, \beta)$ must contain at least $\frac{n}{2} + 1$ elements from the same set. Consequently, $|\langle \text{supp}(\theta, \beta) \rangle| \geq n+3 > n+2$, this implies that $d_{cr}(\theta, \beta) > n+2$, which is a contradiction. If $\text{supp}(\nu, \beta)$ contains $\frac{n}{2}$ elements from the set $\{n+1, n+2, \dots, 2n\}$ and $n \equiv 2 \pmod{4}$, then $\pi = \sigma^{\frac{n}{2}}$ is the sole permutation within the set C_1 that is a product of transpositions, with the number of transpositions being odd. Consider, $\nu = \pi$ for such π . Then, the support of (θ, β) contains elements from set $\{n+1, n+2, \dots, 2n\}$ in at least $\frac{n}{2} + 1$ instances. Consequently, the cardinality of $\langle \text{supp}(\theta, \beta) \rangle$ is at least $n+3$, which exceeds $n+2$. This implies that $d_{cr}(\theta, \beta) > n+2$, leading to a contradiction. In this case, the second choice for ν is σ . Suppose $\text{supp}(\nu, \alpha) = \{n+1, \dots, n+\frac{n}{2}\}$. This implies that $\text{supp}(\theta, \beta) = \{n+\frac{n}{2}, n+\frac{n}{2}+1, \dots, 2n\}$. Therefore, $|\langle \text{supp}(\theta, \beta) \rangle| \geq n+3 > n+2$, It implies that $d_{cr}(\theta, \beta) > n+2$ which is a contradiction. Hence, it can be concluded that the $B(\nu, n+2) \cap B(\theta, n+2) = \phi$. Therefore, Packing radius of C_1 is $n+2$ when $n \equiv 2 \pmod{4}$.

By considering all the cases we can conclude that packing radius of C_1 is $n+1$, $n+1$ and $n+2$ when $n \in \mathbb{N} \setminus \{1\}$ is odd, $n \equiv 0 \pmod{4}$ and $n \equiv 2 \pmod{4}$ respectively under the crown poset metric. $\square$

Theorem 5.3. *Covering radius of C_1 is $2n$.*

Proof. Consider the permutation $\beta = (1\ 2n)(2\ 2n-1) \cdots (n\ n+1) \in S_{2n}$. For any $\pi \in C_1$, it follows that $\text{supp}(\pi, \beta) = \{1, 2, 3, \dots, 2n\}$, thereby indicating that its ideal encompasses $\{1, 2, 3, \dots, 2n\}$. This implies that $d_{cr}(\pi, \beta) = 2n$. Therefore, the covering radius of C_1 is at least $2n$. Furthermore, according to Theorem 4.12, the covering radius is less than or equal to $2n$. Hence, the covering radius of C_1 is $2n$. $\square$

Theorem 5.4. *Cyclic group generated by the $(n+1\ n+2\ \cdots\ 2n)$ permutation denoted as, C_1 is not perfect code for any values of $n \in \mathbb{N} \setminus \{1\}$ under the crown poset metric.*

Proof. The packing and covering radii of C_1 are not the same. Utilizing (4.10), we can deduce that C_1 does not constitute perfect code. $\square$

Example 5.5. Consider $n = 3$, then permutation code C_1 is generated by the permutation $(4\ 5\ 6)$. $C_1 = \{(4\ 5\ 6), (4\ 6\ 5), \text{Id}(C_1)\}$. Relation of crown poset are defined as follows: $1 \preceq_{cr} 4, 1 \preceq_{cr} 5, 2 \preceq_{cr} 5, 2 \preceq_{cr} 6, 3 \preceq_{cr} 4, 3 \preceq_{cr} 6$. The support of any two distinct permutations belonging to C_1 comprises the elements $\{4, 5, 6\}$. By applying the definitions of an ideal and crown poset, it can be inferred that $d_{cr}(\sigma, \pi) = 6$ for all $\sigma, \pi \in C_1$ where, $\sigma \neq \pi$. Consequently, the minimum distance of C_1 is determined to be 6 for $n = 3$. By constructing program in Magma computational algebra system software [26], it is verified that balls of radius 0, 1, 2, 3, and 4 are disjoint, with the first overlap occurring at radius 5. For instance, consider $\sigma = \text{Id}(C_1)$ and $\pi = (4\ 5\ 6)$; the common element in both balls is $(1\ 6\ 4)$. Thus, by employing the definition of the packing radius, we can conclude that the packing radius is 4. Furthermore, using Magma software, it is verified that the covering radius of C_1 is 6 when n is 3. Since, the packing radius and covering radius of C_1 are distinct. Utilizing (4.10), it can be deduced that C_1 does not constitute a perfect code under the crown poset metric.

Theorem 5.6. If $C_2 = \langle \langle \sigma, \pi \rangle \rangle$ where, $\sigma = (1\ 2\ 3 \cdots n)$ and $\pi = (n+1\ n+2\ n+3 \cdots 2n)$ then minimum distance is n under the crown poset metric for $n > 1$.

Proof. We have, $\sigma = (1\ 2\ 3 \cdots n)$ and $\pi = (n+1\ n+2\ n+3 \cdots 2n)$. Consider, $A = \{1, 2, \dots, n\}$ and $B = \{n+1, n+2, \dots, 2n\}$. Here σ acts only on A and π acts only on B so each permutation $\tau_1 = \sigma^{i_1} \pi^{j_1}$ and $\tau_2 = \sigma^{i_2} \pi^{j_2}$ act as for any $x \in A$, $\tau_1(x) = \sigma^{i_1}(x)$, $\tau_2(x) = \sigma^{i_2}(x)$ and for any $x \in B$, $\tau_1(x) = \pi^{j_1}(x)$, $\tau_2(x) = \pi^{j_2}(x)$ so the comparison $\tau_1(x) \neq \tau_2(x)$ reduces to $x \in A \implies \sigma^{i_1}(x) \neq \sigma^{i_2}(x)$ and $x \in B \implies \pi^{j_1}(x) \neq \pi^{j_2}(x)$. We split the domain $[2n]$ into the disjoint union $A \cup B$. Then $\text{supp}(\tau_1, \tau_2) = \{x \in A | \sigma^{i_1}(x) \neq \sigma^{i_2}(x)\} \cup \{x \in B | \pi^{j_1}(x) \neq \pi^{j_2}(x)\}$. Hence, $\text{supp}(\tau_1, \tau_2) = \text{supp}(\sigma^{i_1}, \sigma^{i_2}) \cup \text{supp}(\pi^{j_1}, \pi^{j_2})$.

$$\begin{aligned} \text{supp}(\tau_1, \tau_2) &= \begin{cases} \phi, & \text{if } \sigma^{i_1} = \sigma^{i_2} \ \& \ \pi^{j_1} = \pi^{j_2} \\ [n], & \text{if } \sigma^{i_1} \neq \sigma^{i_2} \ \& \ \pi^{j_1} = \pi^{j_2} \\ [n+1, 2n], & \text{if } \sigma^{i_1} = \sigma^{i_2} \ \& \ \pi^{j_1} \neq \pi^{j_2} \\ [2n], & \text{if } \sigma^{i_1} \neq \sigma^{i_2} \ \& \ \pi^{j_1} \neq \pi^{j_2} \end{cases} \\ \langle \text{supp}(\tau_1, \tau_2) \rangle &= \begin{cases} \phi, & \text{if } \sigma^{i_1} = \sigma^{i_2} \ \& \ \pi^{j_1} = \pi^{j_2} \\ [n], & \text{if } \sigma^{i_1} \neq \sigma^{i_2} \ \& \ \pi^{j_1} = \pi^{j_2} \\ [2n], & \text{if } \sigma^{i_1} = \sigma^{i_2} \ \& \ \pi^{j_1} \neq \pi^{j_2} \\ [2n], & \text{if } \sigma^{i_1} \neq \sigma^{i_2} \ \& \ \pi^{j_1} \neq \pi^{j_2} \end{cases} \\ d_{cr}(\tau_1, \tau_2) &= \begin{cases} 0, & \text{if } \sigma^{i_1} = \sigma^{i_2} \ \& \ \pi^{j_1} = \pi^{j_2} \\ n, & \text{if } \sigma^{i_1} \neq \sigma^{i_2} \ \& \ \pi^{j_1} = \pi^{j_2} \\ 2n, & \text{if } \sigma^{i_1} = \sigma^{i_2} \ \& \ \pi^{j_1} \neq \pi^{j_2} \\ 2n, & \text{if } \sigma^{i_1} \neq \sigma^{i_2} \ \& \ \pi^{j_1} \neq \pi^{j_2} \end{cases} \end{aligned} \quad (12)$$

By using the definition of the minimum distance and (12) we can deduce that minimum distance of C_2 is n . $\square$

Theorem 5.7. *The packing radii of C_2 are $\frac{n}{2} - 1$, 1 and $\frac{n-1}{2}$ when $n > 2$ is even, $n = 2$ and $n \in \mathbb{N} \setminus \{1\}$ is odd respectively, under the crown poset metric.*

Proof. **Case 1:** If $n > 2$ is even

According to the Theorem 4.11 packing radius of C_2 lies between $\frac{n}{2} - 1$ and $n - 1$.

To challenge the assertion that the packing radius of C_2 for n is even is not $\frac{n}{2}$ it is sufficient to provide a counterexample.

Consider, $\sigma = (1 \ 2 \ \cdots \ n) \in C_2$, $\pi = \sigma^2 = (1 \ 3 \ 5 \ 7 \ \cdots \ n-1)(2 \ 4 \ 6 \ \cdots \ n) \in C_2$, I as the identity permutation and $\alpha = (1 \ 3 \ 5 \ 7 \ \cdots \ n-1) \in S_{2n}$ then,

$$\begin{aligned} \text{supp}(I, \alpha) &= \{1, 3, 5, 7, \dots, n-1\} \\ \implies \langle \text{supp}(I, \alpha) \rangle &= \{1, 3, 5, 7, \dots, n-1\} \\ \implies d_{cr}(I, \alpha) &= \frac{n}{2}. \\ \text{Hence, } \alpha &\in B\left(I, \frac{n}{2}\right). \end{aligned} \tag{13}$$

$$\begin{aligned} \text{supp}(\pi, \alpha) &= \{2, 4, 6, \dots, n\} \\ \implies \langle \text{supp}(\pi, \alpha) \rangle &= \{2, 4, 6, \dots, n\} \\ \implies d_{cr}(\pi, \alpha) &= \frac{n}{2} \\ \therefore \alpha &\in B\left(\pi, \frac{n}{2}\right) \end{aligned} \tag{14}$$

By using (13) and (14), it can be deduced that $\alpha \in B(I, \frac{n}{2}) \cap B(\pi, \frac{n}{2})$. Consequently, based on the definition of the packing radius, it can be concluded that the packing radius of C_2 is less than $\frac{n}{2}$. Furthermore, the packing radius is greater than or equal to $\frac{n}{2} - 1$ from the Theorem 4.11. Therefore, the packing radius of C_2 is $\frac{n}{2} - 1$ when $n > 2$ is even.

Case 2: For $n = 2$

If n is 2 then, C_2 includes identity permutation I and a transposition $(1 \ 2)$. Consider, $\sigma = (1 \ 2)$, I = Identity permutation and $\alpha = (1 \ 2)$

$$\begin{aligned} \text{supp}(\sigma, \alpha) &= \emptyset = \langle \text{supp}(\sigma, \alpha) \rangle \implies d_{cr}(\sigma, \alpha) = 0 \\ \text{supp}(I, \alpha) &= \{1, 2\} = \langle \text{supp}(I, \alpha) \rangle \implies d_{cr}(I, \alpha) = 2 \\ \therefore \alpha &\in B(\sigma, 2) \text{ and } \alpha \in B(I, 2). \end{aligned}$$

Hence, $\alpha \in B(\sigma, 2) \cap B(I, 2)$.

By using the definition of the packing radius we can deduce that the packing radius of C_2 is 1 when $n = 2$.

Case 3: If n is odd and $n > 1$

According to the Theorem 4.11 packing radius of C_2 constrained between $\frac{n-1}{2}$ and $n - 1$ when n is odd. To demonstrate packing radius of C_2 for n is odd is less than $\frac{n+1}{2}$ it is sufficient to prove by counterexample.

Consider, $\sigma = (1\ 2\ \cdots\ n) \in C_2$, I as the identity permutation of C_2 and $\alpha = \left(1\ 2\ \cdots\ \frac{n+1}{2}\right) \in S_{2n}$ then,

$$\begin{aligned} \text{supp}(\sigma, \alpha) &= \left\{ \frac{n+1}{2}, \dots, n \right\} \\ \implies \langle \text{supp}(\sigma, \alpha) \rangle &= \left[\frac{n+1}{2}, n \right] \\ \implies d_{cr}(\sigma, \alpha) &= \frac{n+1}{2}. \end{aligned} \tag{15}$$

$$\text{Hence, } \alpha \in B\left(\sigma, \frac{n+1}{2}\right).$$

$$\begin{aligned} \text{supp}(I, \alpha) &= \left\{ 1, 2, \dots, \frac{n+1}{2} \right\} \\ \implies \langle \text{supp}(I, \alpha) \rangle &= \left[1, \frac{n+1}{2} \right] \\ \implies d_{cr}(I, \alpha) &= \frac{n+1}{2} \end{aligned} \tag{16}$$

$$\text{Therefore, } \alpha \in B\left(I, \frac{n+1}{2}\right)$$

By using (15) and (16), it can concluded that the $\alpha \in B\left(\sigma, \frac{n+1}{2}\right) \cap B\left(I, \frac{n+1}{2}\right)$. Consequently, by applying the definition of packing radius, it can be inferred that the packing radius of C_2 is less than $\frac{n+1}{2}$ when n is odd. According to Theorem 4.11, the packing radius is greater than or equal to $\frac{n-1}{2}$. Therefore, the packing radius of C_2 is $\frac{n-1}{2}$ when $n > 1$ is odd.

Hence, packing radii of C_2 are $\frac{n}{2} - 1, 1$ and $\frac{n-1}{2}$ when $n \in \mathbb{N} \setminus \{2\}$ is even, $n = 2$ and $n \in \mathbb{N} \setminus \{1\}$ is odd respectively, under the crown poset metric. $\square$

Theorem 5.8. *Covering radius of C_2 is $2n$.*

Proof. Consider, the permutation $\beta = (1\ n+1)(2\ n+2)\cdots(n\ 2n) \in S_{2n}$. For any $\pi \in C_2$, it follows that $\text{supp}(\pi, \beta) = \{1, 2, 3, \dots, 2n\}$, indicating that its ideal encompasses $\{1, 2, 3, \dots, 2n\}$. Consequently, this implies that $d_{cr}(\pi, \beta) = 2n$. Therefore, the covering radius of C_2 is at least $2n$. Furthermore, according to Theorem 4.12, the covering radius is bounded above by $2n$. Thus, the covering radius of C_2 is precisely $2n$. $\square$

Theorem 5.9. *If $C_2 = \langle \langle \sigma, \pi \rangle \rangle$ where, $\sigma = (1\ 2\ 3\ \cdots\ n)$ and $\pi = (n+1\ n+2\ n+3\ \cdots\ 2n)$ then C_2 is not perfect code for any values of $n \in \mathbb{N} \setminus \{1\}$ under the crown poset metric.*

Proof. The packing radius and covering radius of C_2 are not identical it follows from (4.10) that C_2 cannot be classified as a perfect code. $\square$

Theorem 5.10. *Minimum distance of the cyclic group generated by $(1\ 3\ 5\ \cdots\ 2n-1)(2\ 4\ 6\ \cdots\ 2n)$ permutation denoted as, C_3 is $2n$ under the crown poset metric.*

Proof. The support of any two distinct permutations belonging to C_3 encompasses elements $\{1, 2, 3, 4, \dots, 2n\}$. Utilizing the definitions of an ideal and a crown poset, it can be concluded that $d_{cr}(\tau, \pi) = 2n$ for all $\tau, \pi \in C_3$ where, $\tau \neq \pi$. Hence, minimum distance of the C_3 is $2n$ under the crown poset metric. $\square$

Theorem 5.11. *The packing radius of C_3 is $n, n+1$ when $n \in \mathbb{N} \setminus \{1\}$ is even and odd respectively, under the crown poset metric.*

Proof. Case 1: If n is even

According to the Theorem 4.11 packing radius of C_3 lies between $\lfloor \frac{2n-1}{2} \rfloor$ and $2n-1$. To demonstrate packing radius of C_3 for n is even is less than $n+1$ it is sufficient to prove by counterexample.

Consider, I as the identity permutation, $\sigma = (1\ 3\ 5\ \cdots\ 2n-1)(2\ 4\ 6\ \cdots\ 2n)$, $\pi = \sigma^{n/2} = (1\ n+1)(2\ n+2)\ \cdots\ (\frac{n}{2}\ n+\frac{n}{2})\ \cdots\ (n\ 2n) \in C_3$, and $\alpha = (1\ n+1)(2\ n+2)\ \cdots\ (\frac{n}{2}\ n+\frac{n}{2}) \in S_{2n}$ then,

$$\begin{aligned} \text{supp}(I, \alpha) &= \left\{1, 2, \dots, \frac{n}{2}, n+1, n+2, \dots, n+\frac{n}{2}\right\} \\ \implies \langle \text{supp}(I, \alpha) \rangle &= \text{supp}(I, \alpha) \cup \{n\} \\ \implies d_{cr}(I, \alpha) &= n+1. \end{aligned} \tag{17}$$

Hence, $\alpha \in B(I, n+1)$.

$$\begin{aligned} \text{supp}(\pi, \alpha) &= \left\{\frac{n}{2}+1, \frac{n}{2}+2, \dots, n, n+\frac{n}{2}+1, \dots, 2n\right\} \\ \implies \langle \text{supp}(\pi, \alpha) \rangle &= \text{supp}(\pi, \alpha) \cup \left\{\frac{n}{2}\right\} \\ \implies d_{cr}(\pi, \alpha) &= n+1 \end{aligned} \tag{18}$$

$\therefore \alpha \in B(\pi, n+1)$

By employing (17) and (18), it can be deduced that $\alpha \in B(I, n+1) \cap B(\pi, n+1)$. Consequently, utilizing the definition of packing radius, it can be inferred that the packing radius of C_3 is less than $n+1$.

Now, we will prove that packing radius of C_3 is n when n is even. Let, $\nu, \theta \in C_3$ then, $\text{supp}(\nu, \theta) = \{1, 2, \dots, n, n+1, n+2, \dots, 2n\}$ and $d_{cr}(\nu, \theta) = 2n$. Assume that $\beta \in B(\nu, n) \cap B(\theta, n)$ then $d_{cr}(\nu, \beta) \leq n$ and $d_{cr}(\theta, \beta) \leq n$. If there are $\frac{n}{2}$ elements in $\text{supp}(\nu, \beta)$ from the set $\{n+1, n+2, \dots, 2n\}$, then its ideal must contain at least $n+1$ elements. Consequently, $d_{cr}(\nu, \beta) \geq n+1 > n$, which leads to a

contradiction. Therefore, $\text{supp}(\nu, \beta)$ contains at most $\frac{n}{2} - 1$ elements from the set $\{n+1, n+2, \dots, 2n\}$. If $\text{supp}(\nu, \beta)$ contains fewer than $\frac{n}{2}$ elements from set $\{n+1, n+2, \dots, 2n\}$, it follows that $\text{supp}(\theta, \beta)$ must include at least $\frac{n}{2} + 1$ elements from this set. Consequently, $|\langle \text{supp}(\theta, \beta) \rangle| \geq n+3 > n$. This leads to the conclusion that $d_{cr}(\theta, \beta) > n$, which is a contradiction. Therefore, we can conclude that the $B(\nu, n) \cap B(\theta, n) = \phi$. Hence, Packing radius of C_3 is n when n is even.

Case 2: If $n \equiv 1 \pmod{4}$ and $n > 1$

To demonstrate that the packing radius of C_3 for $n \equiv 1 \pmod{4}$ is less than $n+2$ it is sufficient to provide a counterexample.

Consider, I as the identity permutation, $\pi = (1 \ n \ 2n-1 \ n-2 \ 2n-3 \ \dots \ 3 \ n+2)(2 \ n+1 \ 2n \ n-1 \ 2n-2 \ n-3 \ 2n-4 \ n-5 \ 2n-6 \ \dots \ 4 \ n+3) \in C_3$ and $\alpha = (m \ 2n-1 \ n-2 \ 2n-3 \ n-4 \ \dots \ 2n-m \ [m-1] \ m+1 \ 2n \ n-1 \ 2n-2 \ n-3 \ 2n-4 \ \dots \ 2n-m \ n) \in S_{2n}$ where, $m = \frac{n-1}{2}$ then,

$$\begin{aligned} \text{supp}(I, \alpha) &= \{m, m+1, \dots, n, n+m+1, n+m+2, \dots, 2n\} \\ \implies \langle \text{supp}(I, \alpha) \rangle &= \text{supp}(I, \alpha) \\ \implies d_{cr}(I, \alpha) &= n+2. \end{aligned} \tag{19}$$

Hence, $\alpha \in B(I, n+2)$.

$$\begin{aligned} \text{supp}(\pi, \alpha) &= \{1, 2, \dots, m+1, n, n+1, n+2, \dots, n+m+1\} \\ \implies \langle \text{supp}(\pi, \alpha) \rangle &= \text{supp}(\pi, \alpha) \\ \implies d_{cr}(\pi, \alpha) &= n+2 \end{aligned} \tag{20}$$

$$\therefore \alpha \in B(\pi, n+2)$$

By using (19) and (20), it can be concluded that the $\alpha \in B(I, n+2) \cap B(\pi, n+2)$. Hence, By using definition of packing radius, we can deduce that the packing radius of C_3 is less than $n+2$.

Let, $\nu, \theta \in C_3$ then, $\text{supp}(\nu, \theta) = \{1, 2, \dots, n, n+1, n+2, \dots, 2n\}$ and $d_{cr}(\nu, \theta) = 2n$. Suppose that $\beta \in B(\nu, n+1) \cap B(\theta, n+1)$ then $d_{cr}(\nu, \beta) \leq n+1$ and $d_{cr}(\theta, \beta) \leq n+1$. If there are $\frac{n}{2} + 1$ elements in $\text{supp}(\nu, \beta)$ from the set $\{n+1, n+2, \dots, 2n\}$, then its ideal must contain at least $n+3$ elements. Consequently, $d_{cr}(\nu, \beta) > n+1$, which leads to a contradiction. Therefore, $\text{supp}(\nu, \beta)$ contains at most $\frac{n-1}{2}$ elements from the set $\{n+1, n+2, \dots, 2n\}$. If $\text{supp}(\nu, \beta)$ contains an element less than or equal to $\frac{n-1}{2}$ from the set $\{n+1, n+2, \dots, 2n\}$, it follows that $\text{supp}(\theta, \beta)$ must include at least $\frac{n+1}{2}$ elements from the same set. Consequently, $|\langle \text{supp}(\theta, \beta) \rangle| \geq n+2 > n+1$. This implies that $d_{cr}(\theta, \beta) > n+1$, which leads to a contradiction. Hence, we can conclude that $B(\nu, n+1) \cap B(\theta, n+1) = \phi$. Therefore, Packing radius of C_3 is $n+1$ when $n \equiv 1 \pmod{4}$.

Case 3: If $n \equiv 3 \pmod{4}$

To establish that the packing radius of C_3 for $n \equiv 3 \pmod{4}$ is less than $n + 2$ it is sufficient to give a counterexample.

Consider, I as the identity permutation, $\pi = (1\ n\ 2n-1\ n-2\ 2n-3\ \cdots\ 3\ n+2)(2\ n+1\ 2n\ n-1\ 2n-2\ n-3\ 2n-4\ n-5\ 2n-6\ \cdots\ 4\ n+3) \in C_3$ and $\alpha = (m\ 2n-1\ n-2\ 2n-3\ n-4\ \cdots\ m+2\ n+m+1\ 2n\ n-1\ 2n-2\ n-3\ 2n-4\ \cdots\ n+m+2\ m+1\ n) \in S_{2n}$ where, $m = \frac{n-1}{2}$ then,

$$\begin{aligned} \text{supp}(I, \alpha) &= \{m, m+1, \dots, n, n+m+1, n+m+2, \dots, 2n\} \\ \implies \langle \text{supp}(I, \alpha) \rangle &= \text{supp}(I, \alpha) \\ \implies d_{cr}(I, \alpha) &= n+2. \end{aligned} \tag{21}$$

Hence, $\alpha \in B(I, n+2)$.

$$\begin{aligned} \text{supp}(\pi, \alpha) &= \{1, 2, \dots, m+1, n, n+1, n+2, \dots, n+m+1\} \\ \implies \langle \text{supp}(\pi, \alpha) \rangle &= \text{supp}(\pi, \alpha) \\ \implies d_{cr}(\pi, \alpha) &= n+2 \\ \therefore \alpha &\in B(\pi, n+2) \end{aligned} \tag{22}$$

By using (21) and (22), we can conclude that $\alpha \in B(I, n+2) \cap B(\pi, n+2)$. Hence, By using definition of packing radius, it can be deduced that the packing radius of C_3 is less than $n + 2$.

Let, $\nu, \theta \in C_3$ then, $\text{supp}(\nu, \theta) = \{1, 2, \dots, n, n+1, n+2, \dots, 2n\}$ and $d_{cr}(\nu, \theta) = 2n$. Suppose that $\beta \in B(\nu, n+1) \cap B(\theta, n+1)$ then $d_{cr}(\nu, \beta) \leq n+1$ and $d_{cr}(\theta, \beta) \leq n+1$. If there are $\frac{n}{2} + 1$ elements in $\text{supp}(\nu, \beta)$ from the set $\{n+1, n+2, \dots, 2n\}$, then its ideal must contain at least $n+3$ elements. Consequently, $d_{cr}(\nu, \beta) > n+1$, which leads to a contradiction. Therefore, $\text{supp}(\nu, \beta)$ contains at most $\frac{n-1}{2}$ elements from the set $\{n+1, n+2, \dots, 2n\}$. If $\text{supp}(\nu, \beta)$ contains an element less than or equal to $\frac{n-1}{2}$ from the set $\{n+1, n+2, \dots, 2n\}$, it follows that $\text{supp}(\theta, \beta)$ must include at least $\frac{n+1}{2}$ elements from the same set. Consequently, $|\langle \text{supp}(\theta, \beta) \rangle| \geq n+2 > n+1$. This implies that $d_{cr}(\theta, \beta) > n+1$, which is a contradiction. Hence, we can conclude that $B(\nu, n+1) \cap B(\theta, n+1) = \phi$. Therefore, Packing radius of C_3 is $n+1$ when $n \equiv 3 \pmod{4}$.

By considering all the cases, we can conclude that packing radius of C_3 is n and $n+1$ when $n \in \mathbb{N} \setminus \{1\}$ is even and odd respectively, under the crown poset metric. $\square$

Theorem 5.12. *Covering radius of C_3 is $2n$.*

Proof. Consider, $\beta = (1\ 2\ 3\ 4\ \cdots\ 2n) \in S_{2n}$. For any $\pi \in C_3$, we observe that $\text{supp}(\pi, \beta) = \{1, 2, 3, \dots, 2n\}$, indicating that its ideal encompasses $\{1, 2, 3, \dots, 2n\}$. Consequently, this implies that $d_{cr}(\pi, \beta) = 2n$. Therefore, the covering radius of C_3 is at least $2n$. Furthermore, as established in 4.12, the covering radius is less than or equal to $2n$. Thus, the covering radius of C_3 is conclusively $2n$. $\square$

Theorem 5.13. If $C_3 = \langle \langle \sigma \rangle \rangle$ where, $\sigma = (1\ 3\ \cdots\ 2n-1)(2\ 4\ 6\ \cdots\ 2n)$ then C_3 is not a perfect code under the crown poset metric.

Proof. The packing radius and covering radius of C_3 are not equal. By using (4.10) it can be concluded that C_3 is not perfect code. $\square$

Theorem 5.14. Minimum distance of the cyclic group generated by $(1\ n+1\ 2\ n+2\ \cdots\ n\ 2n)$ permutation, denoted as, C_4 is $2n$ under the crown poset metric.

Proof. The support of any two distinct permutations belong to C_4 contains $\{1, 2, 3, 4, \dots, 2n\}$ elements. By applying the concept of ideal and crown poset it can be deduced that $d_{cr}(\sigma, \pi) = 2n$ for all $\sigma, \pi \in C_4$ where, $\sigma \neq \pi$. Hence, we can conclude that the minimum distance of the C_4 is $2n$ under the crown poset metric. $\square$

Theorem 5.15. Packing radius of C_4 is $n+1$ when $n \in \mathbb{N} \setminus \{1\}$ under the crown poset metric.

Proof. Case 1: If n is even

According to the Theorem 4.11 packing radius of C_4 lies between $\lfloor \frac{2n-1}{2} \rfloor$ and $2n-1$. To examine whether the packing radius of C_4 for n is even smaller than $n+2$ it is sufficient to provide a counterexample.

Consider, $\sigma = (1\ n+1\ 2\ n+2\ \cdots\ n\ 2n) \in C_4$, I as the identity permutation and $\alpha = (1\ n+1\ 2\ n+2\ \cdots\ \frac{n}{2}\ n + \frac{n}{2}\ \frac{n}{2} + 1) \in S_{2n}$ then,

$$\begin{aligned} \text{supp}(\sigma, \alpha) &= \left\{ \frac{n}{2} + 1, \frac{n}{2} + 2, \dots, n, n + \frac{n}{2} + 1, \dots, 2n \right\} \\ \implies \langle \text{supp}(\sigma, \alpha) \rangle &= \left[\frac{n}{2}, n \right] \cup \left[n + \frac{n}{2} + 1, 2n \right]. \\ \implies d_{cr}(\sigma, \alpha) &= n + 1. \end{aligned} \tag{23}$$

Hence, $\alpha \in B(\sigma, n+1)$.

$$\begin{aligned} \text{supp}(I, \alpha) &= \left\{ 1, 2, \dots, \frac{n}{2} + 1, n+1, n+2, \dots, n + \frac{n}{2} \right\} \\ \implies \langle \text{supp}(I, \alpha) \rangle &= \left[1, \frac{n}{2} + 1 \right] \cup \left[n, n + \frac{n}{2} \right] \\ \implies d_{cr}(I, \alpha) &= n + 2 \\ \therefore \alpha &\in B(I, n+2) \end{aligned} \tag{24}$$

By using (23) and (24), we can conclude that $\alpha \in B(\sigma, n+2) \cap B(I, n+2)$. Hence, By using definition of packing radius, we can conclude that packing radius of C_4 is less than $n+2$. Now, we will prove that packing radius of C_4 is $n+1$ when n is even. Let, $\tau, \nu \in C_4$ then, $d_{cr}(\tau, \nu) = 2n$. Suppose that $\beta \in B(\tau, n+1) \cap B(\nu, n+1)$ then $d_{cr}(\tau, \beta) \leq n+1$ and $d_{cr}(\nu, \beta) \leq n+1$. If $\text{supp}(\tau, \beta)$ contains fewer than $\frac{n}{2} - 1$ elements from the set $\{n+1, n+2, \dots, 2n\}$, then $\text{supp}(\nu, \beta)$ must contain more than $\frac{n}{2} + 1$ elements from this set. Assume that $\text{supp}(\nu, \beta)$ includes $\frac{n}{2} + 2$ elements

from $\{n+1, n+2, \dots, 2n\}$. This assumption implies that $|\langle \text{supp}(\nu, \beta) \rangle| \geq n+5$; consequently, $d_{cr}(\nu, \beta) > n+1$, which leads to a contradiction. Therefore, it follows that $\beta \notin B(\nu, n+1)$. Consider the scenario where, $\text{supp}(\tau, \beta)$ contains exactly $\frac{n}{2} - 1$ elements from the set $\{n+1, n+2, \dots, 2n\}$ of the crown poset. Assume that the elements $\{n+1, n+2, \dots, n+\frac{n}{2}-1\}$ are the only ones from the set $\{n+1, n+2, \dots, 2n\}$ present in $\text{supp}(\tau, \beta)$. Consequently, it follows that $\{n+\frac{n}{2}, \dots, 2n\} \subseteq \text{supp}(\nu, \alpha)$. This implies that $\text{supp}(\nu, \beta)$ must contain at least $(\frac{n}{2} + 1)$ elements from the set $\{n+1, n+2, \dots, 2n\}$. Therefore, the cardinality $|\langle \text{supp}(\nu, \beta) \rangle|$ equals $n+3$. Consequently, $d_{cr}(\nu, \beta) = n+3$, which leads to a contradiction. Hence, it can be concluded that $\beta \notin B(\nu, n+1)$. Therefore, we can deduce that, $B(\tau, n+1) \cap B(\nu, n+1) = \phi$, $\forall \tau, \nu \in C_4$. Hence, Packing radius of C_4 is $n+1$ when n is even.

Case 2: If n is odd and $n > 1$

To examine packing radius of C_4 for n is odd is less than $n+2$ it is sufficient to prove by counterexample. Consider I as the Identity permutation, $\sigma = (1 \ n+1 \ 2 \ n+2 \ \dots \ n \ 2n) \in C_4$ and $\alpha = (1 \ n+1 \ 2 \ n+2 \ \dots \ m \ n+m \ m+1) \in S_{2n}$ where, $m = \frac{n-1}{2}$

$$\begin{aligned} \text{supp}(I, \alpha) &= \left[1, \frac{n+1}{2}\right] \cup \left[n+1, n+\frac{n-1}{2}\right] \\ \implies \langle \text{supp}(I, \alpha) \rangle &= \left[1, \frac{n+1}{2}\right] \cup \left[n, n+\frac{n-1}{2}\right] \\ \implies d_{cr}(I, \alpha) &= n+1 \\ \implies \alpha &\in B(I, n+2) \end{aligned} \tag{25}$$

$$\begin{aligned} \text{supp}(\sigma, \alpha) &= \left[\frac{n+1}{2}, n\right] \cup \left[n+\frac{n+1}{2}, 2n\right] \\ \implies \langle \text{supp}(\sigma, \alpha) \rangle &= \left[\frac{n-1}{2}, n\right] \cup \left[n+\frac{n+1}{2}, 2n\right] \\ \implies d_{cr}(\sigma, \alpha) &= n+2. \end{aligned} \tag{26}$$

$$\therefore \alpha \in B(\sigma, n+2).$$

By using (25) and (26), we can conclude that $\alpha \in B(\sigma, n+2) \cap B(I, n+2)$. Hence, By using definition of packing radius, we can conclude that packing radius of C_4 is less than $n+2$. Now, we have to prove that packing radius of C_4 is $n+1$ when n is odd. Let, $\tau, \nu \in C_4$ then $\nu(x) \neq \tau(x), \forall x \in [2n] \implies d_{cr}(\tau, \nu) = 2n$. Suppose that $\beta \in B(\tau, n+1) \cap B(\nu, n+1)$ then $d_{cr}(\beta, \tau) \leq n+1$, $d_{cr}(\beta, \nu) \leq n+1$. If there are $\frac{n+1}{2}$ elements in $\text{supp}(\tau, \beta)$ from the set $\{n+1, n+2, \dots, 2n\}$, then it follows that $|\langle \text{supp}(\tau, \beta) \rangle| = n+2 > n+1$. Consequently, the number of elements in $\text{supp}(\tau, \beta)$ from the set $\{n+1, n+2, \dots, 2n\}$ must be less than or equal to $\frac{n-1}{2}$. Assume that $\text{supp}(\tau, \beta)$ contains less than or equal to $\frac{n-1}{2}$ elements from set $\{n+1, n+2, \dots, 2n\}$ within the crown poset. Consequently, $\text{supp}(\nu, \beta)$ must contain more than $\frac{n-1}{2}$ elements in the same set. Further, suppose that $\text{supp}(\nu, \beta)$ comprises $\frac{n+1}{2}$ elements from $\{n+1, n+2, \dots, 2n\}$. This implies that $|\langle \text{supp}(\nu, \beta) \rangle| \geq n+2 > n+1$, indicating that $d_{cr}(\nu, \beta) > n+1$.

Therefore, $\beta \notin B(\nu, n+1)$. Therefore, we can conclude that, $B(\tau, n+1) \cap B(\nu, n+1) = \emptyset$, $\forall \tau, \nu \in C_4$. Hence, Packing radius of C_4 is $n+1$ when $n > 1$ is odd.

By considering all cases, we can conclude that packing radius of C_4 is $n+1$ when $n \in \mathbb{N} \setminus \{1\}$ under the crown poset metric. $\square$

Theorem 5.16. *Covering radius of C_4 is $2n-1$.*

Proof. Let, $i \in \{1, 2, \dots, n-1, n, n+1, \dots, 2n-1, 2n\}$. Consider, $\alpha \in S_{2n}$ such that $\alpha(i) = j$. According to the definition of C_4 , it is evident that there exist $\pi \in C_4$ such that $\pi(i) = j$. Let, $i = n+1$ so, $\alpha(n+1) = j$ and $\pi(n+1) = j$ it implies that $n+1 \notin \text{supp}(\alpha, \pi)$. Hence, $d_{cr}(\alpha, \pi) \leq 2n-1 \forall \pi \in C_4$.

Clearly, $\beta = (1, n+1) \in S_{2n}$. Let, $\sigma = (1 \ n+1 \ 2 \ n+2 \ 3 \ n+3 \ \dots \ n \ 2n)$ then by using the definition of C_4 we can obtain that $C_4 = \{\sigma^i | 1 \leq i \leq 2n\}$ so, $\sigma(1) = n+1$ & $\sigma^{-1}(n+1) = 1$. Hence, $\sigma^i(1) \neq n+1 \forall i \in [2n] \setminus (1)$ & $\sigma^i(n+1) \neq 1 \forall i \in [2n] \setminus (-1)$. Therefore, $\text{supp}(\sigma^i, \beta) = [2n] \forall i \in [2n] \setminus A$ where, $A = \{1, -1\}$.

$\text{supp}(\sigma, \beta) = [2n] \setminus \{1\}$ then, $d_{cr}(\sigma, \beta) = 2n$ and $\text{supp}(\sigma^{-1}, \beta) = [2n] \setminus \{n+1\}$ then, $d_{cr}(\sigma^{-1}, \beta) = 2n-1$. Therefore, $\min\{d_{cr}(\sigma^i, \beta)\} = 2n-1$. Hence, $\max \min\{d_{cr}(\sigma^i, \beta)\} = 2n-1$. By using (4.9), we can conclude that covering radius of C_4 is $2n-1$. $\square$

Theorem 5.17. *Cyclic group generated by the $(1 \ n+1 \ 2 \ n+2 \ \dots \ 2n)$ permutation denoted as, C_4 is not perfect code for any values of $n \in \mathbb{N} \setminus \{1\}$ under the crown poset metric.*

Proof. The packing radius and covering radius of C_4 are not identical. By implying (4.10) it can be inferred that C_4 is not perfect code. $\square$

Theorem 5.18. *Minimum distance of the cyclic group generated by $(1 \ 2n \ 2 \ 2n-1 \ \dots \ n \ n+1)$ permutation, denoted as, C_5 is $2n$ under the crown poset metric.*

Proof. The support of any two distinct permutations belong to C_5 contains $\{1, 2, 3, 4, \dots, 2n\}$ elements. By using the definition of the ideal and crown poset we can conclude that $d_{cr}(\sigma, \pi) = 2n$ for all $\sigma, \pi \in C_5$ where, $\sigma \neq \pi$. Hence, minimum distance of the C_5 is $2n$ under the crown poset metric. $\square$

Theorem 5.19. *The packing radii of C_5 are $n+1$, $n+1$ and n when $n \in \mathbb{N} \setminus \{1\}$ is odd, $n \equiv 0 \pmod{4}$ and $n \equiv 2 \pmod{4}$ respectively, under the crown poset metric.*

Proof. According to the Theorem 4.11 packing radius of C_5 bounded between $\lfloor \frac{2n-1}{2} \rfloor$ and $2n-1$.

Case 1: If n is even

Case 1.1 If $n \equiv 0 \pmod{4}$

To demonstrate packing radius of C_5 for $n \equiv 0 \pmod{4}$ is less than $n+2$ it is sufficient to prove by counterexample.

Consider, $\sigma = (1 \ 2n \ 2 \ 2n - 1 \ \cdots \ q \ 2n - q + 1 \ q + 1 \ 2n - q \ \cdots \ n - q \ n + q + 1 \ \cdots \ n \ n + 1) \in C_5$, I as the identity permutation and $\alpha = (q + 1 \ 2n - q \ q + 2 \ 2n - q - 1 \ \cdots \ n - q \ n + q + 1 \ n - q + 1) \in S_{2n}$ where, $m = \frac{n}{2}$ and $q = \frac{n}{4}$ then,

$$\begin{aligned} \text{supp}(\sigma, \alpha) &= \{1, 2, \dots, q, n - q + 1, \dots, n, n + 1, \dots, n + q, 2n, 2n - 1, \dots, 2n - q + 1\} \\ \implies \langle \text{supp}(\sigma, \alpha) \rangle &= \text{supp}(\sigma, \alpha) \cup \{n - q\}. \\ \implies d_{cr}(\sigma, \alpha) &= n + 1. \end{aligned} \tag{27}$$

Hence, $\alpha \in B(\sigma, n + 2)$.

$$\begin{aligned} \text{supp}(I, \alpha) &= \{q + 1, q + 2, \dots, m, m + 1, \dots, n - q, n - q + 1, 2n - q, \dots, n + q + 1\} \\ \implies \langle \text{supp}(I, \alpha) \rangle &= \text{supp}(I, \alpha) \cup \{q\} \\ \implies d_{cr}(I, \alpha) &= n + 2 \end{aligned} \tag{28}$$

$$\therefore \alpha \in B(I, n + 2)$$

By using (27) and (28), we can conclude that $\alpha \in B(\sigma, n + 2) \cap B(I, n + 2)$. Hence, By using definition of packing radius, we can conclude that packing radius of C_5 is less than $n + 2$. Now, we will prove that packing radius of C_5 is $n + 1$ when $n \equiv 0(\text{mod } 4)$. Let, $\tau, \nu \in C_5$ then, $d_{cr}(\tau, \nu) = 2n$. Suppose that $\beta \in B(\tau, n + 1) \cap B(\nu, n + 1)$ then, $d_{cr}(\tau, \beta) \leq n + 1$ and $d_{cr}(\nu, \beta) \leq n + 1$. If $\text{supp}(\tau, \beta)$ contains fewer than $\frac{n}{2} + 1$ elements from set $\{n + 1, n + 2, \dots, 2n\}$, then its ideal must include at least $n + 3$ elements, which leads to a contradiction. Therefore, the number of elements in $\text{supp}(\tau, \beta)$ from the set $\{n + 1, n + 2, \dots, 2n\}$ is less than or equal to $\frac{n}{2}$. If $\text{supp}(\tau, \beta)$ contains fewer than $\frac{n}{2}$ elements from the set $\{n + 1, n + 2, \dots, 2n\}$, then $\text{supp}(\nu, \beta)$ must contain more than $\frac{n}{2} + 1$ elements from the same set. This condition implies that $|\langle \text{supp}(\nu, \beta) \rangle| \geq n + 3$; consequently, $d_{cr}(\nu, \beta) > n + 1$, which results in a contradiction. Therefore, it follows that $\beta \notin B(\nu, n + 1)$. If $\text{supp}(\tau, \beta)$ precisely contains $\frac{n}{2}$ elements from the set $\{n + 1, n + 2, \dots, 2n\}$, then $\text{supp}(\nu, \beta)$ comprises $\frac{n}{2}$ elements from $\{n + 1, n + 2, \dots, 2n\}$. Assuming that $\{n + 1, n + 2, \dots, n + \frac{n}{2}\} \subseteq \text{supp}(\tau, \beta)$, it follows that the ideal encompasses $[1, \frac{n}{2}] \cup [n, n + \frac{n}{2}]$, then the $[\frac{n}{2}, n] \cup [n + \frac{n}{2} + 1, 2n]$ is contained within the ideal of $\text{supp}(\nu, \beta)$. Furthermore, in a scenario where $n \equiv 0(\text{mod } 4)$, it is possible to identify a $\gamma \in \{1, 2, \dots, n\}$ such that either $\gamma \in \text{supp}(\tau, \beta)$ or $\gamma \in \text{supp}(\nu, \beta)$. If $\gamma \in \text{supp}(\tau, \beta)$ and $\gamma \notin \{1, 2, \dots, \frac{n}{2}, n\}$, this implies that $|\langle \text{supp}(\tau, \beta) \rangle| \geq n + 2$. Consequently, $d_{cr}(\tau, \beta) > n + 1$, which constitutes a contradiction. Thus, $\beta \notin B(\tau, n + 1)$. Therefore, we can conclude that, $B(\tau, n + 1) \cap B(\nu, n + 1) = \phi$, $\forall \tau, \nu \in C_5$. Hence, Packing radius of C_5 is $n + 1$ when $n \equiv 0(\text{mod } 4)$.

Case 1.2 If $n \equiv 2(\text{mod } 4)$

To establish packing radius of C_5 for $n \equiv 2(\text{mod } 4)$ is less than $n + 1$ it is sufficient to prove by counterexample.

Consider, $\sigma = (1 \ 2n \ 2 \ 2n-1 \ \cdots \ q \ 2n-q+1 \ q+1 \ 2n-q \ \cdots \ n-q \ n+q+1 \ n-q+1 \ n+q \ \cdots \ n \ n+1) \in C_5$, I as the identity permutation and $\alpha = (q \ 2n-q+1 \ q+1 \ 2n-q \ \cdots \ n-q \ n+q+1 \ n-q+1) \in S_{2n}$ where, $m = \frac{n}{2}$ and $q = \frac{n+2}{4}$ then,

$$\begin{aligned} \text{supp}(\sigma, \alpha) &= \{1, 2, \dots, q-1, n-q+1, \dots, n, n+1, n+2, \dots, n+q\} \\ &\quad \cup \{2n, 2n-1, \dots, 2n-q+2\} \\ \implies \langle \text{supp}(\sigma, \alpha) \rangle &= \text{supp}(\sigma, \alpha) \cup \{q\}. \\ \implies d_{cr}(\sigma, \alpha) &= n+1. \end{aligned} \tag{29}$$

Hence, $\alpha \in B(\sigma, n+1)$.

$$\begin{aligned} \text{supp}(I, \alpha) &= \{q, q+1, q+2, \dots, n-q, n-q+1, 2n-q, 2n-q+1, \dots, n+q+1\} \\ \implies \langle \text{supp}(I, \alpha) \rangle &= \text{supp}(I, \alpha) \\ \implies d_{cr}(I, \alpha) &= n+1 \\ \therefore \alpha &\in B(I, n+1) \end{aligned} \tag{30}$$

By using (29) and (30), we can conclude that $\alpha \in B(\sigma, n+1) \cap B(I, n+1)$. Hence, By using definition of packing radius, we can conclude that packing radius of C_5 is less than $n+1$. Now, we will prove that packing radius of C_5 is n when $n \equiv 2 \pmod{4}$. Let, $\tau, \nu \in C_5$ then, $d_{cr}(\tau, \nu) = 2n$. Suppose that $\beta \in B(\tau, n) \cap B(\nu, n)$ then $d_{cr}(\tau, \beta) \leq n$ and $d_{cr}(\nu, \beta) \leq n$. If $\text{supp}(\tau, \beta)$ contains $\frac{n}{2}$ elements from the set $\{n+1, n+2, \dots, 2n\}$, then its ideal must include at least $n+1$ elements, which leads to a contradiction. Therefore, the number of elements in $\text{supp}(\tau, \beta)$ from the set $\{n+1, n+2, \dots, 2n\}$ must be less than or equal to $\frac{n}{2} - 1$. If $\text{supp}(\tau, \beta)$ contains fewer than $\frac{n}{2} - 1$ elements from set $\{n+1, n+2, \dots, 2n\}$, then $\text{supp}(\nu, \beta)$ must contain more than $\frac{n}{2} + 1$ elements from the same set. This condition implies that $|\langle \text{supp}(\nu, \beta) \rangle| \geq n+3$. Consequently, $d_{cr}(\nu, \beta) > n$, which leads to a contradiction. So, it follows that $\beta \notin B(\nu, n+1)$. Therefore, we can conclude that, $B(\tau, n) \cap B(\nu, n) = \phi$, $\forall \tau, \nu \in C_5$. Hence, Packing radius of C_5 is n when $n \equiv 2 \pmod{4}$.

Case 2: If n is odd and $n > 1$

To examine packing radius of C_5 for n is odd is less than $n+2$ it is sufficient to prove by counterexample. Consider, I as the Identity permutation, $\sigma = (1 \ 2n \ 2 \ 2n-1 \ \cdots \ n \ n+1)$, $\sigma^{-2} = (1 \ n \ n-1 \ \cdots \ 3 \ 2)(n+1 \ n+2 \ \cdots \ 2n)$ and $\alpha = (m \ m-1 \ \cdots \ 3 \ 2 \ 1 \ n)(n+1 \ n+2 \ \cdots \ n+m) \in S_{2n}$ where, $m = \frac{n+1}{2}$

$$\begin{aligned} \text{supp}(I, \alpha) &= \left[1, \frac{n+1}{2}\right] \cup \left[n, n + \frac{n+1}{2}\right] \\ \implies \langle \text{supp}(I, \alpha) \rangle &= \left[1, \frac{n+1}{2}\right] \cup \left[n, n + \frac{n+1}{2}\right] \\ \implies d_{cr}(I, \alpha) &= n+2 \\ \implies \alpha &\in B(I, n+2) \end{aligned} \tag{31}$$

$$\begin{aligned}
\text{supp}(\sigma^{-2}, \alpha) &= \left[\frac{n+1}{2} + 1, n \right] \cup \left[n + \frac{n+1}{2}, 2n \right] \\
\langle \text{supp}(\sigma^{-2}, \alpha) \rangle &= \left[\frac{n-1}{2}, n \right] \cup \left[n + \frac{n+1}{2}, 2n \right] \\
\therefore d_{cr}(\sigma^{-2}, \alpha) &= n + 2. \\
\implies \alpha &\in B(\sigma^{-2}, n + 2).
\end{aligned} \tag{32}$$

By using (31) and (32), we can conclude that $\alpha \in B(\sigma^{-2}, n + 2) \cap B(I, n + 2)$. Hence, By using definition of packing radius, we can conclude that packing radius of C_5 is less than $n + 2$. Now, we have to prove that packing radius of C_5 is $n + 1$ when n is odd. Let, $\tau, \nu \in C_5$ then, $\nu(x) \neq \tau(x), \forall x \in [2n] \implies d_{cr}(\tau, \nu) = 2n$. Suppose that $\beta \in B(\tau, n + 1) \cap B(\nu, n + 1)$ then $d_{cr}(\beta, \tau) \leq n + 1$, $d_{cr}(\beta, \nu) \leq n + 1$. If there are $\frac{n+1}{2}$ elements in $\text{supp}(\tau, \beta)$ from the set $\{n + 1, n + 2, \dots, 2n\}$ of the crown poset, then it follows that $|\langle \text{supp}(\tau, \beta) \rangle| = n + 2 > n + 1$. Consequently, the number of elements in $\text{supp}(\tau, \beta)$ from the set $\{n + 1, n + 2, \dots, 2n\}$ of the crown poset is less than or equal to $\frac{n-1}{2}$. Assume that $\text{supp}(\tau, \beta)$ contains less than or equal to $\frac{n-1}{2}$ elements from set $\{n + 1, n + 2, \dots, 2n\}$ within the crown poset. Consequently, $\text{supp}(\nu, \beta)$ must contain more than $\frac{n-1}{2}$ elements in the same set. Further, suppose that $\text{supp}(\nu, \beta)$ comprises $\frac{n+1}{2}$ elements from $\{n + 1, n + 2, \dots, 2n\}$. This implies that $|\langle \text{supp}(\nu, \beta) \rangle| \geq n + 2 > n + 1$, which indicates that $d_{cr}(\nu, \beta) > n + 1$. So, $\beta \notin B(\nu, n + 1)$. Therefore, we can conclude that, $B(\tau, n + 1) \cap B(\nu, n + 1) = \phi, \forall \tau, \nu \in C_5$. Hence, Packing radius of C_5 is $n + 1$ when $n > 1$ is odd.

By considering the all the cases we can deduce that the packing radii of C_5 are $n + 1, n + 1$ and n when $n \in \mathbb{N} \setminus \{1\}$ is odd, $n \equiv 0 \pmod{4}$ and $n \equiv 2 \pmod{4}$ respectively, under the crown poset metric. $\square$

Theorem 5.20. *Covering radius of C_5 is $2n - 1$.*

Proof. Consider, $i \in \{1, 2, \dots, n, \dots, 2n\}$. Let $\alpha \in S_{2n}$ such that $\alpha(i) = j$. Based on the definition of C_5 , it is clear that there exist $\pi \in C_5$ such that $\pi(i) = j$. Let, $i = n + 1$ so, $\alpha(n + 1) = j$ and $\pi(n + 1) = j$ it implies that $n + 1 \notin \text{supp}(\alpha, \pi)$. Therefore, $d_{cr}(\alpha, \pi) \leq 2n - 1 \forall \pi \in C_5$.

We have, $\beta = (1, 2n) \in S_{2n}$. Consider, $\sigma = (1 \ 2n \ 2 \ 2n - 1 \dots n \ n + 1)$ then by applying the definition of C_5 we can get that $C_5 = \{\sigma^i | 1 \leq i \leq 2n\}$ so, $\sigma(1) = 2n$ & $\sigma^{-1}(2n) = 1$. Therefore, $\sigma^i(1) \neq 2n \forall i \in [2n] \setminus (1)$ & $\sigma^i(2n) \neq 1 \forall i \in [2n] \setminus (-1)$. Hence, $\text{supp}(\sigma^i, \beta) = [2n] \forall i \in [2n] \setminus Q$ where, $Q = \{1, -1\}$.

$\text{supp}(\sigma, \beta) = [2n] \setminus \{1\}$ then, $d_{cr}(\sigma^{n+1}, \beta) = 2n$ and $\text{supp}(\sigma^{-1}, \beta) = [2n] \setminus \{2n\}$ then, $d_{cr}(\sigma^{-1}, \beta) = 2n - 1$. Therefore, $\min\{d_{cr}(\sigma^i, \beta)\} = 2n - 1$. Hence, $\max \min\{d_{cr}(\sigma^i, \beta)\} = 2n - 1$. By applying (4.9), we can deduce that the covering radius of C_5 is $2n - 1$. $\square$

Theorem 5.21. *Cyclic group generated by $(1 \ 2n \ 2 \ 2n - 1 \dots n \ n + 1)$ permutation denoted as, C_5 is not perfect code for any values of $n \in \mathbb{N} \setminus \{1\}$ under the crown poset metric.*

Proof. The packing radius and covering radius of C_5 are distinct. By applying (4.10) it can be deduced that C_5 is not perfect code. $\square$

6. CONCLUSION

In this study, we have worked on the nonexistence of perfect permutation code under crown poset metrics for different types of permutation codes such as C_1 , C_2 , C_3 , C_4 and C_5 where, C_1 is the permutation code generated by the permutation $(n+1\ n+2\ \cdots\ 2n)$, C_2 is the permutation code generated by two permutations such as $(1\ 2\ 3\ \cdots\ n)$ and $(n+1\ n+2\ n+3\ \cdots\ 2n)$, C_3 is the permutation code generated by the permutation $(1\ 3\ 5\ \cdots\ 2n-1)(2\ 4\ 6\ \cdots\ 2n)$, C_4 is the permutation code generated by permutation $(1\ n+1\ 2\ n+2\ \cdots\ n\ 2n)$ and C_5 is the permutation code generated by permutation $(1\ 2n\ 2\ 2n-1\ \cdots\ n\ n+1)$.

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