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EVOLUTION ALGEBRAS AND 3-JORDAN ALGEBRAS

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Abstract. This paper is devoted to the study of the finite-dimensional evolution algebras that are 3-Jordan algebras. We show that a non nil-algebra of this class admits a nonzero idempotent. According to this idempotent, we prove that its Peirce decomposition is a direct sum of algebras. In passing, we show that, if the nil-radical of the algebra is of nil-index < 5, then the algebra is an almost generalized Jordan algebra. A characterization of the derivations is given. We conclude with a classification, up to isomorphism in dimension ≤ 6, of the nil-indecomposable 3-Jordan algebras that are evolution algebras.

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1. Introduction

Let F be a commutative field of characteristic different from $2, 3$. An algebra over the field F is said to be a pseudo-composition algebra if there exists a nonzero symmetric bilinear form $\phi\ :\ A \times A \longrightarrow F$ such that

$$x^3 = \phi(x, x)x, \text{ for all } x \in A. \tag{1}$$

In [13] Myeberg and Osborn characterize the pseudo-composition algebras with an unit element. For more on the pseudo-composition algebras, see [4–6, 8, 13]. Giuliani and Peresi, in [6], determine identities of small degree satisfied by the pseudo-compositions algebras. The identity

$$(x^3, y, x) = (x^3y)x - x^3(yx) = 0, \tag{2}$$

so-called 3-*Jordan* is one of these identities. They also prove that the class of the polynomials identities of degree ≤ 5 of the pseudo-composition algebras is a consequence of the commutativity and the 3-Jordan identity. A commutative algebra verifying the 3-Jordan identity is called 3-*Jordan* algebra. The variety of the 3-Jordan algebras contains that of the Jordan algebras and that of the pseudo-composition algebras. In [8], Hentzel and Peresi show that a simple 3-Jordan algebra with an idempotent is either a

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Jordan algebra or a pseudo-composition algebra. They also give the Peirce decomposition and using this decomposition, they give a necessary and sufficient condition for a 3-Jordan algebra to be a Jordan algebra on the one hand, and for it to be a pseudo-composition algebra on the other hand. In [7], Hentzel and Labra show that for characteristic different from $2, 3$, commutative algebras satisfying the identity $(x^2y)x + tx^3y = 0$ with $t \neq 1, -1$ are 3-Jordan algebras. Using the Albert program, they find a 17-dimensional and a 13-dimensional algebra that verifies the identities $(x^2y)x + x^3y = 0$ and $(x^2y)x - x^3y = 0$ respectively and that are not 3-Jordan algebras. Unlike 3-Jordan algebras, so-called evolution algebras, introduced for the purpose of modeling non-Mendelian genetics, are not defined by a polynomial identity (see [2,14,15]). A finite-dimensional evolution algebras are characterized by a basis $B = (e_1, \ldots, e_n)$ so-called *a natural basis* which multiplication table is given by

$$e_ie_j = 0 \text{ and } e_i^2 = \sum_{k=1}^{n} a_{ik}e_k \text{ with } 1 \leq i \neq j \leq n. \tag{3}$$

The matrix $M = (a_{ik})_{1\leq i,j\leq n}$ is the *matrix of structural constants* of A relative to the natural basis B. In [11], the authors characterize finite-dimensional associative and power-associative evolution algebras. They show that power-associative evolution algebras are Jordan algebras. In [12], a characterization of finite-dimensional evolution algebras that are Lie triple algebra is given. Our aim in the present paper is the study of finite-dimensional evolution algebras that are 3-Jordan algebras.

In Section 2, we recall some results on the nil evolution algebras. Section 3 is devoted to the presentation of our main result. We show that a non nil 3-Jordan evolution algebra has a nonzero idempotent and relative to this idempotent, we prove that its Peirce decomposition is a direct sum of algebras. A characterization of the Lie algebra of derivations is also given. In the fourth and last section, we give a classification up to isomorphism, in dimension at most 6 of nil-indecomposable 3-Jordan evolution algebras. In passing, we show that, in dimension at most 4, the 3-Jordan evolution algebras are almost generalized Jordan algebras.

2. Preliminaries

Let A be a commutative algebra over a commutative field F. The *principal power of* $x \in A$ and that of the algebra A are defined respectively by

$$x^1 = x,\ x^{k+1} = x^kx \text{ and } A^1 = A,\ A^{k+1} = A^kA \text{ with } k \geq 1.$$

Definition 2.1. *We say that the algebra A is*

(1) *nilpotent if there exists an integer n such that $A^n = 0$ and the smallest such integer is called the nilpotency-index of A,*

(2) *nil, if there is an integer $n(x)$ such that $x^{n(x)} = 0$ for all $x \in A$. The smallest such integer is called the nil-index of the algebra A.*

Theorem 2.1. [1, *Theorem 2.2*] *Let A be a nil evolution algebra with a natural basis $B = (e_1, \dots, e_n)$. If $M = (a_{ij})$ is the matrix of structural constants of the algebra A relative to the natural basis B, then*

$$a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1} = 0 \tag{4}$$

for all $i_1, \dots, i_k \in \{1, \cdots, n\}$ and $k \in \{1, \cdots, n\}$ with $i_p \neq i_q$ for $p \neq q$.

In [1, Theorem 2.7], The authors show that in finite dimension, nilpotent evolution algebras are equivalent to nil evolution algebras. They also show that any finite-dimensional nil evolution algebra admits a natural basis which matrix of structural constants is strictly upper triangular. Furthermore, in [10, Lemma 2.3], the authors show that the nil-index of finite-dimensional nil evolution algebra is equal to its nilpotency-index.

Notation 2.1. *Let A be an algebra over the field F and $(x_1, \dots, x_p)$ be a family of p elements of A. The set $Alg(x_1, \dots, x_p)$ denoted the subalgebra of A generated by the family $(x_1, \dots, x_p)$ and the set $vect(x_1, \dots, x_p)$ denoted the linear subspace of A generated by the family $(x_1, \dots, x_p)$.*

Definition 2.2. [3, *Definition 3.3*] *Let A be an algebra over the commutative field F. We define the ideals $ann^i(A)$ by*

(i) $ann^0(A) = \{0\}$,

(ii) $ann^1(A) = ann(A) = \{x \in A \mid xA = Ax = 0\}$,

(iii) $ann^i(A) = ann(A/ann^{i-1}A)$ *for all* $i \geq 2$.

Let $B = (e_1, \dots, e_n)$ be a natural basis of a finite-dimensional evolution algebra A. In [3, Page 16], the authors show that $ann^i(A) = Alg(e \in B \mid e^2 \in ann^{i-1}(A)$ with $i \geq 1)$. Moreover, they prove that $\mathcal{U}_i \oplus \mathcal{U}_1 = \{x \in ann^i(A) \mid x ann^{i-1}(A) = 0\}$ is an invariant of the algebra A for all $i \geq 1$. *The type of A* is the sequence $[n_1, \dots, n_r]$ where $n_i = \dim(ann^i(A)) - \dim(ann^{i-1}(A))$ and r is the smallest integer such that $ann^r(A) = A$. Furthermore, $n_1 + \cdots + n_r = \dim(A)$.

Definition 2.3. *An algebra A is said to be:*

(i) *power-associative if $x^i x^j = x^{i+j}$ for any x in A and for all integer $i, j \geq 1$;*

(ii) *Jordan if it is commutative and $(x^2, y, x) = (x^2 y)x - x^2(yx) = 0$, for all $x, y \in A$;*

(iii) *almost generalized Jordan if it is commutative and $\beta(x^2 y)x + \gamma x^3 y - (\beta + \gamma)((yx)x)x = 0$, for $x, y \in A$ and $(\beta, \gamma) \neq (0, 0)$.*

3. Structure of 3-Jordan evolution algebras

In the following, F denotes an infinite commutative field of characteristic different from $2, 3$.

3.1. **Characterisation theorem.** A partial linearization of the identity (2) gives us

$$((zx^2+2x(zx))y)x+(x^3y)z-(zx^2+2x(zx))(yx)-x^3(zy)=0 \tag{5}$$

and

$$\begin{aligned} &2((z(xh)+h(xz)+x(hz))y)x+((zx^2+2x(zx))y)h+((hx^2+2x(xh))y)z \\ &-2(z(xh)+h(zx)+x(zh))(yx)-(zx^2+2x(xz))(yh)-(hx^2+2x(xh))(yz)=0 \end{aligned} \tag{6}$$

with $x, y, z, h \in A$.

Proposition 3.1. [8, *Proposition 1*] *Let A be a 3-Jordan algebra with a nonzero idempotent e. Then, the algebra admits the following Peirce decomposition:*

$$A = A_1 \oplus A_{-1} \oplus A_{\frac{1}{2}} \oplus A_0$$

with $A_i = \{x \in A \mid ex = ix\}$ where $i \in \{1, -1, 0, \frac{1}{2}\}$. Furthermore, we have

$$\begin{aligned} A_1^2 \subset A_1,\ A_1A_{-1} \subset A_{-1},\ A_1A_{\frac{1}{2}} \subset A_{\frac{1}{2}},\ A_1A_0 = 0, \\ A_{\frac{1}{2}}^2 \subset A_{-1} \oplus A_0 \oplus A_1,\ A_{\frac{1}{2}}A_0 \subset A_{\frac{1}{2}},\ A_{\frac{1}{2}}A_{-1} = A_0 \oplus A_1 \oplus A_{\frac{1}{2}}, \\ A_{-1}^2 \subset A_1 \oplus A_0,\ A_{-1}A_0 = 0 \\ A_0^2 \subset A_0. \end{aligned}$$

Theorem 3.1 (Characterization theorem)**.** *Let A be a finite-dimensional evolution F-algebra with natural basis $B = (e_1, \ldots, e_n)$. Then, A is a 3-Jordan algebra if, and only if, for all $1 \le i, j, k, t \le n$ pairwise distinct, the following statements hold:*

(a) $e_i^3e_i^2 - e_i^5 = 0$;

(b) $(e_i^3e_j)e_i = 0$;

(c) $(e_i^2e_j)e_i^2 - ((e_i^2e_j)e_i)e_i - e_i^4e_j = 0$;

(d) $e_i^3e_j^2 - (e_i^3e_j)e_j - ((e_i^2e_j)e_j)e_i = 0$;

(e) $(e_i^2e_j)e_j^2 - ((e_i^2e_j)e_j)e_j - ((e_j^2e_i)e_j)e_i = 0$;

(f) $(e_i^3e_k)e_j + ((e_i^2e_j)e_k)e_i = 0$;

(g) $((e_i^2e_j)e_i)e_k + ((e_i^2e_k)e_i)e_j = 0$ *with* $j < k$;

(h) $((e_i^2e_j)e_k)e_j + ((e_j^2e_i)e_k)e_i = 0$ *with* $i < j$;

(i) $(e_i^2e_j)e_k^2 - ((e_i^2e_j)e_k)e_k - ((e_i^2e_k)e_k)e_j = 0$;

(j) $((e_i^2e_j)e_t)e_k + ((e_i^2e_k)e_t)e_j = 0$ *with* $j < k$.

Proof. Assumed that A is a 3-Jordan algebra.

- The assertions (a) and (b) follow from the identity (2) taking $x = y = e_i$; $x = e_i,\ y = e_j$ respectively.
- The assertions (c), (d) and (f) result from identity (5) by taking respectively $x = y = e_i, z = e_j$; $x = e_i, y = z = e_j$ and $x = e_i, y = e_k, z = e_j$.

- The assertions (e), (g) to (j) follow from the identity (6), taking $x = e_i, y = z = h = e_j$; $x = y = e_i, z = e_j, h = e_k$; $x = e_i, y = e_k, z = h = e_j$; $x = e_i, y = h = e_k, z = e_j$ and $x = e_i, y = e_t, z = e_j, h = e_k$ respectively

Reciprocally, suppose that the statements (a) to (j) are satisfied. Let $x = \sum_{i=1}^{n} x_i e_i, y = \sum_{i=1}^{n} y_i e_i \in A$. We have

$$xy = \sum_{i=1}^{n} x_i y_i e_i^2 \;;\; x^2 = \sum_{i=1}^{n} x_i^2 e_i^2 \;;\; x^3 = \sum_{i,j=1}^{n} x_i^2 x_j e_i^2 e_j = \sum_{i=1}^{n} x_i^3 e_i^3 + \sum_{1\le i\neq j\le n} x_i^2 x_j e_i^2 e_j;$$

$$x^3(xy) = \sum_{i,j=1}^{n} x_i^3 x_j y_j e_i^3 e_j^2 + \sum_{k=1}^{n} \sum_{1\le i\neq j\le n} x_i^2 x_j x_k y_k (e_i^2 e_j) e_k^2 = \sum_{i=1}^{n} x_i^4 y_i e_i^3 e_i^2 +$$
$$\sum_{1\le i\neq j\le n} [x_i^3 x_j y_j e_i^3 e_j^2 + x_i^3 x_j y_i (e_i^2 e_j) e_i^2 + x_i^2 x_j^2 y_j (e_i^2 e_j) e_j^2] + \sum_{k=1}^{n} \sum_{\substack{1\le i\neq j\le n\\ i\neq k, j\neq k}} x_i^2 x_j x_k y_k (e_i^2 e_j) e_k^2;$$

$$x^3 y = \sum_{i,j=1}^{n} x_i^3 y_j e_i^3 e_j + \sum_{k=1}^{n} \sum_{1\le i\neq j\le n} x_i^2 x_j y_k (e_i^2 e_j) e_k = \sum_{i=1}^{n} x_i^3 y_i e_i^4 +$$
$$\sum_{1\le i\neq j\le n} [x_i^3 y_j e_i^3 e_j + x_i^2 x_j y_i (e_i^2 e_j) e_i + x_i^2 x_j y_j (e_i^2 e_j) e_j] + \sum_{k=1}^{n} \sum_{\substack{1\le i\neq j\le n\\ i\neq k, j\neq k}} x_i^2 x_j y_k (e_i^2 e_j) e_k;$$

$$(x^3 y)x = \sum_{i,j=1}^{n} x_i^3 x_j y_i e_i^4 e_j + \sum_{k=1}^{n} \sum_{1\le i\neq j\le n} [x_i^3 x_k y_j (e_i^3 e_j) e_k + x_i^2 x_j x_k y_i ((e_i^2 e_j) e_i) e_k +$$
$$x_i^2 x_j x_k y_j ((e_i^2 e_j) e_j) e_k] + \sum_{k,t=1}^{n} \sum_{\substack{1\le i\neq j\le n\\ i\neq k, j\neq k}} x_i^2 x_j x_t y_k ((e_i^2 e_j) e_k) e_t = \sum_{i=1}^{n} x_i^4 y_i e_i^5 +$$
$$\sum_{1\le i\neq j\le n} [x_i^3 x_j y_i (e_i^4 e_j + ((e_i^2 e_j) e_i) e_i) + x_i^4 y_j (e_i^3 e_j) e_i + x_i^3 x_j y_j (((e_i^2 e_j) e_j) e_i + (e_i^3 e_j) e_j) +$$
$$x_i^2 x_j^2 y_i (((e_i^2 e_j) e_i) e_j + ((e_j^2 e_i) e_i) e_i)] + \sum_{k=1}^{n} \sum_{\substack{1\le i\neq j\le n\\ i\neq k, j\neq k}} [x_i^3 x_k y_j ((e_i^3 e_j) e_k + ((e_i^2 e_k) e_j) e_i) +$$
$$x_i^2 x_j x_k y_i ((e_i^2 e_j) e_i) e_k + x_i^2 x_j x_k y_j (((e_i^2 e_j) e_j) e_k + ((e_i^2 e_k) e_j) e_j) + x_i^2 x_j^2 y_k ((e_i^2 e_j) e_k) e_j] +$$
$$\sum_{1\le k\neq t\le n} \sum_{\substack{1\le i\neq j\le n\\ i,j\notin\{k,t\}}} x_i^2 x_j x_t y_k ((e_i^2 e_j) e_k) e_t = \sum_{i=1}^{n} x_i^4 y_i e_i^3 e_i^2 + \sum_{1\le i\neq j\le n} [x_i^3 x_j y_i (e_i^2 e_j) e_i^2 + x_i^3 x_j y_j e_i^3 e_j^2 +$$
$$x_i^2 x_j^2 y_i (e_j^2 e_i) e_i^2] + \sum_{i=1}^{n} \sum_{\substack{1\le j< k\le n\\ j\neq i, k\neq i}} x_i^2 x_j x_k y_i (((e_i^2 e_j) e_i) e_k + ((e_i^2 e_k) e_i) e_j) +$$
$$\sum_{k=1}^{n} \sum_{\substack{1\le i\neq j\le n\\ i\neq k, j\neq k}} x_i^2 x_j x_k y_j (e_i^2 e_k) e_j^2 + \sum_{k=1}^{n} \sum_{\substack{1\le i< j\le n\\ i\neq k, j\neq k}} x_i^2 x_j^2 y_k (((e_i^2 e_j) e_k) e_j + ((e_j^2 e_i) e_k) e_i) +$$
$$\sum_{1\le j<t\le n} \sum_{\substack{1\le i\neq k\le n\\ i,k\notin\{j,t\}}} x_i^2 x_j x_t y_k (((e_i^2 e_j) e_k) e_t + ((e_i^2 e_t) e_k) e_j)$$

It follows that $x^3(yx) = (x^3y)x$ and A is a 3-Jordan algebra. $\square$

In the example above, we show that a finite-dimensional 3-Jordan evolution algebra is not necessarily a Jordan algebra or an almost generalized Jordan algebra.

Example 3.1. *Let A be a 7-dimensional evolution algebra and $B = (e_1, \ldots, e_7)$ a natural basis of A. The multiplication table of A is given by:*

$$e_1^2 = e_2, e_2^2 = e_1, e_3^2 = e_4, e_4^2 = e_5 + e_6, e_5^2 = -e_6^2 = e_7, e_7^2 = 0.$$

The algebra A is a 3-Jordan algebra. It is neither a Jordan algebra, nor an almost generalized Jordan algebra. Indeed, let $x = x_1e_1 + \cdots + x_7e_7,\ y = y_1e_1 + \cdots + y_7e_7 \in A$. We show by direct calculation that $(x^3y)x = x^3(yx) = x_1^2x_2^2(y_1e_1 + y_2e_2)$. It follows that the algebra A is a 3-Jordan algebra. Set $x = e_1 + e_3 + e_4$ and $y = e_2 + e_5 - e_6$, we obtain $\beta(x^2y)x + \gamma x^3y - (\beta + \gamma)((xy)x)x = \beta e_2 + 2\gamma e_7 \neq 0$, i.e. the algebra A is not an almost generalized Jordan algebra. Since $0 = e_1^4 \neq e_1^2e_1^2 = e_1$, it follows that the algebra A is not power-associative, so it is not a Jordan algebra. Moreover, the vector $e = e_1 + e_2$ is a nonzero idempotent of the algebra A.

3.2. **Nil 3-Jordan evolution algebras.** In the following, we describe finite-dimensional nil 3-Jordan evolution algebras.

Theorem 3.2. *Let A be a finite-dimensional nil evolution algebra over the field F. Then, A is a 3-Jordan algebra if, and only if, the following statements hold:*
(1) $(e_i^2e_j)e_k^2 = 0$ for all $i, j, k \in \{1, \cdots, n\}$.
(2) The nil-index of the algebra A is at most 5.

Proof. Since the algebra A being a nil-algebra, it admits a natural basis $B = (e_1, \ldots, e_n)$ which multiplication table is given by: $e_i^2 = \sum_{k=i+1}^{n} a_{ik}e_k$ for all $i \in \{1, \ldots, n\}$. Assumed that A is a 3-Jordan algebra. The assertion (1) follows from the Theorem 2.1 and from the assertions (c), (e) and (i) of the Theorem 3.1. The assertion (2) can be deduced from the Theorem 2.1 and from the assertion (j) of the Theorem 3.1. Reciprocally, we suppose that the algebra A verifies the assertions (1) and (2). Let $x = \sum_{k=1}^{n} x_ke_k,\ y = \sum_{k=1}^{n} y_ke_k$. Since A is a nil-algebra of nil-index at most 5, it is nilpotent with nilpotency-index at most 5. It follows that $(x^3y)x = 0$. We also have $x^3 = \sum_{i=1}^{n}\sum_{j=i+1}^{n} x_i^2x_je_i^2e_j$ and $x^3(yx) = \sum_{i,k=1}^{n}\sum_{j=i+1}^{n} x_i^2x_jx_ky_k(e_i^2e_j)e_k^2 = 0 = (x^3y)x$. We deduce that A is a 3-Jordan algebra. $\square$

Proposition 3.2. *Let A be a finite-dimensional nil evolution algebra of nil-index at most 4. Then, A is a 3-Jordan algebra.*

Proof. The algebra A is nilpotent with nilpotency-index at most 4. For all $x, y \in A$, we have $x^3y = 0$ and $x^3(yx) = 0$ since $xy \in A$. It follows that $(x^3y)x = x^3(xy) = 0$ and A is 3-Jordan algebra. $\square$

As the following example shows, a nil 3-Jordan evolution algebra with a nil-index at most 4 is not necessarily a Jordan algebra:

Example 3.2. *Let A be a 3-dimensional nil evolution algebra in the natural basis $B = (e_1, e_2, e_3)$ which multiplication is given by:*

$$e_1^2 = e_2,\ e_2^2 = e_3,\ e_3^2 = 0.$$

The nil-index of the algebra A is 4, so, the Proposition 3.2 tells us that it is a 3-Jordan algebra. But, It is not a Jordan algebra [10, Theorem 4.4].

However, as the example above shows, a nil-algebra with a nil-index at least 5 is not necessarily a 3-Jordan algebra:

Example 3.3. *Let A be a 5-dimensional nil evolution algebra in the natural basis $B = (e_1, \ldots, e_5)$ which multiplication table is given by:*

$$e_1^2 = e_2^2 = e_3,\ e_3^2 = e_4,\ e_4^2 = e_5,\ e_5^2 = 0.$$

The nil-index of A is 5 [3, Theorem 6.4 (iv)]. Since $(e_1^2 e_3)e_3^3 = e_3^2 e_3^2 = e_4^2 = e_5 \neq 0$, it is not a 3-Jordan algebra.

3.3. **Particular form of Peirce decomposition.**

Lemma 3.1. *Let A be a n-dimensional non nil 3-Jordan evolution algebra with $n \geq 2$. Assume that the algebra A admits a natural basis $B = (e_1, \ldots, e_n)$ such that for any $e_i \in B$, we have $e_i^3 = 0$. Then, there are $i_0, i_1 \in \{1, \ldots, n\}$ distinct such that $(e_{i_0}^2 e_{i_1})e_{i_0} \neq 0$.*

Proof. Consider a natural basis $B = (e_1, \ldots, e_n)$ of A such that for any $e_i \in B$, we have $e_i^3 = 0$. Suppose that for all $i, j \in \{1, \ldots, n\}$ distinct, we have $(e_i^2 e_j)e_i = 0$. The assertions (f) and (j) of Theorem 3.1 lead respectively to $((e_i^2 e_j)e_k)e_i = 0$ for i, j, k pairwise distinct and to $((e_i^2 e_j)e_t)e_k = ((e_i^2 e_k)e_t)e_j = 0$ for i, j, k, t pairwise distinct with $j < k$. It follows that the algebra A is nil of nil-index at most 5 and we get the lemma. □

Proposition 3.3. *Let A be a n-dimensional non nil 3-Jordan evolution algebra with $n \geq 2$. Assume that the algebra A admits a natural basis $B = (e_1, \ldots, e_n)$ such that for any $e_i \in B$, we have $e_i^3 = 0$. Then, the algebra A admits a nonzero idempotent.*

Proof. Since the algebra A being non nil, the Lemma 3.1 tells us that there are $i_0, i_1 \in \{1, \ldots n\}$ distinct such that $(e_{i_0}^2 e_{i_1})e_{i_0} \neq 0$, i.e. $a_{i_0 i_1} a_{i_1 i_0} \neq 0$. In the following, we will consider such integers.

- Suppose that $\dim(A) = 2$, then $e_{i_0}^2 = a_{i_0 i_1} e_{i_1}$ and $e_{i_1}^2 = a_{i_1 i_0} e_{i_0}$ with $a_{i_0 i_1}, a_{i_1 i_0} \in F^*$. The algebra A is a 3-Jordan algebra. Set $u_0 = (a_{i_0 i_1}^2 a_{i_1 i_0})^{-\frac{1}{3}} e_{i_0}$ and $u_1 = (a_{i_0 i_1} a_{i_1 i_0}^2)^{-\frac{1}{3}} e_{i_1}$; the family (u_0, u_1) is a natural basis of the algebra A and its multiplication table is defined by $u_0^2 = u_1$ and $u_1^2 = u_0$. It follows that the vector $e = u_0 + u_1$ is a nonzero idempotent of A.

- Assume that $\dim(A) \geq 3$ and let $k \in \{1, \ldots, n\}$ distinct from i_0 and i_1.

  (a) The assertion (f) of Theorem 3.1 leads to relations

$$\begin{aligned} 0 &= ((e_{i_0}^2 e_{i_1})e_k)e_{i_0} = a_{i_0i_1}a_{i_1k}a_{ki_0}e_{i_0}^2, \text{ i.e. } a_{i_1k}a_{ki_0} = 0 \\ 0 &= ((e_{i_0}^2 e_k)e_{i_1})e_{i_0} = a_{i_0k}a_{ki_1}a_{i_1i_0}e_{i_0}^2, \text{ i.e. } a_{i_0k}a_{ki_1} = 0. \end{aligned}$$

  (b) The assertion (h) of Theorem 3.1 gives the following equalities

$$\begin{aligned} 0 &= ((e_{i_0}^2 e_k)e_{i_1})e_k + ((e_k^2 e_{i_0})e_{i_1})e_{i_0} = a_{ki_0}a_{i_0i_1}a_{i_1i_0}e_{i_0}^2, \text{ i.e. } a_{ki_0} = 0, \\ 0 &= ((e_{i_1}^2 e_k)e_{i_0})e_k + ((e_k^2 e_{i_1})e_{i_0})e_{i_1} = a_{ki_1}a_{i_1i_0}a_{i_0i_1}e_{i_1}^2, \text{ i.e. } a_{ki_1} = 0. \end{aligned}$$

  (c) The assertion (g) of Theorem 3.1 involves the following identities

$$\begin{aligned} 0 &= ((e_{i_0}^2 e_{i_1})e_{i_0})e_k + ((e_{i_0}^2 e_k)e_{i_0})e_{i_1} = a_{i_0i_1}a_{i_1i_0}a_{i_0k}e_k^2, \text{ i.e. } a_{i_0k} = 0 \text{ for } e_k^2 \neq 0, \\ 0 &= ((e_{i_1}^2 e_{i_0})e_{i_1})e_k + ((e_{i_1}^2 e_k)e_{i_1})e_{i_0} = a_{i_1i_0}a_{i_0i_1}a_{i_1k}e_k^2, \text{ i.e. } a_{i_1k} = 0 \text{ for } e_k^2 \neq 0. \end{aligned}$$

  We deduce that $e_{i_0}^2 = a_{i_0i_1}e_{i_1} + z_0$, $e_{i_1}^2 = a_{i_1i_0}e_{i_0} + z_1$ with $z_0, z_1 \in ann(A)$. Set $u_0 = a(e_{i_0} + a_{i_1i_0}^{-1}z_1)$, $u_1 = b(e_{i_1} + a_{i_0i_1}^{-1}z_0)$ and determine the nonzero scalars a and b such that $u_0^2 = u_1$ and $u_1^2 = u_0$. The equality $u_1 = u_0^2 = a^2 a_{i_0i_1}(e_{i_1} + a_{i_0i_1}^{-1}z_0) = a^2b^{-1}a_{i_0i_1}u_1$ leads to $b = a^2 a_{i_0i_1}$ and the equality $u_0 = u_1^2 = b^2 a_{i_1i_0}(e_{i_0} + a_{i_1i_0}^{-1}z_1) = b^2a^{-1}a_{i_1i_0}u_0$ involves $1 = a^3a_{i_0i_1}^2 a_{i_1i_0}$. Thus, $a = (a_{i_0i_1}^{-2}a_{i_1i_0}^{-1})^{\frac{1}{3}}$ and $b = a_{i_0i_1}(a_{i_0i_1}^{-2}a_{i_1i_0}^{-1})^{\frac{2}{3}} = (a_{i_0i_1}^{-1}a_{i_1i_0}^{-2})^{\frac{1}{3}}$. Consequently, the family $(u_0, u_1, e_k)_{k \notin \{i_0, i_1\}}$ is a natural basis of the algebra A and the multiplication table of A in this natural basis is of form

$$u_0^2 = u_1,\ u_1^2 = u_0,\ e_k^2 = \sum_{\substack{1 \leq j \leq n \\ j \notin \{k, i_0, i_1\}}} a_{kj}e_j.$$

  It follows that the vector $e = u_0 + u_1$ is a nonzero idempotent of the algebra A.

□

Theorem 3.3. *Let A be a n-dimensional non nil 3-Jordan evolution algebra with $n \geq 2$. Suppose that the algebra A admits a natural basis $B = (e_1, \ldots, e_n)$ such that for all $e_i \in B$, we have $e_i^3 = 0$. Then, the algebra A admits the following decomposition into a direct sum of algebras given by:*

$$A = A_{ss} \oplus N \tag{7}$$

where N is a finite-dimensional nil 3-Jordan evolution algebra and
$A_{ss} = \bigoplus_{k=0}^{s} Alg(u_{2k}, u_{2k+1})$. For $0 \leq k \leq s$, the algebra $Alg(u_{2k}, u_{2k+1})$ is a 2-dimensional evolution algebra in the natural basis (u_{2k}, u_{2k+1}) which multiplication table is given by: $u_{2k}^2 = u_{2k+1}$ and $u_{2k+1}^2 = u_{2k}$.

Proof. If the algebra A is of dimension 2, we then obtain the theorem with $s = 0$ and $N = 0$. In the following, assume that $\dim(A) > 2$. The proof of the Proposition 3.3, tells us that the algebra A admits a natural basis $B' = (u_0, u_1, e_k)_{1\leq k\leq n-2}$ which multiplication table is of form

$$u_0^2 = u_1,\ u_1^2 = u_0, e_k^2 = \sum_{k\neq j=1}^{n-2} a_{kj}e_j \text{ with } 1 \leq k \leq n-2.$$

Moreover, the vector $e = u_0 + u_1$ is a nonzero idempotent of A. Let us determine Peirce subspaces relative to the idempotent e. Let $x = au_0 + bu_1 + \sum_{k=1}^{n-2} x_k e_k \in A$, we have $ex = au_0^2 + bu_1^2 = au_1 + bu_0$.

- For $x \in A_0$, we have $0 = ex = au_1 + bu_0$, i.e. $a = b = 0$ and $x = \sum_{k=1}^{n-2} x_k e_k$. So $A_0 \subset Alg(e_1, \ldots, e_{n-2})$. Reciprocally, we have $ee_k = 0$ for all $1 \leq k \leq n-2$. We deduce that $A_0 = Alg(e_1, \ldots, e_{n-2})$ is a 3-Jordan evolution algebra in the natural basis $(e_1, \ldots, e_{n-2})$.
- For $x \in A_{\frac{1}{2}}$, we have $\frac{1}{2}x = ex = au_1 + bu_0$, i.e. $a = b = x_k = 0$ for all $1 \leq k \leq n-2$. So $x = 0$ and $A_{\frac{1}{2}} = 0$.
- For $x \in A_{-1}$, we have $-x = ex = au_1 + bu_0$, $a = -b$ and $x_k = 0$ for all $1 \leq k \leq n-2$. So $x = a(u_0 - u_1) \in F(u_0 - u_1)$ and reciprocally, we have $e(u_0 - u_1) = u_0^2 - u_1^2 = -(u_0 - u_1)$. We deduce that $A_{-1} = F(u_0 - u_1)$.
- For $x \in A_1$, we have $x = ex = au_1 + bu_0$, $a = b$ and $x_k = 0$ for all $1 \leq k \leq n-2$. So $x = a(u_0 + u_1) \in F(u_0 + u_1)$ and reciprocally, we have $e(u_0 + u_1) = u_0^2 + u_1^2 = u_0 + u_1$. We deduce that $A_1 = F(u_0 + u_1)$.

Consequently, the Peirce decomposition of A is of form

$$A = F(u_0 + u_1) \oplus F(u_0 - u_1) \oplus Alg(e_1, \ldots, e_{n-2}).$$

The subspaces $F(u_0 + u_1) \oplus F(u_0 - u_1)$ is an evolution algebra with a natural basis (u_0, u_1) and $A_0 = Alg(e_1, \ldots, e_{n-2})$ is a 3-Jordan evolution algebra. Furthermore, $Alg(u_0, u_1) \oplus A_0$ is a direct sum of algebras. If A_0 is a nil algebra, it is finished and we get the theorem with $s = 0$. Otherwise, we repeat the process on the algebra A_0. Since A is a finite-dimensional algebra, we obtain the result after a finite number of operations. □

3.4. **General form of Peirce decomposition.** Assumed that A is a finite-dimensional 3-Jordan non nil evolution algebra admitting a natural basis $B = (e_1, \ldots, e_n)$ such that there is $i_0 \in \{1, \ldots, n\}$ verifying $e_{i_0}^3 \neq 0$. By putting $u_{i_0} = a_{i_0 i_0}^{-1} e_{i_0}$, we can take $a_{i_0 i_0} = 1$ and we will get $e_{i_0}^3 = e_{i_0}^2$. So, in the following, we take $a_{i_0 i_0} = 1$. The assertion (a) of Theorem 3.1 tells us that the vector $e_{i_0}^2$ is a nonzero idempotent of A. Let $j \in \{1, \ldots, n\}$ distinct from i_0 such that $e_j^2 \neq 0$.

- The assertion (b) of Theorem 3.1 gives us $0 = (e_{i_0}^2 e_j)e_{i_0} = a_{i_0 j} a_{j i_0} e_{i_0}^2$, i.e.

$$a_{i_0 j} a_{j i_0} = 0. \tag{8}$$

- The assertion (d) of Theorem 3.1 leads to

$$e_{i_0}^2 e_j^2 = a_{i_0 j} a_{jj} e_j^2. \tag{9}$$

- The assertion (c) of Theorem 3.1 involves $0=(e_j^2e_{i_0})e_j^2-((e_j^2e_{i_0})e_j)e_j-e_j^4e_{i_0}=a_{ji_0}e_{i_0}^2e_j^2-a_{jj}^2a_{ji_0}e_{i_0}^2=a_{ji_0}a_{i_0j}a_{jj}e_j^2-a_{jj}^2a_{ji_0}e_{i_0}^2=-a_{jj}^2a_{ji_0}e_{i_0}^2$, i.e.

$$a_{jj}a_{ji_0}=0. \tag{10}$$

- The assertion (d) of Theorem 3.1 gives us $0=e_j^3e_{i_0}^2-(e_j^3e_{i_0})e_{i_0}-((e_j^2e_{i_0})e_{i_0})e_j=a_{jj}e_j^2e_{i_0}^2=a_{jj}^2a_{i_0j}e_j^2$, i.e.

$$a_{jj}a_{i_0j}=0 \tag{11}$$

  It follows that $e_{i_0}^2e_j^2=a_{i_0j}a_{jj}e_j^2=0$.
- The assertion (c) of Theorem 3.1 leads to $0=(e_{i_0}^2e_j)e_{i_0}^2-((e_{i_0}^2e_j)e_{i_0})e_{i_0}-e_{i_0}^2e_j=-a_{i_0j}e_j^2$, i.e. $a_{i_0j=0}$. Moreover, the relation $0=e_{i_0}^2e_j^2=\sum_{k=1}^n a_{i_0k}a_{jk}e_k^2=a_{i_0i_0}a_{ji_0}e_{i_0}^2$, i.e. $a_{ji_0}=0$.

We deduce that $e_{i_0}^2=e_{i_0}+z_0$ with $z_0\in ann(A)$. The vector $v_1=e_{i_0}^2=e_{i_0}+z_0$ is a nonzero idempotent of A and the Peirce decomposition of the algebra A relative to v_1 is given by:

$$A=kv_1\oplus A_0 \tag{12}$$

where A_0 is a $(n-1)$-dimensional 3-Jordan evolution algebra generated by the family $\{e_k \mid 1\leq k\neq i_0\leq n\}$. This Peirce decomposition is a direct sum of the algebras. We deduce that A_0 is a finite-dimensional 3-Jordan evolution algebra.

Theorem 3.4 (Decomposition theorem)**.** *Let A be a finite-dimensional 3-Jordan evolution algebra. Then, the algebra A admits a decomposition into a direct sum of algebras given by:*

$$A=Fv_1\oplus\cdots\oplus Fv_p\oplus A_{ss}\oplus N \tag{13}$$

where the family $(v_1,\ldots,v_p)$ is empty or a pairwise orthogonal idempotents family; A_{ss} is a $(2s)$-dimensional evolution algebra defined in the Theorem 3.3 and its nil-radical N is zero or a finite-dimensional nil 3-Jordan evolution algebra.

Proof. If the algebra A admits a natural basis B such that for all $e\in B$, we have $e^3=0$, then, A admits a decomposition into a direct sum of algebras given by the identity (7). Thus, we obtain the Theorem with the family$(v_1,\ldots,v_p)$ which is empty. Otherwise, it admits a decomposition given by the identity (12). If the algebra A_0 admits a natural basis B_0 such that for all $e\in B_0$, we have $e^3=0$, then, we get the theorem with $p=1$. Otherwise, the process is repeated on A_0 and since A is a finite-dimensional, the process will end for a certain integer p. We then obtain the theorem. □

Proposition 3.4. *Let A be a finite-dimensional 3-Jordan evolution algebra. If the nil-radical of A is of nil-index at most 4, then, A is almost generalized Jordan algebra. In particular, it verifies the identity $x^3y-((xy)x)x=0$ with $x,y\in A$.*

Proof. Suppose that the algebra A has a nil-index at most 4, and consider the decomposition of A into a direct sum of algebras given by the identity (13). Let $x = \sum_{k=1}^{p} x_k v_k + \sum_{j=0}^{s}(x'_{2j+1}u_{2j+1} + x'_{2j}u_{2j}) + z_x,\ y = \sum_{k=1}^{p} y_k v_k + \sum_{j=0}^{s}(y'_{2j+1}u_{2j+1} + y'_{2j}u_{2j}) + z_y \in A$ with $z_x, z_y \in N$. We have

$$
\begin{aligned}
x^2 &= \sum_{k=1}^{p} x_k^2 v_k + \sum_{j=0}^{s}(x'^2_{2j+1}u_{2j} + x'^2_{2j}u_{2j+1}) + z_x^2 \\
x^3 &= \sum_{k=1}^{p} x_k^3 v_k + \sum_{j=0}^{s} x'_{2j+1}x'_{2j}(x'_{2j}u_{2j} + x'_{2j+1}u_{2j+1}) + z_x^3 \\
x^3 y &= \sum_{k=1}^{p} x_k^3 y_k v_k + \sum_{j=0}^{s} x'_{2j+1}x'_{2j}(x'_{2j}y_{2j}u_{2j+1} + x'_{2j+1}y'_{2j+1}u_{2j}) \\
xy &= \sum_{k=1}^{p} x_k y_k v_k + \sum_{j=0}^{s}(x'_{2j+1}y'_{2j+1}u_{2j} + x'_{2j}y'_{2j}u_{2j+1}) + z_x z_y \\
(xy)x &= \sum_{k=1}^{p} x_k^2 y_k v_k + \sum_{j=0}^{s} x'_{2j+1}x'_{2j}(y'_{2j+1}u_{2j+1} + y'_{2j}u_{2j}) + (z_x z_y)z_x \\
((xy)x)x &= \sum_{k=1}^{p} x_k^3 y_k v_k + \sum_{j=0}^{s} x'_{2j+1}x'_{2j}(x'_{2j+1}y'_{2j+1}u_{2j} + x'_{2j}y'_{2j}u_{2j+1}) \\
((xy)x)x &= x^3 y.
\end{aligned}
$$

It follows that A is almost generalized Jordan algebra. □

Remark 3.1. *The nil* 3*-Jordan evolution algebras of dimension* 2 *and* 3 *are almost generalized Jordan algebras.*

3.5. **Derivations.** *A derivation* on the F-algebra A is a linear operator $d : A \longrightarrow A$ such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in A$. The set $Der_K(A)$ of all derivations of A is a Lie algebra where the multiplication of two derivations d and d' of A is defined by $[d, d'] = dod' - d'od$.

Let A be a finite-dimensional evolution algebra with the natural basis $B = (e_1, \ldots, e_n)$ and d be a derivation on A; set $d(e_i) = \sum_{k=1}^{n} d_{ik}e_k$. In [9, page 27] the authors show that for $1 \le i \ne j \le n$, if the family (e_i^2, e_j^2) is linear independent, then $d_{ij} = d_{ji} = 0$ and if $e_i^2 = \alpha_i e_j^2$ then $d_{ij} = -\alpha_i d_{ji}$. It follows that for $e_i \in ann(A)$, we have $d(e_i) \in ann(A)$.

Theorem 3.5. *Let A be a finite-dimensional* 3*-Jordan evolution algebra. Consider the decomposition of A into a direct sum of algebras given by the identity* (13). *Then $Der_K(A) = Der_K(N)$.*

Proof. Let d be a derivation of A and $v = \sum_{k=1}^{p} a_k v_k + \sum_{j=0}^{s}(b_{2j+1}u_{2j+1} + b_{2j}u_{2j})$. [9, Proof of Proposition 2.4] tells us that $d(v_k) = 0$ for all $1 \le k \le p$. For all $1 \le j \le s$, we have $d(u_{2j+1}) = a_{2j+1}u_{2j+1} + z_{2j+1}$ and $d(u_{2j}) = a_{2j}u_{2j} + z_{2j}$ with $a_{2j+1},\ a_{2j} \in F$ and $z_{2j+1}, z_{2j} \in ann(A)$. The equality $d(u_{2j}) = d(u_{2j+1}^2) = 2u_{2j+1}d(u_{2j+1}) = 2a_{2j+1}u_{2j+1}^2 = 2a_{2j+1}u_{2j}$, leads to $a_{2j} = 2a_{2j+1}$ and $z_{2j} = 0$. Also, the equality $d(u_{2j+1}) = d(u_{2j}^2) = 2u_{2j}d(u_{2j}) = 2a_{2j}u_{2j}^2 = 2a_{2j}u_{2j+1}$, involves $a_{2j+1} = 2a_{2j}$; $z_{2j+1} = 0$. We deduce that $a_{2j} = a_{2j+1} = 0$ and $d(u_{2j}) = d(u_{2j+1}) = 0$, i.e. $Der_K(A) = Der_K(N)$. □

Example 3.4. *Consider the algebra defined in the Example 3.1. Let* $N = Alg(e_3, \ldots, e_7)$ *and* $d \in Der_K(N)$.

- *The family* (e_3^2, e_j^2) *is linear independent for* $4 \leq j \leq 6$, *then* $d(e_3) = d_{33}e_3 + d_{37}e_7$.
- *The relation* $e_3^2 = e_4$ *gives us* $d(e_4) = d(e_3^2) = 2e_3d(e_3) = 2d_{33}e_3^2 = 2d_{33}e_4$.
- *The relation* $e_5^2 = -e_6^2$ *and the linear independence of the families* (e_5^2, e_j^2) *and* (e_6^2, e_j^2) *for* $j \in \{3, 4\}$ *lead to* $d(e_5) = d_{55}e_5 + d_{56}e_6 + d_{57}e_7$ *and* $d(e_6) = -d_{56}e_5 + d_{66}e_6 + d_{67}e_7$.
- *The relation* $e_5 + e_6 = e_4^2$ *gives us* $d(e_5) + d(e_6) = d(e_4^2) = 2e_4d(e_4) = 4d_{33}e_4^2 = 4d_{33}(e_5 + e_6)$, *i.e.* $d_{55} = 4d_{33} + d_{56}$, $d_{66} = 4d_{33} - d_{56}$ *and* $d_{67} = -d_{57}$. *We deduce that* $d(e_5) = (4d_{33} + d_{56})e_5 + d_{56}e_6 + d_{57}e_7$ *and* $d(e_6) = -d_{56}e_5 + (4d_{33} - d_{56})e_6 - d_{57}e_7$.
- *The relation* $e_7 = e_5^2$ *involves* $d(e_7) = d(e_5^2) = 2e_5d(e_5) = 2(4d_{33} + d_{56})e_7$.

Set

$$d_1 = diag(1, 2, 4, 4, 8)\ ;\ d_2 = E_{51}\ ;\ d_3 = E_{53} - E_{54}\ ;\ d_4 = E_{33} - E_{44} + E_{43} - E_{34} + 2E_{55}$$

where E_{ij} *denotes the elementary matrix of order* 5 *and* $diag(1, 2, 4, 4, 8)$ *denotes a diagonal matrix. We have* $Der_K(A) = Alg(d_1, d_2, d_3, d_4)$ *and the multiplication table is defined by :*

$$[d_1, d_2] = 7d_2\ ;\ [d_1, d_3] = 4d_3\ ;\ [d_2, d_4] = -2d_2\ ;\ [d_3, d_4] = -d_3.$$

The multiplication not mentioned are vanish. As consequence, the Lie algebra $Der_K(A)$ *is not abelian.*

4. Classification of the nil 3-Jordan evolution algebras

In finite dimension, the knowledge of the nil 3-Jordan evolution algebras is sufficient to construct 3-Jordan evolution algebras according to the Theorem 3.4. In this section, we give the classification, up to isomorphism and in dimension at most 6, of nil 3-Jordan indecomposable evolution algebra that are not an almost generalized Jordan algebra. The Theorem 3.2 and the Proposition 3.4, tell us that such algebras are of nil-index 5. Let N be an indecomposable nil 3-Jordan evolution algebra of nil-index 5; its type is of form $[n_1, n_2, n_3, n_4]$ where n_1, n_2, n_3, n_4 are the nonzero integers such that $n_1 + \cdots + n_4 = \dim(A)$. Necessarily, $\dim(N) \geq 4$ and let $B = (e_1, \ldots, e_n)$ be a natural base of A, we have $((e_i^2e_j)e_k)e_t = 0$ and $(e_i^2e_j)e_k^2 = 0$ for all $1 \leq i, j, k, t \leq n$ by the Theorem 3.2.

Lemma 4.1. *There is no nil* 3*-Jordan evolution algebra of type* $[n_1, 1, n_3, n_4]$ *where* n_1, n_3, n_4 *are the nonzero integers.*

Proof. Assume that N is a nil 3-Jordan evolution algebra of type $[n_1, 1, n_3, n_4]$ where n_1, n_3, n_4 are the nonzero integers. Consider a natural basis B of N. According to [3, Page 16], it admits a disjoint union decomposition given by:

$$B = B_1 \cup B_2 \cup B_3 \cup B_4 \text{ with } B_i = \{e \in B \ \mid \ e^2 \in ann^{i-1}(N) \text{ and } e \notin ann^{i-1}(N)\}.$$

Since $card(B_2) = 1$, let e_{i_0} be the unique element of B_2 ; we have $e_{i_0}^2 \neq 0$. Set $B_3 = (e_{k_1}, \ldots, e_{k_{n_3}})$, we have $e_{k_s}^2 = a_{k_s i_0} e_{i_0} + z_k$ where $z_k \in ann(N)$, $a_{k_s i_0} \neq 0$ and $s \in \{1, \ldots, n_3\}$. Consider $e_j \in B_4$; for all $e_{k_s} \in B_3$, the equality $0 = (e_j^2 e_{k_s}) e_{k_s}^2 = a_{jk_s} e_{k_s}^2 e_{k_s}^2 = a_{jk_s} a_{k_s i_0}^2 e_{i_0}^2$ leads to $a_{jk_s} = 0$ for all $s \in \{1, \ldots, n_3\}$. It follows that $e_j^2 \in ann^2(N)$ and $e_j \in ann^3(N)$: impossible. We deduce that N cannot be of the type $[n_1, 1, n_3, n_4]$. □

Proposition 4.1. [3, *Corollary 2.6*] *Let A be a finite-dimensional evolution algebra such that* $\dim(ann(A)) \geq \frac{1}{2}\dim(A) \geq 1$. *Then, the algebra A is decomposable.*

4.1. **Classification in dimension 4 and 5.**

Proposition 4.2. *Any 4-dimensional indecomposable nil 3-Jordan evolution algebra is almost generalized Jordan algebra.*

Proof. Let N be an indecomposable nil 3-Jordan algebra. Its nil-index is at most 5. If the nil-index is $2, 3$ or 4, then, the algebra is almost generalized Jordan algebra. If the nil-index is 5, then its type is $[1, 1, 1, 1]$: impossible by the Lemma 4.1. □

Proposition 4.3. *Let N be a 5-dimensional indecomposable nil evolution algebra that is not almost generalized Jordan algebra. Then, A is isomorphic to one and only one, of the following algebras:*

(1) $N_{5,1}\ :\ e_1^2 = e_2,\ e_2^2 = e_3 + e_4,\ e_3^2 = e_5,\ e_4^2 = -e_5,\ e_5^2 = 0.$

(2) $N_{5,2}(\alpha)\ :\ e_1^2 = e_2 + \alpha(e_3 + e_4),\ e_2^2 = e_3 + e_4,\ e_3^2 = e_5,\ e_4^2 = -e_5,\ e_5^2 = 0$ *with* $\alpha \in F^*$.

Proof. We have $\dim(ann(N)) < \frac{1}{2}\dim(N)$, i.e. $\dim(ann(N)) = 1$ or 2. By Lemma 4.1 the only possible type of N is $[1, 2, 1, 1]$. Let $B = (e_1, \ldots, e_5)$ be a natural basis of N such that $ann(N) = Alg(e_5)$, $ann^2(N) = Alg(e_3, e_4, e_5)$, $ann^3(N) = Alg(e_2, e_3, e_4, e_5)$. The multiplication table of N in this basis is of form $e_1^2 = a_{12}e_2 + a_{13}e_3 + a_{14}e_4 + a_{15}e_5,\ e_2^2 = a_{23}e_3 + a_{24}e_4 + a_{25}e_5,\ e_3^2 = a_{35}e_5,\ e_4^2 = a_{45}e_5, e_5^2 = 0$ with $(a_{23}, a_{24}) \neq 0$ and $a_{12}, a_{35}, a_{45} \in F^*$. The equality $0 = (e_1^2 e_2)e_2^2 = a_{12}(a_{23}^2 e_3^2 + a_{24}^2 e_4^2)$ involves $a_{23}, a_{24} \in F^*$ otherwise $a_{23} = a_{24} = 0$: impossible. It follows that

$$e_4^2 = -a_{23}^2 a_{24}^{-2} e_3^2.$$

The equality $0 = (e_1^2 e_2)e_1^2 = a_{12}(a_{13}a_{23}e_3^2 + a_{14}a_{24}e_4^2) = a_{12}a_{23}(a_{13} - a_{14}a_{23}a_{24}^{-1})e_3^2$ leads to

$$a_{13} = a_{14}a_{23}a_{24}^{-1}.$$

Set $u_2 = a_{12}e_2 + (a_{15} - a_{14}a_{24}^{-1}a_{25})e_5,\ u_3 = a_{12}^2(a_{23}e_3 + a_{25}e_5),\ u_4 = a_{12}^2 a_{24}e_4,\ u_5 = a_{12}^4 a_{23}^2 a_{35}e_5$. The family $(e_1, u_2, u_3, u_4, u_5)$ is a natural basis of N and the multiplication table of N in this basis is given by:

$$e_1^2 = u_2 + \alpha(u_3 + u_4),\ u_2^2 = u_3 + u_4,\ u_3^2 = u_5,\ u_4^2 = -u_5,\ u_5^2 = 0.$$

We have $\mathcal{U}_3 \oplus \mathcal{U}_1 = vect(u_2, u_5); \mathcal{U}_4 \oplus \mathcal{U}_1 = vect(e_1, u_5)$ and $(\mathcal{U}_4 \oplus \mathcal{U}_1)^2 = vect(u_2 + \alpha(u_3 + u_4))$.

- For $(\mathcal{U}_4 \oplus \mathcal{U}_1)^2 \subseteq \mathcal{U}_4 \oplus \mathcal{U}_1$, then $\alpha = 0$ and N is isomorphic to $N_{5,1}$.
- For $(\mathcal{U}_4 \oplus \mathcal{U}_1)^2 \nsubseteq \mathcal{U}_4 \oplus \mathcal{U}_1$, then $\alpha \neq 0$ and N is isomorphic to $N_{5,2}(\alpha)$.

□

4.2. **Classification in dimension 6.** We have $\dim(ann(N)) < \frac{1}{2}\dim(N)$, i.e. $\dim(ann(N)) = 1$ or 2. By the Lemma 4.1, we deduce that the possible types of N are $[1,2,1,2]$, $[1,2,2,1]$, $[2,2,1,1]$.

Proposition 4.4. *Let N be an indecomposable nil evolution algebra of type $[1,2,1,2]$ that is not be an almost generalized Jordan algebra. Then, A is isomorphic to one and only one, of the following algebras:*

(1) $N_{6,1}(\alpha)\ :\ e_1^2 = e_3,\ e_2^2 = \alpha e_3,\ e_3^2 = e_4 + e_5,\ e_4^2 = e_5,\ e_5^5 = -e_6, e_6^2 = 0$ *with* $\alpha \neq 0$;

(2) $N_{6,2}(\alpha,\beta)\ :\ e_1^2 = e_3 + \alpha(e_4+e_5),\ e_2^2 = \beta e_1^2,\ e_3^2 = e_4 + e_5,\ e_4^2 = e_5,\ e_5^5 = -e_6,\ e_6^2 = 0$ *with* $\alpha\beta \neq 0$;

(3) $N_{6,3}(\alpha,\beta)\ :\ e_1^2 = e_3,\ e_2^2 = \alpha e_3 + \beta e_6,\ e_3^2 = e_4 + e_5,\ e_4^2 = e_5,\ e_5^5 = -e_6, e_6^2 = 0$ *with* $\alpha\beta \neq 0$;

(4) $N_{6,4}(\alpha,\beta,\gamma,\delta)\ :\ e_1^2 = e_3 + \alpha(e_4+e_5),\ e_2^2 = \beta e_3 + \gamma(e_4+e_5) + \delta e_6,\ e_3^2 = e_4 + e_5,\ e_4^2 = e_5,\ e_5^5 = -e_6, e_6^2 = 0$ *with* $(\alpha,\gamma) \neq 0$ *;* $\{\delta \neq 0$ *or* $\gamma \neq \alpha\beta\}$ *and* $\beta \in F^*$.

Proof. N is of type $[1,2,1,2]$. It admits a natural basis $B = (e_1,\ldots,e_6)$ such that $ann(N) = Alg(e_6)$, $ann^2(N) = Alg(e_6,e_5,e_4)$ and $ann^3(N) = Alg(e_6,e_5,e_4,e_3)$. The multiplication table of N in this natural basis is of form: $e_1^2 = a_{13}e_3 + \cdots + a_{16}e_6$, $e_2^2 = a_{23}e_3 + \cdots + a_{26}e_6$, $e_3^2 = a_{34}e_4 + a_{35}e_5 + a_{36}e_6$, $e_4^2 = a_{46}e_6, e_5^2 = a_{56}e_6$, $e_6^2 = 0$ with $a_{13}, a_{23}, a_{46}, a_{56} \in F^*$ and $(a_{34}, a_{35}) \neq 0$. The equality $0 = (e_1^2 e_3)e_3^2 = a_{13}e_3^2 e_3^2 = a_{13}(a_{34}^2 e_4^2 + a_{35}^2 e_5^2)$ gives us $a_{34}, a_{35} \in F^*$ and

$$e_5^2 = -a_{34}^2 a_{35}^{-2} e_4^2.$$

We also have

$$0 = (e_1^2 e_3)e_1^2 = a_{13}(a_{34}a_{14}e_4^2 + a_{35}a_{15}e_5^2) = a_{13}a_{34}(a_{14} - a_{34}a_{35}^{-1}a_{15})e_4^2,\ \text{i.e. } a_{14} = a_{34}a_{35}^{-1}a_{15}$$

and

$$0 = (e_1^2 e_3)e_2^2 = a_{13}(a_{34}a_{24}e_4^2 + a_{35}a_{25}e_5^2) = a_{13}a_{34}(a_{24} - a_{34}a_{35}^{-1}a_{25})e_4^2,\ \text{i.e. } a_{24} = a_{34}a_{35}^{-1}a_{25}.$$

Set $u_3 = a_{13}e_3 + (a_{16} - a_{15}a_{36}a_{35}^{-1})e_6$, $u_4 = a_{13}^2(a_{34}e_4 + a_{36}e_6)$, $u_5 = a_{13}^2 a_{35}e_5$, $u_6 = a_{13}^4 a_{34}^2 a_{46}e_6$. The family $(e_1, e_2, u_3, u_4, u_5, u_6)$ is a natural basis of A and the multiplication table is of form:

$$e_1^2 = u_3 + \alpha_1(u_4+u_5),\ e_2^2 = \alpha_2 u_3 + \alpha_3(u_4+u_5) + \alpha_4 u_6,\ u_3^2 = u_4 + u_5,\ u_4^4 = u_6,\ u_5^2 = -u_6, u_6^2 = 0$$

with $\alpha_2 \in F^*$ and $\alpha_1, \alpha_3, \alpha_4 \in F$. We have $\mathcal{U}_3 \oplus \mathcal{U}_1 = Alg(u_6, u_3)$, $\mathcal{U}_4 \oplus \mathcal{U}_1 = Alg(u_6, e_2, e_1)$ and $(\mathcal{U}_4 \oplus \mathcal{U}_1)^2 = Alg(\alpha_2 u_3 + \alpha_3(u_4+u_5) + \alpha_4 u_6, u_3 + \alpha_1(u_4+u_5))$.

- $(\mathcal{U}_4 \oplus \mathcal{U}_1)^2 \subseteq \mathcal{U}_3 \oplus \mathcal{U}_1$ and $\dim(\mathcal{U}_4 \oplus \mathcal{U}_1)^2 = 1$ then $\alpha_1 = \alpha_3 = \alpha_4 = 0$ and N is isomorphic to the algebra $N_{6,1}(\alpha_2)$.

- $(\mathcal{U}_4 \oplus \mathcal{U}_1)^2 \nsubseteq \mathcal{U}_3 \oplus \mathcal{U}_1$ and $\dim(\mathcal{U}_4 \oplus \mathcal{U}_1)^2 = 1$, then $\alpha_3 = \alpha_1\alpha_2$, $\alpha_4 = 0$ and $\alpha_1 \neq 0$. N is isomorphic to $N_{6,2}(\alpha_1, \alpha_2)$
- $(\mathcal{U}_4 \oplus \mathcal{U}_1)^2 \subseteq \mathcal{U}_3 \oplus \mathcal{U}_1$ and $\dim(\mathcal{U}_4 \oplus \mathcal{U}_1)^2 = 2$ then $\alpha_1 = \alpha_3 = 0$ and $\alpha_4 \neq 0$. N is isomorphic to the algebra $N_{6,3}(\alpha_2, \alpha_4)$;
- $(\mathcal{U}_4 \oplus \mathcal{U}_1)^2 \nsubseteq \mathcal{U}_3 \oplus \mathcal{U}_1$ and $\dim(\mathcal{U}_4 \oplus \mathcal{U}_1)^2 = 2$, then $(\alpha_1, \alpha_3) \neq 0$ and $\{\alpha_4 \neq 0$ or $\alpha_3 \neq \alpha_1\alpha_2\}$. The algebra N is isomorphic to $N_{6,4}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$

□

Proposition 4.5. *Let N be an indecomposable nil evolution algebra of type $[1, 2, 2, 1]$ that is not be an almost generalized Jordan algebra. Then, A is isomorphic to one and only one, of the following algebras:*

(1) $N_{6,5}(\alpha)\ :\ e_1^2 = e_2 + \alpha e_3,\ e_2^2 = e_4 + e_5,\ e_3^2 = -\frac{1}{\alpha^2}(e_4 + e_5), e_4^2 = e_6, e_5^2 = -e_6, e_6^2 = 0$ *with* $\alpha \in F^*$;

(2) $N_{6,6}(\alpha, \beta)\ :\ e_1^2 = e_2 + \alpha e_3 + \beta(e_4 + e_5),\ e_2^2 = e_4 + e_5,\ e_3^2 = -\frac{1}{\alpha^2}(e_4 + e_5), e_4^2 = e_6, e_5^2 = -e_6, e_6^2 = 0$ *with* $\alpha, \beta \in F^*$;

(3) $N_{6,7}(\alpha, \beta)\ :\ e_1^2 = e_2 + \alpha e_3,\ e_2^2 = e_4 + e_5,\ e_3^2 = \beta(e_4 + e_5), e_4^2 = e_6, e_5^2 = -e_6, e_6^2 = 0$ *with* $1 + \alpha^2\beta \neq 0$, $\beta \in F^*$ *and* $\alpha \in F$;

(4) $N_{6,8}(\alpha, \beta, \gamma)\ :\ e_1^2 = e_2 + \alpha e_3 + \beta(e_4 + e_5),\ e_2^2 = e_4 + e_5,\ e_3^2 = \gamma(e_4 + e_5), e_4^2 = e_6, e_5^2 = -e_6, e_6^2 = 0$ *with* $1 + \alpha^2\gamma \neq 0$ *and* $\beta, \gamma \in F^*$;

(5) $N_{6,9}(\alpha, \beta, \gamma)\ :\ e_1^2 = e_2 + \alpha e_3,\ e_2^2 = e_4 + e_5,\ e_3^2 = \beta(e_4 + e_5) + \gamma e_6, e_4^2 = e_6, e_5^2 = -e_6, e_6^2 = 0$ *with* $\alpha \in F$ *and* $\gamma, \beta \in F^*$;

(6) $N_{6,10}(\alpha, \beta, \gamma, \delta)\ :\ e_1^2 = e_2 + \alpha e_3 + \beta(e_4 + e_5),\ e_2^2 = e_4 + e_5,\ e_3^2 = \delta(e_4 + e_5) + \gamma e_6, e_4^2 = e_6, e_5^2 = -e_6, e_6^2 = 0$ *with* $\alpha \in F$ *and* $\beta, \gamma, \delta \in F^*$.

Proof. N is of type $[1, 2, 2, 1]$. It admits a natural basis $B = (e_1, \ldots, e_6)$ such that $ann(N) = Alg(e_6)$, $ann^2(N) = Alg(e_6, e_5, e_4)$ and $ann^3(N) = Alg(e_6, e_5, e_4, e_3, e_2)$. The multiplication table of N in this natural basis is given by $e_1^2 = a_{12}e_2 + a_{13}e_3 + a_{14}e_4 + a_{15}e_5 + a_{16}e_6,\ e_2^2 = a_{24}e_4 + a_{25}e_5 + a_{26}e_6,\ e_3^2 = a_{34}e_4 + a_{35}e_5 + a_{36}e_6,\ e_4^2 = a_{46}e_6,\ e_5^2 = a_{56}e_6,\ e_6^2 = 0$ with $a_{46}, a_{56} \in F^*$. The pairs $(a_{34}, a_{35}),\ (a_{24}, a_{25})$ and (a_{12}, a_{13}) are nonzero. By permitting the vectors e_3 and e_4 of the natural basis, we can take $a_{12} \neq 0$. The equality $0 = (e_1^2 e_2)e_2^2 = a_{12}(a_{24}^2 e_4^2 + a_{25}^2 e_5^2)$ gives $a_{24}a_{25} \neq 0$. Otherwise $a_{24} = a_{25} = 0$ this is impossible. Thus,

$$e_5^2 = -a_{24}^2 a_{25}^{-2} e_4^2.$$

The equality $0 = (e_1^2 e_2)e_1^2 = a_{12}(a_{14}a_{24}e_4^2 + a_{15}a_{25}e_5^2) = a_{12}a_{24}(a_{14} - a_{24}a_{25}^{-1}a_{15})e_5^2$ leads to

$$a_{14} = a_{24}a_{25}^{-1}a_{15}.$$

Also, the equality $0 = (e_1^2 e_2)e_1^3 = a_{12}(a_{24}a_{34}e_4^2 + a_{25}a_{35}e_5^2) = a_{12}a_{24}(a_{34} - a_{24}a_{25}^{-1}a_{35})e_5^2$ involves

$$a_{34} = a_{24}a_{25}^{-1}a_{35}.$$

It follows that $a_{34}, a_{35} \in F^*$. Set $u_2 = a_{12}e_2 + (a_{16} - a_{15}a_{26}a_{25}^{-1})e_6,\ u_3 = e_3,\ u_4 = a_{12}^2(a_{24}e_4 + a_{26}e_6),\ u_5^2 = a_{12}^2 a_{25}e_5,\ u_6 = a_{12}^4 a_{24}^2 a_{46}e_6$. Then, the family $(e_1, u_2, u_3, u_4, u_5, u_6)$ is a natural basis of N and its multiplication table is of form:

$$e_1^2 = u_2 + \alpha_1 u_3 + \alpha_2(u_4 + u_5),\ u_2^2 = u_4 + u_5,\ u_3^2 = \alpha_3(u_4 + u_5) + \alpha_4 u_6,\ u_4^2 = u_6,\ u_5^5 = -u_6,\ u_6^2 = 0$$

with $\alpha_3 \in F^*$ and $\alpha_1, \alpha_2, \alpha_4 \in F$. We have $\mathcal{U}_3 \oplus \mathcal{U}_1 = vect(u_6, u_3, u_2)$; $(\mathcal{U}_3 \oplus \mathcal{U}_1)^2 = vect(u_4 + u_5, \alpha_4 u_6)$; $\mathcal{U}_4 \oplus \mathcal{U}_1 = vect(u_6, e_1)$; $(\mathcal{U}_4 \oplus \mathcal{U}_1)^2 = vect(u_2 + \alpha_1 u_3 + \alpha_2(u_4 + u_5))$ and $((\mathcal{U}_4 \oplus \mathcal{U}_1)^2)^2 = vect\{(1 + \alpha_1^2\alpha_3)(u_4 + u_5) + \alpha_1^2\alpha_4 u_6\}$.

- For $\dim(\mathcal{U}_3 \oplus \mathcal{U}_1)^2 = 1$; $((\mathcal{U}_4 \oplus \mathcal{U}_1)^2)^2 = 0$ and $(\mathcal{U}_4 \oplus \mathcal{U}_1)^2 \subseteq \mathcal{U}_3 \oplus \mathcal{U}_1$, we have $\alpha_4 = \alpha_2 = 0$ and $\alpha_3 = -\frac{1}{\alpha_1^2}$ with $\alpha_1 \in F^*$. Then N is isomorphic to $N_{6,5}(\alpha_1)$.
- For $\dim(\mathcal{U}_3 \oplus \mathcal{U}_1)^2 = 1$; $((\mathcal{U}_4 \oplus \mathcal{U}_1)^2)^2 = 0$ and $(\mathcal{U}_4 \oplus \mathcal{U}_1)^2 \nsubseteq \mathcal{U}_3 \oplus \mathcal{U}_1$, we have $\alpha_4 = 0$, $\alpha_2 \in F^*$ and $\alpha_3 = -\frac{1}{\alpha_1^2}$ with $\alpha_1 \in F^*$. Then N is isomorphic to $N_{6,6}(\alpha_1, \alpha_2)$.
- For $\dim(\mathcal{U}_3 \oplus \mathcal{U}_1)^2 = 1$; $((\mathcal{U}_4 \oplus \mathcal{U}_1)^2)^2 \neq 0$ and $(\mathcal{U}_4 \oplus \mathcal{U}_1)^2 \subseteq \mathcal{U}_3 \oplus \mathcal{U}_1$, we have $\alpha_2 = \alpha_4 = 0$ and $1 + \alpha_1^2\alpha_3 \neq 0$. Then, N is isomorphic to $N_{6,7}(\alpha_1, \alpha_3)$
- For $\dim(\mathcal{U}_3 \oplus \mathcal{U}_1)^2 = 1$; $((\mathcal{U}_4 \oplus \mathcal{U}_1)^2)^2 \neq 0$ and $(\mathcal{U}_4 \oplus \mathcal{U}_1)^2 \nsubseteq \mathcal{U}_3 \oplus \mathcal{U}_1$, we have $\alpha_4 = 0$, $\alpha_2 \in F^*$ and $1 + \alpha_1^2\alpha_3 \neq 0$. Then N is isomorphic to $N_{6,8}(\alpha_1, \alpha_2, \alpha_3)$
- For $\dim(\mathcal{U}_3 \oplus \mathcal{U}_1)^2 = 2$ and $(\mathcal{U}_4 \oplus \mathcal{U}_1)^2 \subseteq \mathcal{U}_3 \oplus \mathcal{U}_1$, we have $\alpha_2 = 0$ and $\alpha_4 \in F^*$. Then, N is isomorphic to $N_{6,9}(\alpha_1, \alpha_3, \alpha_4)$
- For $\dim(\mathcal{U}_3 \oplus \mathcal{U}_1)^2 = 2$ and $(\mathcal{U}_4 \oplus \mathcal{U}_1)^2 \nsubseteq \mathcal{U}_3 \oplus \mathcal{U}_1$, we have $\alpha_2, \alpha_4 \in F^*$. Then, N is isomorphic to $N_{6,10}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$.

□

Proposition 4.6. *Let N be an indecomposable nil evolution algebra of type $[2, 2, 1, 1]$ that is not be an almost generalized Jordan algebra. Then, A is isomorphic to one and only one, of the following algebras:*

(1) $N_{611}\ :\ e_1^2 = e_2,\ e_2^2 = e_3 + e_4,\ e_3^2 = -e_4^2 = e_5,\ e_5^2 = e_6^2 = 0;$

(2) $N_{612}(\alpha)\ :\ e_1^2 = e_2 + \alpha(e_3 + e_4),\ e_2^2 = e_3 + e_4,\ e_3^2 = -e_4^2 = e_5,\ e_5^2 = e_6^2 = 0$ *with* $\alpha \in F^*$.

Proof. N is of type $[2, 2, 1, 1]$. Then, it admits a natural basis $B = (e_1, \ldots, e_6)$ such that $ann(N) = Alg(e_6, e_5)$, $ann^2(N) = Alg(e_6, e_5, e_4, e_3)$ and $ann^3(N) = Alg(e_6, e_5, e_4, e_3, e_2)$. The multiplication table of N in this natural basis is given by: $e_1^2 = a_{12}e_2 + a_{13}e_3 + a_{14}e_4 + a_{15}e_5 + a_{16}e_6,\ e_2^2 = a_{23}e_3 + a_{24}e_4 + a_{25}e_5 + a_{26}e_6,\ e_3^2 = a_{35}e_5 + a_{36}e_6,\ e_4^2 = a_{45}e_5 + a_{46}e_6,\ e_5^2 = e_6^2 = 0$ with $a_{12} \in F^*$. The pairs (a_{23}, a_{24}), (a_{35}, a_{36}) and (a_{45}, a_{46}) are nonzero. The equality $0 = (e_1^2 e_2)e_2^2 = a_{12}(a_{23}^2 e_3^2 + a_{24}^2 e_4^2)$ tells us that $a_{23}, a_{24} \in F^*$. We deduce that

$$e_4^2 = -a_{23}^2 a_{24}^{-2} e_3^2.$$

The equality $0 = (e_1^2 e_2)e_1^2 = a_{12}(a_{13}a_{23}e_3^2 + a_{14}a_{24}e_4^2) = a_{12}a_{23}(a_{13} - a_{14}a_{23}a_{24}^{-1})e_3^2$ gives

$$a_{13} = a_{14}a_{23}a_{24}^{-1}.$$

Set $u_2 = a_{12}e_2 + (a_{15} - a_{14}a_{25}a_{24}^{-1})e_5 + (a_{26} - a_{14}a_{26}a_{24}^{-1})e_6,\ u_3 = a_{12}^2 a_{23}e_3 + a_{12}^2(a_{25}e_5 + a_{26}e_6),\ u_4 = a_{12}^2 a_{24}e_4,\ u_5 = a_{12}^4 a_{23}^2(a_{35}e_5 + a_{36}e_6)$ and $u_6 = e_6$ for $a_{35} \neq 0$ otherwise we take $u_6 = e_5$. The family $(e_1, u_2, u_3, u_4, u_5, u_6)$ is a natural basis of N and its multiplication table is given by:

$$e_1^2 = u_2 + \alpha(u_3 + u_4),\ u_2^2 = u_3 + u_4,\ u_3^2 = u_5,\ u_4^2 = -u_5,\ u_5^2 = u_6^2 = 0 \text{ with } \alpha \in F.$$

We have $\mathcal{U}_3 \oplus \mathcal{U}_1 = vect(u_6, u_5, u_2)$; $\mathcal{U}_4 \oplus \mathcal{U}_1 = vect(u_6, u_5, e_1)$ and $(\mathcal{U}_4 \oplus \mathcal{U}_1)^2 = vect(u_2 + \alpha(u_3 + u_4))$.

- For $(\mathcal{U}_4 \oplus \mathcal{U}_1)^2 \subseteq \mathcal{U}_3 \oplus \mathcal{U}_1$, we have $\alpha = 0$ and N is isomorphic to $N_{6,11}$.
- For $(\mathcal{U}_4 \oplus \mathcal{U}_1)^2 \nsubseteq \mathcal{U}_3 \oplus \mathcal{U}_1$, we have $\alpha \neq 0$ and N is isomorphic to $N_{6,12}(\alpha)$.

□

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