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NEW APPROACH FOR SOLVING FUZZY MULTIOBJECTIVE QUADRATIC OPTIMIZATION PROBLEMS BASED ON CONJUGATE DIRECTIONS

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Abstract. This work focuses on studying fuzzy multiobjective quadratic optimization. It proposes a transformation that associates each fuzzy multiobjective quadratic optimization problem with a corresponding multiobjective quadratic optimization problem. This approach is based on the use of conjugate directions. These are vectors that are conjugate with respect to all symmetric, pairwise commutative matrices derived from the quadratic part of a multiobjective quadratic optimization problem. A discussion is presented on the set of solutions to the multiobjective quadratic optimization problem depending on the membership degree parameter α. It has also been proven that any Pareto-optimal solution to a multiobjective quadratic optimization problem is an efficient solution to the original fuzzy multiobjective quadratic optimization problem. The validity of the proposed method is established through the demonstration of its convergence to an optimal solution in a finite number of iterations. Then, the method is used to solve several fuzzy multiobjective quadratic optimization problems numerically.

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1. Introduction

A multiobjective optimization problem is defined as the simultaneous optimization of multiple objective functions. Improving one objective generally results in the deterioration of one or more of the others, making these functions conflicting. These types of problems are commonly encountered in fields such as engineering, resource management, and economics. Multiobjective optimization problems play a crucial role in decision-making processes. In contrast to single-objective optimization, which yields a unique optimal solution, multiobjective optimization produces a set of non-dominated solutions. This set is commonly referred to as the Pareto optimal solutions.

To determine Pareto optimal solutions, one technique involves the use of conjugate directions. Conjugate direction methods were initially developed for solving linear systems [1, 2], and later

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extended to unconstrained optimization of nonlinear functions with multiple variables [3–7]. In recent years, researchers have revisited these methods from a new perspective, particularly by extending them to multiobjective quadratic optimization. Notably, Fukuda *et al.* [8] proposed a conjugate direction-type method for solving unconstrained multiobjective quadratic optimization problems and proved convergence to the optimal solution within at most n iterations. Jian Chen *et al.* [9] employed the Barzilai-Borwein subspace to construct conjugate directions applicable to both quadratic and non-quadratic multiobjective problems. Tian *et al.* [10] incorporated conjugate direction strategies within evolutionary algorithms for solving multiobjective optimization problems. Chen Wang *et al.* [11] introduced a novel approach to conjugate gradient methods for multiobjective optimization by removing the conventional line search procedure. Ruifen Cao *et al.* [12] developed a hybrid algorithm combining conjugate gradient techniques with evolutionary strategies to optimize spot placement in intensity-modulated proton therapy. Kumar *et al.* [13] proposed a nonlinear conjugate gradient method for uncertain multiobjective optimization problems. Recently, B. B. Upadhyay *et al.* [14] proposed a conjugate direction-type method for solving interval-valued multiobjective quadratic optimization problems and demonstrated its convergence to the optimal solution in at most $n(n-1)$ iterations.

From these studies, it is clear that conjugate direction-type methods have been extensively investigated in the context of deterministic multiobjective optimization. However, such methods remain largely unexplored in the framework of fuzzy optimization despite their numerous advantages. The primary objective of this paper is, therefore to contribute to the literature by proposing a conjugate direction-type algorithm specifically designed to solve fuzzy multiobjective quadratic optimization problems.

The results of this work build upon those of [8] and [14]. In particular, any fuzzy multiobjective quadratic optimization problem is transformed into an equivalent unconstrained deterministic multiobjective quadratic optimization problem. It is verified that the matrices associated with the quadratic parts of the objective functions commute. Accordingly, we propose a conjugate direction-type algorithm for solving fuzzy multiobjective quadratic problems. This method exploits the relationship between the Pareto optimal solutions of the deterministic problem and the efficient solutions of the fuzzy counterpart. The proposed algorithm generates a set of conjugate directions with respect to the quadratic matrices of the objective functions. We also demonstrate that the algorithm converges to an optimal solution of the fuzzy multiobjective quadratic optimization problem. To validate the efficiency of the proposed method, several numerical examples are provided.

For a clear exposition of our results, the remainder of this article is organized as follows. Section 2 introduces essential concepts and definitions. Section 3 presents the main theoretical results. In Section 4, we provide a convergence analysis of the proposed method. Section 5 illustrates the practical performance of the algorithm through numerical examples. Finally, Section 6 concludes the paper.

2. Preliminaries and Basic Definitions

This section introduces the fundamental concepts used throughout this work.

Definition 2.1 ([15,16]). *Let $\mathcal{R}$ be a reference set. A fuzzy set $\tilde{A}$ in $\mathcal{R}$ is defined as:*

$$\tilde{A} = \{(t, \pi_{\tilde{A}}(t)) \mid t \in \mathcal{R}\}, \tag{1}$$

where $\pi_{\tilde{A}} : \mathcal{R} \to [0,1]$ is the membership function of the fuzzy number $\tilde{A}$.

Definition 2.2 ([16,17]). *Let $\tilde{a}$ be a fuzzy number. The α-cut of $\tilde{a}$ is an interval of the form $[a^L(\alpha), a^R(\alpha)]$, where $a^L(\alpha)$ and $a^R(\alpha)$ are upper semi-continuous functions with compact support.*

Definition 2.3 ([18]). *A triangular fuzzy number $\tilde{v} = (v - x_1, v, v + x_2)$, with $v - x_1$, v and $v + x_2$ real numbers, is called symmetric if $x_1 = x_2$.*

Definition 2.4 ([19]). *Let $\tilde{u}$ and $\tilde{v}$ be two arbitrary fuzzy numbers. A partial ordering between them is defined as follows:*

(*i*) $\tilde{u} \preceq \tilde{v}$ *if and only if* $\mathtt{D}(\tilde{u}) \leq \mathtt{D}(\tilde{v})$,
(*ii*) $\tilde{u} \succeq \tilde{v}$ *if and only if* $\mathtt{D}(\tilde{u}) \geq \mathtt{D}(\tilde{v})$,
(*iii*) $\tilde{u} \cong \tilde{v}$ *if and only if* $\mathtt{D}(\tilde{u}) = \mathtt{D}(\tilde{v})$,

where $\mathtt{D} : \mathbb{F} \to \mathbb{R}$ is a ranking function defined, for a fuzzy number $\tilde{u}$ with α-cut
$\tilde{u}^\alpha = [u^L(\alpha), u^R(\alpha)]$, by:

$$\mathtt{D}(\tilde{u}) = \frac{1}{2}\int_0^1 \tilde{u}^L(\alpha)\, d\alpha + \int_0^1 \tilde{u}^R(\alpha)\, d\alpha.$$

Definition 2.5 ([20]). *Let $\mathcal{R} \subseteq \mathbb{R}^n$ and let $\mathbb{F}$ denote the set of all fuzzy numbers over $\mathcal{R}$. Any function $\tilde{\mathfrak{h}} : \mathcal{R} \to \mathbb{F}$ is called a fuzzy-valued function. For each $\alpha \in [0,1]$, the α-cut of $\tilde{\mathfrak{h}}$ is denoted by $\tilde{\mathfrak{h}}^\alpha = [(\mathfrak{h}^\alpha(v))^L, (\mathfrak{h}^\alpha(v))^R]$.*

Definition 2.6 ([14]). *Let $\mathcal{T} : \mathbb{R}^l \to \mathbb{R}$ be a function of several variables.*

- *$\mathcal{T}$ is said to be weakly increasing if for all $u, v \in \mathbb{R}^l$, we have:*

$$u < v \Rightarrow \mathcal{T}(u) < \mathcal{T}(v). \tag{2}$$

- *$\mathcal{T}$ is said to be strongly increasing if for all $u, v \in \mathbb{R}^l$, we have:*

$$u \leq v \Rightarrow \mathcal{T}(u) < \mathcal{T}(v). \tag{3}$$

Proposition 2.7 ([14,21]). *Let $\Psi : \mathbb{R}^n \to \mathbb{R}^l$ and $\mathcal{T} : \mathbb{R}^l \to \mathbb{R}$, and suppose $v^* \in \arg\min_{v\in\mathbb{R}^n} \mathcal{T}(\Psi(v))$.*

(*i*) *If $\mathcal{T}$ is weakly increasing, then v^* is a weak Pareto optimal solution of $\min_{v\in\mathbb{R}^n} \Psi(v)$.*
(*ii*) *If $\mathcal{T}$ is strongly increasing, then v^* is a Pareto optimal solution of $\min_{v\in\mathbb{R}^n} \Psi(v)$.*

3. Main Results

In this section, we first introduce some definitions and propositions to set the stage. Then, a conjugate-direction-based solution algorithm is presented.

Definition 3.1. *Let $\tilde{\mathcal{N}}$ and $\tilde{\mathcal{P}}$ be two fuzzy matrices defined as follows:*

$$\tilde{\mathcal{N}} := \begin{bmatrix} \tilde{a}_{11} & \cdots & \tilde{a}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{a}_{m1} & \cdots & \tilde{a}_{mn} \end{bmatrix} \quad \textit{and} \quad \tilde{\mathcal{P}} := \begin{bmatrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_m \end{bmatrix}.$$

For each $\alpha \in [0,1]$, define the matrices $\mathcal{N}^\alpha$ and $\mathcal{P}^\alpha$ using the centers of the intervals of the α-cuts:

$$\tilde{\mathcal{N}}^\alpha := \frac{1}{2}\begin{bmatrix} a_{11}^R(\alpha) + a_{11}^L(\alpha) & \cdots & a_{1n}^R(\alpha) + a_{1n}^L(\alpha) \\ \vdots & \ddots & \vdots \\ a_{m1}^R(\alpha) + a_{m1}^L(\alpha) & \cdots & a_{mn}^R(\alpha) + a_{mn}^L(\alpha) \end{bmatrix}, \quad \tilde{\mathcal{P}}^\alpha := \frac{1}{2}\begin{bmatrix} b_1^R(\alpha) + b_1^L(\alpha) \\ \vdots \\ b_m^R(\alpha) + b_m^L(\alpha) \end{bmatrix}.$$

Remark 3.2. *If the components of $\tilde{\mathcal{N}}$ and $\tilde{\mathcal{P}}$ are symmetric fuzzy numbers, then $\tilde{\mathcal{N}}^\alpha$ and $\tilde{\mathcal{P}}^\alpha$ are independent of the membership degree α.*

Proposition 3.3. *A fuzzy multiobjective quadratic optimization problem is formulated as:*

$$\min_{v \in \mathbb{R}^n} \tilde{\Phi}(v) := (\tilde{\varphi}_1(v), \ldots, \tilde{\varphi}_l(v)) \tag{4}$$

where each fuzzy objective function $\tilde{\varphi}_i : \mathbb{R}^n \to \mathbb{F}$, $i = 1, 2, \ldots, l$, is defined by:

$$\tilde{\varphi}_i(v) = \frac{1}{2} v^T \odot \tilde{\mathcal{N}}_i^\alpha \odot v \oplus (\tilde{\mathcal{P}}_i^\alpha)^T \odot v, \quad \forall\, v \in \mathbb{R}^n, \quad \forall\, i = 1, 2, \ldots, l,$$

where $\odot$ and $\oplus$ denote fuzzy multiplication and fuzzy addition, respectively.

We now extend the notion of conjugate directions to the set of real symmetric matrices parameterized by α, denoted by $\tilde{\mathcal{N}}_i^\alpha$, for all $\alpha \in [0,1]$ and $i = 1, 2, \ldots, l$.

Remark 3.4. *If the components of a fuzzy matrix $\tilde{\mathcal{N}}$ are symmetric, this does not necessarily imply that the matrices $\tilde{\mathcal{N}}$ and $\tilde{\mathcal{N}}^\alpha$ are symmetric, and vice versa.*

Definition 3.5. *Let $\tilde{\mathcal{N}}_i^\alpha$ be a symmetric matrix of order $n \times n$. Then, the vectors $v_0, v_1, \ldots, v_q \in \mathbb{R}^n$ are called $\tilde{\mathcal{N}}_i^\alpha$-conjugate vectors if, $\forall\, k \neq q \in \{0, 1, \ldots, n-1\}$, we have: $v_k^T\, \tilde{\mathcal{N}}_i^\alpha\, v_q = 0$, $\forall\, i = 1, 2, \ldots, l$.*

Remark 3.6. *Such vectors exist if the matrices $\tilde{\mathcal{N}}_i^\alpha$ commute.*

Lemma 3.7. *Let $\tilde{\mathfrak{h}} : \mathbb{R}^n \to \mathbb{F}$ be a fuzzy function defined as follows:*
$\tilde{\mathfrak{h}}(v) := v^T \odot \tilde{\mathcal{N}} \odot v + \tilde{\mathcal{P}}^T \odot v$, $\forall\, v \in \mathbb{R}^n$, and let $\left[(\tilde{\mathfrak{h}}^\alpha(v))^L,\, (\tilde{\mathfrak{h}}^\alpha(v))^R\right]$ be its α-cut $\forall\, \alpha \in [0,1]$. Then, we have:

$$[(\tilde{\mathfrak{h}}^\alpha(v))^L + (\tilde{\mathfrak{h}}^\alpha(v))^R] = v^T\, \tilde{\mathcal{N}}^\alpha\, v + \tilde{\mathcal{P}}_\alpha^T\, v, \quad \forall\, v \in \mathbb{R}^n.$$

To prove the lemma, we adopt the following notation: $\bigoplus_{k=1}^{n}(\tilde{z}_k) := \tilde{z}_1 \oplus \tilde{z}_2 \oplus \cdots \oplus \tilde{z}_n$.

Proof 3.8. *We consider:*

$$v^T \odot \tilde{\mathcal{N}} \odot v = (v^T \odot \tilde{\mathcal{N}}) \odot v = \left(\bigoplus_{k=1}^{n}(v_k \odot \tilde{a}_{k1}), \cdots, \bigoplus_{k=1}^{n}(v_k \odot \tilde{a}_{kn})\right) \odot v.$$

Taking the α-cuts of the fuzzy coefficients in $\tilde{\mathcal{N}}$, denoted by $[\tilde{\mathcal{N}}]^\alpha$, we obtain:

$$\begin{aligned} v^T \odot [\tilde{\mathcal{N}}]^\alpha \odot v &= \left(\bigoplus_{k=1}^{n}(v_k \odot [a_{k1}^L(\alpha), a_{k1}^R(\alpha)]), \ldots, \bigoplus_{k=1}^{n}(v_k \odot [a_{kn}^L(\alpha), a_{kn}^R(\alpha)])\right) \odot v \\ &= \left([g_1^L(\alpha), g_1^R(\alpha)], \ldots, [g_n^L(\alpha), g_n^R(\alpha)]\right) \odot v, \end{aligned}$$

where

$$(g_i^\alpha)^L = \sum_{\substack{k=1 \\ v_k>0}}^{n} v_k a_{ki}^L(\alpha) + \sum_{\substack{k=1 \\ v_k\leq 0}}^{n} v_k a_{ki}^R(\alpha) \quad \textit{and} \quad (g_i^\alpha)^R = \sum_{\substack{k=1 \\ v_k>0}}^{n} v_k a_{ki}^R(\alpha) + \sum_{\substack{k=1 \\ v_k\leq 0}}^{n} v_k a_{ki}^L(\alpha). \tag{5}$$

As a result,

$$\begin{aligned} v^T \odot [\tilde{\mathcal{N}}]^\alpha \odot v &= \bigoplus_{i=1}^{n}\left(v_i \odot [(g_i^\alpha)^L,\, (g_i^\alpha)^R]\right) \\ &= \left[\sum_{i=1\ ,v_i>0}^{n} v_i (g_i^\alpha)^L + \sum_{i=1\ ,v_i\leq 0}^{n} v_i (g_i^\alpha)^R\ ,\ \sum_{i=1\ ,v_i>0}^{n} v_i (g_i^\alpha)^R + \sum_{i=1\ ,v_i\leq 0}^{n} v_i (g_i^\alpha)^L\right]. \end{aligned}$$

Using (5), we get:

$$v^T \odot \tilde{\mathcal{N}}^\alpha \odot v = 2\left[\sum_{\substack{i=1 \\ v_k v_i > 0}}^{n}\sum_{k=1}^{n} a_{ki}^L(\alpha) v_k v_i + \sum_{\substack{i=1 \\ v_k v_i \leq 0}}^{n}\sum_{k=1}^{n} a_{ki}^R(\alpha) v_k v_i\ ,\ \sum_{\substack{i=1 \\ v_k v_i > 0}}^{n}\sum_{k=1}^{n} a_{ki}^R(\alpha) v_k v_i + \sum_{\substack{i=0 \\ v_k v_i \leq 0}}^{n}\sum_{k=0}^{n} a_{ki}^L(\alpha) v_k v_i\right]. \tag{6}$$

In the same way, we obtain:

$$(\tilde{\mathcal{P}}^\alpha)^T \odot v = \left[\sum_{k=0\ ,v_k>0}^{n} v_k b_k^L(\alpha) + \sum_{k=0\ ,v_k\leq 0}^{n} v_k b_k^R(\alpha)\ ,\ \sum_{k=1\ ,v_k>0}^{n} v_k b_k^R(\alpha) + \sum_{k=1\ ,v_k\leq 0}^{n} v_k b_k^L(\alpha)\right]. \tag{7}$$

From the equations (6) and (7) and for all $v \in \mathbb{R}^n$, we have:

$$(\tilde{\mathfrak{h}}^\alpha(v))^L = 2\sum_{\substack{i=1 \\ v_k v_i > 0}}^{n}\sum_{k=1}^{n} a_{ki}^L(\alpha) v_k v_i + 2\sum_{\substack{i=1 \\ v_k v_i \leq 0}}^{n}\sum_{k=1}^{n} a_{ki}^R(\alpha) v_k v_i + \sum_{k=1\ ,v_k>0}^{n} v_k b_k^L(\alpha) + \sum_{k=1\ ,v_k\leq 0}^{n} v_k b_k^R(\alpha)\ ; \tag{8}$$

$$(\tilde{\mathfrak{h}}^\alpha(v))^R = 2\sum_{\substack{i=1 \\ v_k v_i > 0}}^{n}\sum_{k=1}^{n} a_{ki}^R(\alpha) v_k v_i + 2\sum_{\substack{i=1 \\ v_k v_i \leq 0}}^{n}\sum_{k=1}^{n} a_{ki}^L(\alpha) v_k v_i + \sum_{k=1\ ,v_k>0}^{n} b_k^R(\alpha) v_k + \sum_{k=1\ ,v_k\leq 0}^{n} b_k^L(\alpha) v_k. \tag{9}$$

Calculating $\left[(\tilde{\mathfrak{h}}^{\alpha}(v))^L + (\tilde{\mathfrak{h}}^{\alpha}(v))^R\right]$*, we get:*

$$\left[(\tilde{\mathfrak{h}}^{\alpha}(v))^L + (\tilde{\mathfrak{h}}^{\alpha}(v))^R\right] = 2\sum_{i=1}^{n}\sum_{k=1}^{n}(a_{ki}^L + a_{ki}^R(\alpha))v_k v_i + \sum_{k=1}^{n}(b_k^L + b_k^R)v_k.$$

Consequently:

$$\left[(\tilde{\mathfrak{h}}^{\alpha}(v))^L + (\tilde{\mathfrak{h}}^{\alpha}(v))^R\right] = v^T\,\tilde{\mathcal{N}}^{\alpha}\,v + \left(\tilde{\mathcal{P}}^{\alpha}\right)^T\ v;\quad \forall\ v\in\mathbb{R}^n\ et\ \forall\ \alpha\in[0,1].$$

Using Lemma 3.7, we define the deterministic quadratic multiobjective optimization problem corresponding to the fuzzy quadratic multiobjective optimization problem.

Proposition 3.9. *The associated deterministic quadratic multiobjective optimization problem for the problem defined in Proposition 3.3 is given by:*

$$\min_{v\in\mathbb{R}^n}\Psi^{\alpha}(v) := (\psi_1^{\alpha}(v),...,\psi_l^{\alpha}(v)) \tag{10}$$

where $\psi_i^{\alpha}:\mathbb{R}^n\to\mathbb{R}$*,* $i=1,2,...,l$*, is defined as follows:*

$$\psi_i^{\alpha}(v) = (\psi_i^{\alpha}(v))^L + (\psi_i^{\alpha}(v))^R = v^T\,\tilde{\mathcal{N}}_i^{\alpha}\,v + (\tilde{\mathcal{P}}_i^{\alpha})^T\,v,\quad \forall\ v\in\mathbb{R}^n\ and\ \forall\ i=1,2,...,l.$$

Let us now discuss the notion of solutions.

Definition 3.10. *Let* $v^*\in X\subseteq\mathbb{R}^n$*.*

- (i) v^* *is said to be a Pareto optimal solution of Problem* (4) *if there does not exist* $v\in X$ *such that* $\tilde{\varphi}_i(v)\preceq\tilde{\varphi}_i(v^*)$ *for all* $i=1,2,...,l$ *and* $\tilde{\varphi}_j(v)\prec\tilde{\varphi}_j(v^*)$ *for at least one index* $j\in\{1,2,...,l\}$*.*
- (ii) v^* *is said to be a weakly Pareto optimal solution of Problem* (4) *if there does not exist* $v\in X$ *such that* $\tilde{\varphi}_i(v)\prec\tilde{\varphi}_i(v^*)$ *for all* $i=1,2,\cdots,l$*.*

Proposition 3.11. *Let* $v^*\in X\subseteq\mathbb{R}^n$ *and* $\alpha\in[0,1]$ *be fixed.*

- (i) v^* *is called a Pareto optimal solution of Problem* (10) *if there does not exist* $v\in X$ *such that* $\psi_i^{\alpha}(v)\le\psi_i^{\alpha}(v^*)$ *for all* $i=1,2,...,l$ *and* $\psi_j^{\alpha}(v)<\psi_j^{\alpha}(v^*)$ *for at least one index* $j\in\{1,2,\cdots,l\}$*.*
- (ii) v^* *is called a weakly Pareto optimal solution of Problem* (10) *if there does not exist* $v\in X$ *such that* $\psi_i^{\alpha}(v)<\psi_i^{\alpha}(v^*)$ *for all* $i=1,2,...,l$*.*

Theorem 3.12. *If* v^* *is a Pareto optimal solution of Problem* (10)*, then* v^* *is an efficient solution of Problem* (4)*.*

Proof. We proceed by contraposition.
Suppose that v^* is not an efficient solution of Problem (4). Then there exists $v\in\mathbb{R}^n$ such that $\tilde{\varphi}_i(v)\preceq\tilde{\varphi}_i(v^*)$, and $\tilde{\varphi}_j(v)\prec\tilde{\varphi}_j(v^*)$ for all $v\in\mathbb{R}^n$.

From Proposition 3.9 and Lemma 3.7, we have:

$$\tilde{\varphi}_i(v) \preceq \tilde{\varphi}_i(v^*) \Rightarrow (\tilde{\varphi}_i^\alpha(v))^L \leq (\tilde{\varphi}_i^\alpha(v^*))^L \text{ and } (\tilde{\varphi}_i^\alpha(v))^R \leq (\tilde{\varphi}_i^\alpha(v^*))^R \tag{11}$$

$$\Rightarrow (\tilde{\varphi}_i^\alpha(v))^L + (\tilde{\varphi}_i^\alpha(v))^R \leq (\tilde{\varphi}_i^\alpha(v^*))^L + (\tilde{\varphi}_i^\alpha(v^*))^R \tag{12}$$

$$\Rightarrow \psi_i^\alpha(v) \leq \psi_i^\alpha(v^*). \tag{13}$$

Similarly, we obtain: $[\psi_j^\alpha(v)] < [\psi_j^\alpha(v^*)]$ for at least one $j \in \{1, 2, \cdots, l\}$.
This contradicts the fact that v is a Pareto optimal solution of Problem (10).
Therefore, v^* is a Pareto optimal solution of Problem (4). □

The conjugate direction algorithm for the fuzzy quadratic multiobjective optimization problem is presented as follows:

Algorithm 1 Conjugate Direction Algorithm for Problem (4)

1: For all $i = 1, 2, ..., l$ and for $\alpha \in [0, 1]$, compute $\tilde{\mathcal{N}}_i^\alpha$ and $\tilde{\mathcal{P}}_i^\alpha$.
2: Determine the set of vectors $\{v_0, \cdots, v_{n-1}\}$ that are $\tilde{\mathcal{N}}_i^\alpha$-conjugate for all $i = 1, 2, \cdots, l$.
3: Consider a strongly increasing continuous function $\mathcal{T} : \mathbb{R}^l \to \mathbb{R}$ such that $\mathcal{T} \circ \Psi^\alpha : \mathbb{R}^n \to \mathbb{R}$ is a strongly convex function.
4: Choose an initial point $\eta_0 \in \mathbb{R}^n$ and set $l = 0$.
5: Compute the step size ζ_l in the direction v_l by solving:

$$\zeta_l = \arg\min_{\tau \in \mathbb{R}} \mathcal{T}(\Psi^\alpha(\eta_l + \tau v_l)) \tag{14}$$

6: Update the current point:

$$\eta_{l+1} = \eta_l + \zeta_l v_l \tag{15}$$

7: Stop if $l = n - 1$. Otherwise, return to Step 5.

Remark 3.13. *Since the function $\mathcal{T} \circ \Psi^\alpha : \mathbb{R}^n \to \mathbb{R}$ is strongly convex, then ζ_l is unique for all k. Therefore, beginning with an arbitrary initial point η_0, the algorithm generates $\eta_1, \cdots, \eta_n \in \mathbb{R}^n$, such that $(\mathcal{T} \circ [\Psi^\alpha(\eta_l)])_{l \in \mathcal{I}}$ forms a decreasing sequence. Indeed, from (14) and (15), we have:*

$$\mathcal{T}([\Psi^\alpha(\eta_{k+1})] = \mathcal{T}([\Psi^\alpha(\eta_k + \zeta_k v_k)]) \leq \mathcal{T}([\Psi^\alpha(\eta_k)]), \ k = 0, 1, \cdots, l - 1. \tag{16}$$

Now, let us consider the composite function $\mathcal{T} \circ \Psi^\alpha$ and the initial point ζ_0.

Theorem 3.14. *Let $\{\eta_1, \cdots, \eta_n\}$ be the sequence generated by Algorithm 1, executed with the $\tilde{\mathcal{N}}_i^\alpha$-conjugate basis $\{u_1, \cdots, u_n\}$ for all $i = \{1, 2, \cdots, l\}$ and an initial point $\eta_0 \in \mathbb{R}^n$. Let $\mathcal{T} : \mathbb{R}^l \to \mathbb{R}$ be a continuous strongly increasing function. Then, there exists $\gamma \geq 0$ and a set $\mathcal{A} \subseteq \mathbb{R}^n$ such that if $\eta_n \in \mathbb{R}^n \setminus \mathcal{A}$, then η_n is an*

effective solution of Problem (4),
where $\mathcal{A} = \left\{ \eta_0 + \sum_{i=0}^{n-1} \rho_i v_i,\ \ |\rho_i| < \delta \ for\ some\ i \in \{0, \cdots, n-1\} \right\}$,
and $\delta = \max\{|\beta_i^k|;\ k = 0, 1, \cdots, l-1,\ i = 1, 2, \cdots, l\}$ *with* $\mathcal{L}_i^k(\beta_i^k) = 0$.

Proof 3.15. *Let* $k \in \{0, \cdots, n-1\}$, $i = 1, 2, \cdots, l$ *and* $\alpha \in [0, 1]$.
Consider a quadratic function $\mathcal{L}$ *defined from* $\mathbb{R} \to \mathbb{R}$ *by*

$$\mathcal{L}_i^k(\rho) = \mathcal{L}_i^k(\eta_0,\ \rho) = \frac{1}{2}\Big[(v_k)^T \tilde{\mathcal{N}}_i^\alpha v_k\Big]\rho^2 + \Big[\Big(\tilde{\mathcal{N}}_i^\alpha \eta_0 + \tilde{\mathcal{P}}_i^\alpha\Big)^T v_k\Big]\rho. \tag{17}$$

Since $\mathcal{L}_i^k$ *is a quadratic function with 0 as a root and* $\lim_{\rho \to \pm\infty} \mathcal{L}_i^k(\rho) = \infty$, *there exists* δ *such that* $\mathcal{L}_i^k(\rho) \geq 0$, *for all* $k \in \{0, \cdots, n-1\}$, $i \in \mathcal{I}$ *and* $|\rho| \geq \delta$.
Thus, there exists a real number δ *such that:*

$$\sum_{k=0}^{n-1} \mathcal{L}_i^k(\rho_k) \geq 0, \forall\ |\rho_k| \geq \delta, k \in \{0, \cdots, n-1\} \ and \ i = 1, 2, \cdots, l. \tag{18}$$

Let us define the set $\mathcal{A} = \left\{ \eta_0 + \sum_{k=0}^{n-1} \rho_k v_k : |\rho_k| < \delta,\ k \in \{0, \cdots, n-1\} \right\}$,
where ρ_k *is a root of the function* $\mathcal{L}^k$.
Let $\kappa \in \mathbb{R}^n \backslash \mathcal{A}$, *there exist* $\rho_0, \cdots, \rho_{n-1}$ *such that* $\kappa = \eta_0 + \sum_{k=0}^{n-1} \rho_k v_k,\ |\rho_k| \geq \delta$ *for all* $k = 0, \cdots, n-1$.
From equation (16),

$$\begin{aligned}
\mathcal{T}(\Psi^\alpha(\eta_l)) &\leq \mathcal{T}(\Psi^\alpha(\eta_0)) \\
&= \mathcal{T}\Bigg(\Bigg[\frac{1}{2}\Big(\eta_0 + \sum_{k=0}^{n-1} \rho_k v_k - \sum_{k=0}^{n-1} \rho_k v_k\Big)^T \tilde{\mathcal{N}}_i^\alpha \Big(\eta_0 + \sum_{k=0}^{n-1} \rho_k v_k - \sum_{k=0}^{n-1} \rho_k v_k\Big) \\
&\quad + (\tilde{\mathcal{P}}_i^\alpha)^T \Big(\eta_0 + \sum_{k=0}^{n-1} \rho_k v_k - \sum_{k=0}^{n-1} \rho_k v_k\Big)\Bigg]_{i=1}^k\Bigg), \\
&= \mathcal{T}\Bigg(\Bigg[\Psi_i^\alpha\Big(\eta_0 + \sum_{k=0}^{n-1} \rho_k v_k\Big) - \Big(\eta_0 + \sum_{k=0}^{n-1} \rho_k v_k\Big)^T \tilde{\mathcal{N}}_i^\alpha \Big(\sum_{k=0}^{n-1} \rho_k v_k\Big) \\
&\quad + \frac{1}{2}\Big(\sum_{k=0}^{n-1} \rho_k v_k\Big)^T \tilde{\mathcal{N}}_i^\alpha \Big(\sum_{k=0}^{n-1} \rho_k v_k\Big) - (\tilde{\mathcal{P}}_i^\alpha)^T \Big(\sum_{k=0}^{n-1} \rho_k v_k\Big)\Bigg]_{i=1}^k\Bigg), \\
&= \mathcal{T}\Bigg(\Bigg[\Psi_i^\alpha\Big(\eta_0 + \sum_{k=0}^{n-1} \rho_k v_k\Big) - \sum_{k=0}^{n-1} \mathcal{L}_k^{(i)}(\rho_k)\Bigg]_{i=1}^k\Bigg) \\
&\leq \mathcal{T}\Bigg(\Bigg[\Psi_i^\alpha\Big(\eta_0 + \sum_{k=0}^{n-1} \rho_k v_k\Big)\Bigg]_{i=1}^k\Bigg), \\
&= \mathcal{T}(\Psi^\alpha(\kappa)).
\end{aligned}$$

This establishes that if $\eta_n \in \arg\min_{\tau \in \mathbb{R}} \mathcal{T}(\Psi^\alpha(\eta_l + \tau v_l))$*, then it is an effective solution of Problem* (4) *on* $\mathbb{R}^n \backslash \mathcal{A}$*. If* $\delta = 0$*, the following inequality holds:*

$$\begin{aligned} \mathcal{T}(\Psi^\alpha(\eta_n)) &\leq \mathcal{T}\left(\Psi^\alpha\left(\eta_0 + \sum_{k=0}^{n-1} \rho_k v_k\right)\right), \quad \forall\, \rho_k \in \mathbb{R}, \\ &\leq \mathcal{T} \circ \Psi^\alpha(\kappa), \quad \forall\, \kappa \in \mathbb{R}^n. \end{aligned} \tag{19}$$

From inequality (19), we observe that Algorithm 1 converges to an optimal solution of problem (4) on $\mathbb{R}^n$ when $\delta = 0$, which is not always guaranteed. Let us illustrate this through a didactic example.

Example 3.16. *Consider the following quadratic multiobjective optimization problem where the matrix and vector components are symmetric triangular fuzzy numbers:*

$$\min_{[v_1, v_2, v_3]^T \in \mathbb{R}^3} \tilde{\Phi}(v) = (\tilde{\varphi}_1(v_1, v_2, v_3), \tilde{\varphi}_2(v_1, v_2, v_3)) \tag{20}$$

where $\tilde{\varphi}_i : \mathbb{R}^3 \to \mathbb{F}$, $i \in \{1, 2\}$ *are defined as follows:*

$$\tilde{\varphi}_1(v_1, v_2, v_3) = \frac{1}{2}(v_1, v_2, v_3) \odot \tilde{\mathcal{N}}_1 \odot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \oplus \tilde{\mathcal{P}}_1 \odot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

$$\tilde{\varphi}_2(v_1, v_2, v_3) = \frac{1}{2}(v_1, v_2, v_3) \odot \tilde{\mathcal{N}}_2 \odot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \oplus \tilde{\mathcal{P}}_2 \odot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

With $\tilde{\mathcal{N}}_1 = \begin{pmatrix} (3,5,7) & (1,2,3) & (-1,0,1) \\ (1,2,3) & (1,5,9) & (-2,0,2) \\ (-1,0,1) & (-2,0,2) & (-1,3,7) \end{pmatrix}$, $\tilde{\mathcal{P}}_1 = \begin{pmatrix} (-2,0,2) \\ (0,1,2) \\ (1,2,3) \end{pmatrix}$,

$\tilde{\mathcal{N}}_2 = \begin{pmatrix} (6,8,10) & (-1,2,5) & (-3,0,3) \\ (-1,2,5) & (5,8,11) & (-1,0,1) \\ (-3,0,3) & (-1,0,1) & (-2,1,4) \end{pmatrix}$ *and* $\tilde{\mathcal{P}}_2 = \begin{pmatrix} (-1,1,3) \\ (1,3,5) \\ (-2,-1,0) \end{pmatrix}$.

Let us compute $[\tilde{\mathcal{N}}_i]^\alpha$ *and* $[\tilde{\mathcal{P}}_i]^\alpha$ *using the* α*-cuts of the fuzzy components of the matrices* $\tilde{\mathcal{N}}_i$ *and* $\tilde{\mathcal{P}}_i$*.*

$$[\tilde{\mathcal{N}}_1]^\alpha = \begin{pmatrix} [2\alpha+3,-2\alpha+7] & [\alpha+1,-\alpha+3] & [\alpha-1,-\alpha+1] \\ [\alpha+1,-\alpha+3] & [4\alpha+1,-4\alpha+9] & [2\alpha-2,-2\alpha+2] \\ [\alpha-1,-\alpha+1] & [2\alpha-2,-2\alpha+2] & [4\alpha-1,-4\alpha+7] \end{pmatrix}; \quad [\tilde{\mathcal{P}}_1]^\alpha = \begin{pmatrix} [2\alpha-2,-2\alpha+2] \\ [\alpha,-\alpha+2] \\ [\alpha+1,-\alpha+3] \end{pmatrix}$$

$$[\tilde{\mathcal{N}}_2]^\alpha = \begin{pmatrix} [2\alpha+6,-2\alpha+10] & [3\alpha-1,-3\alpha+5] & [3\alpha-3,-3\alpha+3] \\ [3\alpha-1,-3\alpha+5] & [3\alpha+5,-3\alpha+11] & [\alpha-1,-\alpha+1] \\ [3\alpha-3,-3\alpha+3] & [\alpha-1,-\alpha+1] & [3\alpha-2,-3\alpha+4] \end{pmatrix}; \quad [\tilde{\mathcal{P}}_2]^\alpha = \begin{pmatrix} [2\alpha-1,-2\alpha+3] \\ [2\alpha+1,-2\alpha+5] \\ [\alpha-2,-\alpha] \end{pmatrix}$$

$$\tilde{\mathcal{N}}_1^\alpha = \begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}; \quad \tilde{\mathcal{N}}_2^\alpha = \begin{pmatrix} 8 & 2 & 0 \\ 2 & 8 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad \tilde{\mathcal{P}}_1^\alpha = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \text{ et } \tilde{\mathcal{P}}_2^\alpha = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$$

Note that, since the fuzzy coefficients are symmetric, the matrices $\tilde{\mathcal{N}}_1^\alpha$, $\tilde{\mathcal{N}}_2^\alpha$, $\tilde{\mathcal{P}}_1^\alpha$, and $\tilde{\mathcal{P}}_2^\alpha$ do not depend on α.

The vectors $\tilde{\mathcal{N}}_i^\alpha$-conjugate for $i \in \{1,2\}$ are

$$v_0 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \text{and} \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Subsequently, we choose the initial point $\eta_0 = (0,0,0)$ and represent the set $\mathcal{A}$.

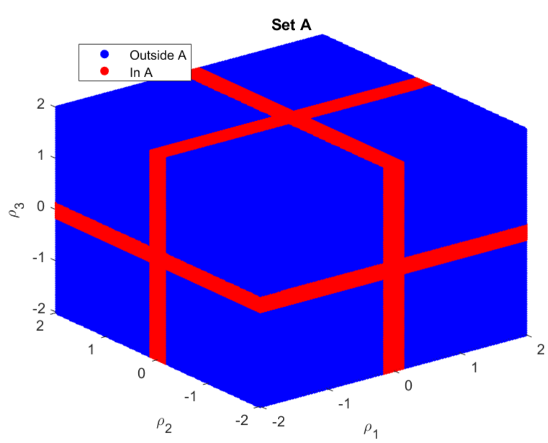

FIGURE 1. 3D representation of the set $\mathcal{A}$

FIGURE 1 shows a representation of the set $\mathcal{A}$, which is a region where the convergence of Algorithm 1 is not guaranteed. Thus, if a solution generated by Algorithm 1 belongs to $\mathcal{A}$, then this solution will not be effective for problem (20).

Let us now express the quadratic function $\mathcal{L}$ in terms of ρ:

$$\mathcal{L}_1^0(\rho) = 7\rho^2 + \rho, \qquad \mathcal{L}_1^1(\rho) = 3\rho^2 - \rho, \qquad \mathcal{L}_1^2(\rho) = \tfrac{3}{2}\rho^2 + 2\rho,$$

$$\mathcal{L}_2^0(\rho) = 10\rho^2 + 4\rho, \qquad \mathcal{L}_2^1(\rho) = 6\rho^2 - 2\rho, \qquad \mathcal{L}_2^2(\rho) = \tfrac{1}{2}\rho^2 - \rho.$$

The functions $\mathcal{L}_i^k(\rho)$, for $i = 1, 2$ and $k = 0, 1, 2$, are computed using relation (17).

We observe that all these functions have 0 as a root.

FIGURE 2 shows a graphical representation of these functions.

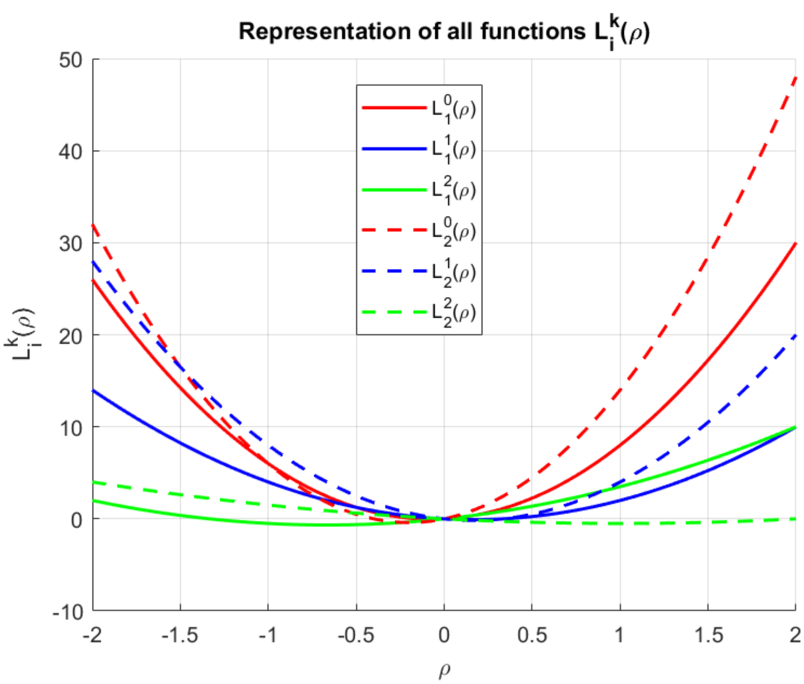

Figure 2. Graphical representation of the functions $\mathcal{L}_i^k$

FIGURE 2 shows that δ is obtained from the function $\mathcal{L}_2^2(\rho)$, and for this example $\delta = 2$ satisfies inequality (18). Consequently, it is not guaranteed that Algorithm 1 will find an optimal solution for the considered problem (4).

To address this limitation, we propose a method for determining optimal solutions of (20) on $\mathbb{R}^n$ that coincides with $\left\{\eta_0 + \sum_{k=0}^{n-1} \rho_k v_k, \quad \forall \rho_k \in \mathbb{R}\right\}$. For this purpose, we introduce a new algorithm that executes Algorithm 1 $(n+1)$ times. In each execution, a new initial point is carefully selected outside $\mathcal{A}$. Subsequently, we compare the previous values of the function $\mathcal{T} \circ \Psi^\alpha$ with the newly obtained ones and retain the solutions that best optimize this function. We present this algorithm in the following lines:

Algorithm 2 Conjugate Direction Algorithm for Problem (4)

1: For all $i = 1, 2, ..., l$ and for $\alpha \in [0, 1]$, compute $\tilde{\mathcal{N}}_i^\alpha$ and $\tilde{\mathcal{P}}_i^\alpha$;

2: Determine the set of $\tilde{\mathcal{N}}_i^\alpha$-conjugate vectors $\{v_0, v_1 ..., v_{n-1}\}, \forall\ i = 1, 2, ..., l$;

3: Consider a strongly increasing continuous function $\mathcal{T} : \mathbb{R}^l \to \mathbb{R}$ such that $\mathcal{T} \circ \Psi^\alpha : \mathbb{R}^n \to \mathbb{R}$ is a strongly convex function;

4: Choose an initial point $\eta_0 \in \mathbb{R}^n$ and for $\nu = 0, ..., n$, execute Algorithm 1 with initial points:

$$\eta_{\nu,0} = \eta_0 + \nu(2\overline{\delta} + 1) \sum_{k=0}^{n-1} v_k; \tag{21}$$

where $\overline{\delta} = \max\{\delta(\eta_{\nu,0}) : \nu = 0, ..., n\}$ and $\delta(\eta_{\nu,0})$ is given by (18). This creates a sequence of elements $\{\eta_{\nu,0}, \eta_{\nu,1}, ..., \eta_{\nu,n}\}$;

5: Let $\nu_* \in \{0, 1, ..., n\}$ such that $\mathcal{T}(\Psi^\alpha(\eta_{\nu_*,0})) = \min\limits_{\nu=0,...,n}\{\mathcal{T}(\Psi^\alpha(\eta_{\nu,n}))\}$ and $\eta_* = \eta_{\nu_*,n}$.

We now show that the optimal solution to problem (4) is obtained in $n(n + 1)$ iterations.

4. Convergence Analysis

This section will validate the solution η_* obtained in Step 5 of Algorithm 2 as an efficient solution to problem (4). We will also demonstrate that the optimal solution is achieved within $n(n + 1)$ iterations.

Theorem 4.1. *Let $\eta_{\nu,1}, ..., \eta_{\nu,n}$ for $\nu = 0, 1, ..., n$ be a sequence generated by Algorithm 2, implemented with:*

- *An initial point $\eta_0 \in \mathbb{R}^n$,*
- *A conjugate basis $v_0, ..., v_{n-1}$ for matrices $\mathcal{N}_i^\alpha\ \forall\ i = 1, 2, ..., l$,*
- *A strongly increasing continuous auxiliary function $\mathcal{T} : \mathbb{R}^l \to \mathbb{R}$.*

Then η_* provided by Algorithm 2 is a Pareto optimal solution to problem (4).

Proof 4.2. *From Remark 3.13, Proposition 2.7, and Theorem 3.12, we have:*

$$\mathcal{T}(\Psi(\eta_{\nu,n})) \leq \mathcal{T}(\Psi(v)); \quad \forall v \in \mathbb{R}^n \backslash \mathcal{A}^\nu \text{ and } \nu = 0, 1, ..., n. \tag{22}$$

where $\mathcal{A}^\nu = \mathcal{A}(\eta_{\nu,0}, \overline{\delta}) = \bigcup_{i=0}^{n-1} H_{\overline{\delta}}^i(\eta_{\nu,0})$ and $H_{\overline{\delta}}^i(\eta_{\nu,0}) = \mathbb{R}^i \times (v_i^{\nu,0} - \delta, v_i^{\nu,0} + \delta) \times \mathbb{R}^{n-i-1}$.

From Step 5 of Algorithm 2 and inequality (22), we obtain:

$$\mathcal{T}(\Psi^\alpha(\eta_*)) \leq \mathcal{T}(\Psi^\alpha(v)); \quad \forall v \in \bigcup_{\nu=0}^{n} (\mathbb{R}^n \backslash \mathcal{A}^\nu). \tag{23}$$

We now show that $\bigcup_{\nu=0}^{n} (\mathbb{R}^n \backslash \mathcal{A}^\nu) = \mathbb{R}^n$.

Considering the definition of $\mathcal{A}^\nu$ as an interval of length 2δ with $\eta_{\nu,0} = \eta_0 + \nu(2\delta + 1) \sum_{i=0}^{n-1} v_i$ for $\nu = 0, ..., n$.

Let $x \in \mathbb{R}^n$. In the basis $\{v_k\}$, we have:

$$x = \eta_0 + \sum_{k=0}^{n-1} \sigma_k v_k, \quad \forall\ \sigma_k \in \mathbb{R}.$$

Using the controlled offset technique, the expression of x relative to $\eta_{\nu,0}$ is:

$$x = \eta_{\nu,0} + \sum_{k=0}^{n-1} (\sigma_k - \nu(2\delta+1))\, v_k.$$

For $x \notin \mathcal{A}^\nu$, it is necessary that $|\sigma_k - \nu(2\delta+1)| \geq \delta$ for all k, i.e., $\sigma_k \notin (\nu(2\delta+1)-\delta, \nu(2\delta+1)+\delta)$.

We seek $\nu \in \{0, ..., n\}$ such that:

$$\nu(2\delta+1) \notin \bigcup_{k=0}^{n-1} (\sigma_k - \delta, \sigma_k + \delta).$$

The total length of the n intervals $(\sigma_k - \delta, \sigma_k + \delta)$ is $2n\delta$.

The points $\{\nu(2\delta+1)\}_{\nu=0}^{n}$, numbering $n+1$ in total, are equally spaced with separation $2\delta+1 > 2\delta$.

We therefore have $n+1$ equally spaced points with separation $2\delta+1$ to place within an interval of total length $2n\delta$. It is thus clear that at least one point $\nu^*(2\delta+1)$ lies outside all intervals $(\sigma_k-\delta, \sigma_k+\delta)$.

For this point ν^*, we have $|\sigma_k - \nu^*(2\delta+1)| \geq \delta$ for all k. Hence, $x \notin \mathcal{A}^{\nu^*}$. Consequently, we conclude that:

$$\bigcup_{\nu=0}^{n} (\mathbb{R}^n \backslash \mathcal{A}^\nu) = \mathbb{R}^n.$$

If the function $\mathcal{T}$ defined in Step 3 of Algorithm 2 is weakly increasing, then using Theorem 4.1 we formulate the following corollary.

Corolla 4.3. *Let $\eta_{\nu,1}, ..., \eta_{\nu,n}$ for $\nu = 0, ..., n$ be a sequence generated by Algorithm 2, implemented with:*

- *An initial point η_0,*
- *A conjugate basis $v_0, ..., v_{n-1}$ for matrices $\tilde{\mathcal{N}}_i^\alpha\ \forall\ i = 1, 2, ..., l$,*
- *A weakly increasing continuous auxiliary function $\mathcal{T} : \mathbb{R}^l \to \mathbb{R}$.*

Then η_* provided by Algorithm 2 is a weakly Pareto optimal solution to problem (4).

Proof 4.4. *The proof of this corollary follows similarly to that of Theorem 4.1.*

The implementation of Algorithm 2 begins by generating initial points $\eta_{\nu,0}$ for all $\nu = 0, ..., n$. Subsequently, using a "for" loop, we compute and store the values $\eta_{\nu,n}$. After exiting the loop, we evaluate the function $\mathcal{T} \circ \Psi^\alpha$ for each solution $\eta_{\nu,n}$ and perform a comparison to retain the optimal solution $\eta_{\nu^*,n}$ satisfying:

$$\mathcal{T}(\Psi^\alpha(\eta_{\nu^*,n})) = \min_{\nu=0,\ldots,n} \mathcal{T}(\Psi^\alpha(\eta_{\nu,n})).$$

We now solve problem (20) using Algorithm 2. We begin by arbitrarily selecting the initial point $\eta_0 = (0,0,0)^T$. For this problem, we consider the strongly increasing function defined by:

$$\mathcal{T} := 0.5\psi_1 + 0.5\psi_2.$$

To determine the value of $\overline{\delta}$, we first compute δ for each initial point and then select the maximum obtained δ value, yielding $\overline{\delta} = 31.33$.

The sequence of solutions is recorded in Table 1. Analysis of the values in Table 2 and Theorem 4.1 leads us to retain $\eta_{0,3} = (0.0196, -0.3137, -0.2500)^T$ as an efficient solution to problem (20).

TABLE 1. Solution sequences for problem (20) generated by Algorithm 2

U_l	$(1,1,0)^T$	$(1,-1,0)^T$	$(0,0,1)^T$	
$\eta_{0,i}$	$(0,0,0)^T$	$(-0.1471,-0.1471,0)^T$	$(0.0196,-0.3137,0)^T$	$(0.0196,-0.3137,-0.2500)^T$
ζ_l	-0.1471	0.1667	-0.2500	
$\eta_{1,i}$	$(12,0,6)^T$	$(5.8529,-6.1471,6.0000)^T$	$(0.0196,-0.3137,6.0000)^T$	$(0.0196,-0.3137,-0.2500)^T$
ζ_l	-6.1471	-5.8333	-6.2500	
$\eta_{2,i}$	$(24.0000,0,12.0000)^T$	$(11.8529,-12.1471,12.0000)^T$	$(0.0196,-0.3137,12.0000)^T$	$(0.0196,-0.3137,-0.2500)^T$
ζ_l	-12.1471	-11.8333	-12.2500	
$\eta_{3,i}$	$(36.0000,0,18.0000)^T$	$(17.8529,-18.1471,18.0000)^T$	$(0.0196,-0.3137,18.0000)^T$	$(0.0196,-0.3137,-0.2500)^T$
ζ_l	-18.1471	-17.8333	-18.2500	

TABLE 2. Evaluation of the function $\mathcal{T} \circ \Psi^\alpha$

ν	$\eta_{\nu,3}$	$\mathcal{T}(\Psi(\eta_{\nu,3}))$
0	$(0.0196,-0.3137,-0.2500)^T$	-0.3713
1	$(0.0196,-0.3137,-0.2500)^T$	-0.3713
2	$(0.0196,-0.3137,-0.2500)^T$	-0.3713
3	$(0.0196,-0.3137,-0.2500)^T$	-0.3713

5. NUMERICAL EXAMPLES

The objective of this section is to test the effectiveness of our method by applying it to solve fuzzy quadratic multiobjective optimization problems. The considered examples differ in their fuzzy number asymmetry.

Example 5.1. *Consider the fuzzy quadratic multiobjective optimization problem where the components of matrices $\tilde{\mathcal{N}}_i$ and vectors $\tilde{\mathcal{P}}_i$ are asymmetric triangular fuzzy numbers:*

$$\min_{[v_1,v_2,v_3]^T \in \mathbb{R}^3} \tilde{\Phi}(v) = (\tilde{\varphi}_1(v_1,v_2,v_3), \tilde{\varphi}_2(v_1,v_2,v_3)), \tag{24}$$

where $\tilde{\varphi}_i : \mathbb{R}^3 \to \mathbb{F}$, $i \in \{1,2\}$ are defined as follows:

$$\tilde{\varphi}_1(v_1,v_2,v_3) = \frac{1}{2}(v_1,v_2,v_3) \odot \tilde{\mathcal{N}}_1 \odot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \oplus \tilde{\mathcal{P}}_1 \odot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\tilde{\varphi}_2(v_1,v_2,v_3)=\frac{1}{2}(v_1,v_2,v_3)\odot\tilde{\mathcal{N}}_2\odot\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}\oplus\tilde{\mathcal{P}}_2\odot\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$$

with

$$\tilde{\mathcal{N}}_1=\begin{pmatrix}(0,1.3,1.6)&(0.4,0,0.6)&(0.1,0.2,1.3)\\(0.4,0,0.6)&(2.5,4.6,4.7)&(0,1,1)\\(0.1,0.2,1.3)&(0,1,1)&(4.1,6.5,7)\end{pmatrix},\quad \tilde{\mathcal{P}}_1=\begin{pmatrix}(0,0,1)\\(0.1,1.1,1.1)\\(0.5,0.7,1.9)\end{pmatrix}$$

and

$$\tilde{\mathcal{N}}_2=\begin{pmatrix}(1.2,3.6,4)&(0,0,2)&(0.2,0.4,2.6)\\(0,0,2)&(4,10.2,12)&(0,2,2)\\(0.2,0.4,2.6)&(0,2,2)&(5.6,12.8,14)\end{pmatrix},\quad \tilde{\mathcal{P}}_2=\begin{pmatrix}(0,0.2,1.4)\\(0.1,0.2,1.3)\\(0,0,1)\end{pmatrix}$$

Let us compute $[\tilde{\mathcal{N}}_i]^\alpha$ and $[\tilde{\mathcal{P}}_i]^\alpha$ whose components are the α-cuts of the components of $\tilde{\mathcal{N}}_i$ and $\tilde{\mathcal{P}}_i$, for $i=1,2$.

$$[\tilde{\mathcal{N}}_1]^\alpha=\begin{pmatrix}[1.3\alpha,-0.3\alpha+1.6]&[0,-\alpha+1]&[0.1\alpha+0.1,-1.1\alpha+1.3]\\[0,-\alpha+1]&[2.1\alpha+2.5,-0.1\alpha+4.7]&[\alpha,1]\\[0.1\alpha+0.1,-1.1\alpha+1.3]&[\alpha,1]&[5.5\alpha+1,-0.5\alpha+7]\end{pmatrix},$$

$$[\tilde{\mathcal{N}}_2]^\alpha=\begin{pmatrix}[2.4\alpha+1.2,-0.4\alpha+4]&[0,-2\alpha+2]&[0.2\alpha+0.2,-2.2\alpha+2.6]\\[0,-2\alpha+2]&[6.2\alpha+4,-1.8\alpha+12]&[2\alpha,2]\\[0.2\alpha+0.2,-2.2\alpha+2.6]&[2\alpha,2]&[7.2\alpha+5.6,-1.2\alpha+14]\end{pmatrix},$$

$$[\tilde{\mathcal{P}}_1]^\alpha=\begin{pmatrix}[0,-\alpha+1]\\[\alpha+0.1,1.1]\\[0.2\alpha+0.5,-1.2\alpha+1.9]\end{pmatrix}\quad \textit{and}\quad [\tilde{\mathcal{P}}_2]^\alpha=\begin{pmatrix}[0.2\alpha,-1.2\alpha+1.4]\\[0.1\alpha+0.1,-1.1\alpha+1.3]\\[0,-\alpha+1]\end{pmatrix}.$$

$$\tilde{\mathcal{N}}_1^\alpha=\begin{pmatrix}\frac{\alpha+1.6}{2}&\frac{-\alpha+1}{2}&\frac{-\alpha+1.4}{2}\\\frac{-\alpha+1}{2}&\frac{2\alpha+7.2}{2}&\frac{\alpha+1}{2}\\\frac{-\alpha+1.4}{2}&\frac{\alpha+1}{2}&\frac{5\alpha+8}{2}\end{pmatrix},\quad \tilde{\mathcal{P}}_1^\alpha=\begin{pmatrix}\frac{-\alpha+1}{2}\\\frac{\alpha+1.2}{2}\\\frac{-\alpha+2.4}{2}\end{pmatrix},$$

$$\tilde{\mathcal{N}}_2^{\alpha} = \begin{pmatrix} \frac{2\alpha+5.2}{2} & \frac{-2\alpha+2}{2} & \frac{-2\alpha+2.8}{2} \\ \frac{-2\alpha+2}{2} & \frac{4\alpha+16.4}{2} & \frac{2\alpha+2}{2} \\ \frac{-2\alpha+2.8}{2} & \frac{2\alpha+2}{2} & \frac{6\alpha+19.6}{2} \end{pmatrix} \text{ and } \tilde{\mathcal{P}}_2^{\alpha} = \begin{pmatrix} \frac{-\alpha+1.4}{2} \\ \frac{2\alpha+2.4}{2} \\ \frac{-\alpha+1}{2} \end{pmatrix}.$$

The two matrices $\tilde{\mathcal{N}}_1^{\alpha}$ and $\tilde{\mathcal{N}}_2^{\alpha}$ commute only when $\alpha = 0.4$. We therefore obtain:

$$\tilde{\mathcal{N}}_1^{0.4} = \begin{pmatrix} 1 & 0.3 & 0.5 \\ 0.3 & 4 & 0.7 \\ 0.5 & 0.7 & 5 \end{pmatrix}, \quad \tilde{\mathcal{N}}_2^{0.4} = \begin{pmatrix} 3 & 0.6 & 1 \\ 0.6 & 9 & 1.4 \\ 1 & 1.4 & 11 \end{pmatrix}, \quad \tilde{\mathcal{P}}_1^{0.4} = \begin{pmatrix} 0.3 \\ 0.8 \\ 1 \end{pmatrix} \text{ and } \tilde{\mathcal{P}}_2^{0.4} = \begin{pmatrix} 0.5 \\ 1.6 \\ 0.3 \end{pmatrix}.$$

Table 3 presents the solution sequence for problem (24) generated by Algorithm 2.

Table 3. Solution sequences for problem (24) generated by Algorithm 2

$\alpha = 0.4$	U_l	$(0.9914, -0.0718, -0.1093)^T$	$(0.0134, 0.8870, -0.4616)^T$	$(0.1301, 0.4561, 0.8804)^T$	.
	$\eta_{0,i}$	$(0, 0, 0)^T$	$(-0.1259, 0.0091, 0.0139)^T$	$(-0.1276, -0.1054, 0.0735)^T$	$(-0.1452, -0.1672, -0.0457)^T$
	ζ_l	-0.1270	-0.1291	-0.1354	
	$\eta_{1,i}$	$(28.3725, 31.7825, 7.7375)^T$	$(3.4617, 33.5866, 10.4839)^T$	$(3.1250, 11.2994, 22.0823)^T$	$(-0.1451, -0.1649, -0.0469)^T$
	ζ_l	-25.1269	-25.1265	-25.1354	
	$\eta_{2,i}$	$(56.7450, 63.5650, 15.4750)^T$	$(7.0494, 67.1641, 20.9539)^T$	$(6.3777, 22.7042, 44.0910)^T$	$(-0.1449, -0.1625, -0.0481)^T$
	ζ_l	-50.1267	-50.1239	-50.1354	
	$\eta_{3,i}$	$(85.1175, 95.3475, 23.2125)^T$	$(10.6370, 100.7416, 31.4238)^T$	$(9.6304, 34.1090, 66.0998)^T$	$(-0.1448, -0.1602, -0.0494)^T$
	ζ_l	-75.1266	-75.1213	-75.1354	

Table 4. Evaluation of the function $\mathcal{T} \circ \Psi^{\alpha}$

	ν	$\eta_{\nu,3}$	$\mathcal{T}(\Psi^{\alpha}(\eta_{\nu,3}))$
$\alpha = 0.4$	0	$(-0.1452, -0.1672, -0.0457)^T$	-0.1442
	1	$(-0.1451, -0.1649, -0.0469)^T$	-0.1442
	2	$(-0.1449, -0.1625, -0.0481)^T$	-0.1441
	3	$(-0.1448, -0.1602, -0.0494)^T$	-0.1440

The results presented in Table 4 demonstrate that $\eta_{0,3} = (-0.1452, -0.1672, -0.0457)^T$ is the efficient solution to problem (24).

Example 5.2. *In the following problem, the matrices $\tilde{\mathcal{N}}_i$ have symmetric components while the vectors $\tilde{\mathcal{P}}_i$ have asymmetric components:*

$$\min_{[v_1,v_2,v_3]^T \in \mathbb{R}^3} \tilde{\Phi}(v) = (\tilde{\varphi}_1(v_1, v_2, v_3), \tilde{\varphi}_2(v_1, v_2, v_3)), \tag{25}$$

where $\tilde{\varphi}_i : \mathbb{R}^3 \to \mathbb{F}$, $i \in \{1, 2\}$ are defined as follows:

$$\tilde{\varphi}_1(v_1, v_2, v_3) = \frac{1}{2}(v_1, v_2, v_3) \odot \tilde{\mathcal{N}}_1 \odot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \oplus \tilde{\mathcal{P}}_1 \odot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

$$\tilde{\varphi}_2(v_1,v_2,v_3)=\frac{1}{2}(v_1,v_2,v_3)\odot\tilde{\mathcal{N}}_2\odot\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}\oplus\tilde{\mathcal{P}}_2\odot\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}.$$

with $\tilde{\mathcal{N}}_1=\begin{pmatrix}(1,4,7)&(2,3,4)&(-1,1,3)\\(2,3,4)&(2,5,8)&(0,2,4)\\(-1,1,3)&(0,2,4)&(1,3,5)\end{pmatrix},\quad\tilde{\mathcal{P}}_1=\begin{pmatrix}(0,3,4)\\(-1,2,6)\\(1,1,3)\end{pmatrix},$

$\tilde{\mathcal{N}}_2=\begin{pmatrix}(1,2,3)&(0,1,2)&(-2,1,4)\\(0,1,2)&(0,2,4)&(-1,1,3)\\(-2,1,4)&(-1,1,3)&(-3,2,7)\end{pmatrix}$ *and* $\tilde{\mathcal{P}}_2=\begin{pmatrix}(2,3,5)\\(-1,0,3)\\(3,4,7)\end{pmatrix}.$

Let us compute $[\tilde{\mathcal{N}}_i]^\alpha$ *and* $[\tilde{\mathcal{P}}_i]^\alpha$ *through the* α*-cuts of the fuzzy components of* $\tilde{\mathcal{N}}_i$ *and* $\tilde{\mathcal{P}}_i$.

$$[\tilde{\mathcal{N}}_1]^\alpha=\begin{pmatrix}[3\alpha+1,-3\alpha+7]&[\alpha+2,-\alpha+4]&[2\alpha-1,-2\alpha+3]\\[\alpha+2,-\alpha+4]&[3\alpha+2,-3\alpha+8]&[2\alpha,-2\alpha+4]\\[2\alpha-1,-2\alpha+3]&[2\alpha,-2\alpha+4]&[2\alpha+1,-2\alpha+5]\end{pmatrix},$$

$$[\tilde{\mathcal{N}}_2]^\alpha=\begin{pmatrix}[\alpha+1,-\alpha+3]&[\alpha,-\alpha+2]&[3\alpha-2,-3\alpha+4]\\[\alpha,\alpha+2]&[2\alpha,-2\alpha+4]&[2\alpha-1,-2\alpha+3]\\[3\alpha-2,-3\alpha+4]&[2\alpha-1,-2\alpha+3]&[5\alpha-3,-5\alpha+7]\end{pmatrix},$$

$[\tilde{\mathcal{P}}_1]^\alpha=\begin{pmatrix}[3\alpha,-\alpha+4]\\[3\alpha-1,-4\alpha+6]\\[1,-2\alpha+3]\end{pmatrix}$ *and* $[\tilde{\mathcal{P}}_2]^\alpha=\begin{pmatrix}[\alpha+2,-2\alpha+5]\\[\alpha-1,-3\alpha+3]\\[\alpha+3,-3\alpha+7]\end{pmatrix}.$

Using the interval centers, we compute $\tilde{\mathcal{N}}_i^\alpha$ and $\tilde{\mathcal{P}}_i^\alpha$ for all $i = 1, 2$.

$$\tilde{\mathcal{N}}_1^\alpha = \begin{pmatrix} 4 & 3 & 1 \\ 3 & 5 & 2 \\ 1 & 2 & 3 \end{pmatrix}, \quad \tilde{\mathcal{N}}_2^\alpha = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \quad \tilde{\mathcal{P}}_1^\alpha = \begin{pmatrix} \alpha+2 \\ \frac{-\alpha+5}{2} \\ -\alpha+2 \end{pmatrix} \text{ and } \quad \tilde{\mathcal{P}}_2^\alpha = \begin{pmatrix} \frac{-\alpha+7}{2} \\ -\alpha+1 \\ -\alpha+5 \end{pmatrix}$$

We observe that the vectors $\tilde{\mathcal{P}}_i^\alpha$ depend on the membership degree α. We now discuss the solution to problem (25) as a function of α using the strongly increasing function $\mathcal{T} = \alpha\psi_1 + (1-\alpha)\psi_2$. The obtained solutions are reported in Table 5.

Table 5. Solution sequences for problem (25) generated by Algorithm 2

$\alpha = 0.3$	U_l	$(1,1,0)^T$	$(1,0,1)^T$	$(0,1,1)^T$	.
	$\eta_{0,i}$	$(0,0,0)^T$	$(-0.4862,-0.4862,0)^T$	$(-1.0188,-0.4862,-0.5326)^T$	$(-1.0188,-0.2794,-0.3258)^T$
	ζ_l	-0.4862	-0.5326	0.2068	
	$\eta_{1,i}$	$(13.4000,13.4000,13.4000)^T$	$(-4.0287,-4.0287,13.4000)^T$	$(-7.6328,-4.0287,9.7959)^T$	$(-7.6328,-4.4767,9.3479)^T$
	ζ_l	-17.4287	-3.6041	-0.4480	
	$\eta_{2,i}$	$(26.8000,26.8000,26.8000)^T$	$(-7.5713,-7.5713,26.8000)^T$	$(-14.2469,-7.5713,20.1244)^T$	$(-14.2469,-8.6740,19.0216)^T$
	ζ_l	-34.3713	-6.6756	-1.1028	
	$\eta_{3,i}$	$(40.2000,40.2000,40.2000)^T$	$(-11.1138,-11.1138,40.2000)^T$	$(-20.8609,-11.1138,30.4528)^T$	$(-20.8609,-12.8713,28.6953)^T$
	ζ_l	-51.3138	-9.7472	-1.7575	
$\alpha = 0.5$	U_l	$(1,1,0)^T$	$(1,0,1)^T$	$(0,1,1)^T$	.
	$\eta_{0,i}$	$(0,0,0)^T$	$(-0.4048,-0.4048,0)^T$	$(-0.7833,-0.4048,-0.3786)^T$	$(-0.7833,-0.2366,-0.2104)^T$
	ζ_l	-0.4048	-0.3786	0.1681	
	$\eta_{1,i}$	$(13.2000,13.2000,13.2000)^T$	$(-3.5476,-3.5476,13.2000)^T$	$(-6.9433,-3.5476,9.8043)^T$	$(-6.9433,-4.1058,9.2461)^T$
	ζ_l	-16.7476	-3.3957	-0.5582	
	$\eta_{2,i}$	$(26.4000,26.4000,26.4000)^T$	$(-6.6905,-6.6905,26.4000)^T$	$(-13.1033,-6.6905,19.9871)^T$	$(-13.1033,-7.9751,18.7026)^T$
	ζ_l	-33.0905	-6.4129	-1.2846	
	$\eta_{3,i}$	$(39.6000,39.6000,39.6000)^T$	$(-9.8333,-9.8333,39.6000)^T$	$(-19.2633,-9.8333,30.1700)^T$	$(-19.2633,-11.8443,28.1591)^T$
	ζ_l	-49.4333	-9.4300	-2.0109	
$\alpha = 1$	U_l	$(1,1,0)^T$	$(1,0,1)^T$	$(0,1,1)^T$	.
	$\eta_{0,i}$	$(0,0,0)^T$	$(-0.3333,-0.3333,0)^T$	$(-0.4074,-0.3333,-0.0741)^T$	$(-0.4074,-0.2222,0.0370)^T$
	ζ_l	-0.3333	-0.0741	0.1111	
	$\eta_{1,i}$	$(17.4800,17.4800,17.4800)^T$	$(-3.8293,-3.8293,17.4800)^T$	$(-7.7879,-3.8293,13.5215)^T$	$(-7.7879,-4.8836,12.4673)^T$
	ζ_l	-21.3093	-3.9585	-1.0542	
	$\eta_{2,i}$	$(34.9600,34.9600,34.9600)^T$	$(-7.3253,-7.3253,34.9600)^T$	$(-15.1683,-7.3253,27.1170)^T$	$(-15.1683,-9.5449,24.8975)^T$
	ζ_l	-42.2853	-7.8430	-2.2196	
	$\eta_{3,i}$	$(52.4400,52.4400,52.4400)^T$	$(-10.8213,-10.8213,52.4400)^T$	$(-22.5487,-10.8213,40.7126)^T$	$(-22.5487,-14.2062,37.3277)^T$
	ζ_l	-63.2613	-11.7274	-3.3849	

Table 6. Evaluation of the function $\mathcal{T} \circ \Psi^\alpha$

	ν	$\eta_{\nu,3}$	$\mathcal{T}(\Psi(\eta_{\nu,3}))$
$\alpha = 0.3$	0	$(-1.0188,-0.2794,-0.3258)^T$	-2.1736
	1	$(-7.6328,-4.4767,9.3479)^T$	141.2141
	2	$(-14.2469,-8.6740,19.0216)^T$	519.9682
	3	$(-20.8609,-12.8713,28.6953)^T$	1134.0887
$\alpha = 0.5$	0	$(-0.7833,-0.2366,-0.2104)^T$	-1.5247
	1	$(-6.9433,-4.1058,9.2461)^T$	146.6825
	2	$(-13.1033,-7.9751,18.7026)^T$	553.7536
	3	$(-19.2633,-11.8443,28.1591)^T$	1219.6886
$\alpha = 1$	0	$(-0.4074,-0.2222,0.0370)^T$	-0.9321
	1	$(-7.7879,-4.8836,12.4673)^T$	288.6444
	2	$(-15.1683,-9.5449,24.8975)^T$	1159.4457
	3	$(-22.5487,-14.2062,37.3277)^T$	2611.4718

An analysis of Table 6 reveals that smaller values of α lead to convergence toward the efficient solution of problem (25). Thus, for $\alpha = 0$, the optimal solution is $\eta_{0,3} = (-1.5417, -0.4653, -0.5069)^T$.

6. Conclusion

This article has presented the first conjugate direction-type method for solving fuzzy quadratic multiobjective optimization problems. The proposed method begins by transforming a fuzzy quadratic multiobjective optimization problem into a quadratic multiobjective optimization problem. It establishes a relationship between the efficient solution of the fuzzy quadratic multiobjective problem and the Pareto optimal solution of the quadratic multiobjective problem. The developed methodology relies on the existence of conjugate vectors for all matrices in the quadratic part of the multiobjective optimization problem's objective functions. The proposed algorithm has been successfully applied to solve unconstrained fuzzy quadratic multiobjective optimization problems effectively. The obtained results demonstrate that, compared to existing approaches, our method provides better control over the optimal solution through the use of a confidence degree α. The numerical experiments confirm the practical applicability and effectiveness of the proposed approach in handling fuzzy uncertainties while maintaining solution quality.

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