Asia Pac. J. Math. 2025 **12**:107

FRACTIONAL POWERS OF OPERATORS AND THE ABSTRACT CAUCHY PROBLEM

WASEEM GHAZI ALSHANTI[1,*], MA′MON ABU HAMMAD[1], ROSHDI KHALIL[2]

[1]Department of Mathematics, Al Zaytoonah University of Jordan, Amman 11733, Jordan

[2]Department of Mathematics, The University of Jordan, Amman 11942, Jordan

[*]Corresponding author: w.alshanti@zuj.edu.jo

Received Oct. 25, 2025

Abstract. In this paper we utilize the theory of fractional powers of operators to introduce an original idea for solving a homogeneous second order abstract Cauchy problem of the form $Au''(t)+Bu'(t)+Cu(t)=0$, where A, B, and C are assumed to be nice linear operators of exponential type.

2020 Mathematics Subject Classification. 34A08; 47D06; 47D03.

Key words and phrases. abstract cauchy problem; fractional powers of operators; nice operator; operators of exponential type.

1. Introduction

Various linear and non-linear dynamical systems in which the unknown function and its derivatives take values in some abstract spaces, such as Hilbert or Banach spaces, are called abstract differential equations. One of the most powerful tools for solving linear abstract differential equations is the method of semigroups of linear operators on Banach spaces. The basics of this method was originated, independently, by both E. Hille in (1948) [2] and K. Yosida in (1948) [6]. The power of the semigroup approach became clear by contribution of W. Feller in (1952, 1954) [10]. One of the classical vector valued differential equations that can be handled via semigroups of operators is the so called abstract Cauchy problem which has the form:

$$\begin{aligned} &\frac{du}{dt} = Au(t), \quad t \geq 0, \\ &u(0) = x, \end{aligned} \tag{1.1}$$

where $A : D(A) \subseteq X \to X$ is a linear operator of an appropriate type, $x \in X$ (Banach space) is given, and $u : [0, \infty) \to X$ is the unknown (solution) function. The solution $u(t)$ is described by the following definition.

DOI: 10.28924/APJM/12-107

Definition 1.1. [13] *A function $u : [0, \infty) \to X$ is said to be a solution of the initial value problem (1.1) if*

$$(i)\ u \in C\left([0,\infty), X\right) \cap C'\left((0,\infty), X\right), \quad u(t) \in D(A) \text{ for all } t \geq 0,$$
$$(ii)\ u \text{ satisfies } (1.1) \text{ in } X.$$

For both linear and nonlinear abstract Cauchy problems, there are many applications in engineering and applied sciences [17], [18].

For any abstract Cauchy problem, one can associate a family of bounded linear operators that is known as a semigroup of operators.

Let X be a Banach space, and $L(X, X)$ be the space of all bounded linear operators on X. A one-parameter semigroup is a family of linear operators, namely, $\{T(t)\}_{t \geq 0} \subseteq L(X, X)$ such that

$$\textbf{(i)}\ T(0) = I, \text{ the identity operator of } X,$$
$$\textbf{(ii)}\ T(s+t) = T(s)T(t) \text{ fore very } t, s \geq 0.$$

If, in addition, for each fixed $x \in X$, $T(t)x \to x$ as $t \to 0+$, then the semigroup is called c_0-semigroup or (strongly continuous semigroup).

The fact that every non-zero continuous real or complex function g that satisfies the property $g(s + t) = g(s)g(t)$ for every $t, s \geq 0$ has the form $g(t) = e^{axt}$, and that g is determined by the number $a = g'(0)$, reveals the association of an operator A to $\{T(t)\}_{t \geq 0}$ such that

$$Ax := \lim_{t \to 0+} \frac{T(0+t)x - T(0)x}{t}, \text{ for all } x \in D(A),$$

and is called the infinitesimal generator of $\{T(t)\}_{t \geq 0}$.

The concept of this operator plays a crucial role in generating the solutions of abstract Cauchy problems via semigroups approach.

Theorem 1.1. [13] *Let $A : D(A) \subseteq X \to X$ be a densely defined linear operator with a non-empty resolvent set $\rho(A)$. Then the initial value problem (1.1) has a unique solution $u(t)$, which is continuously differentiable on $[0, \infty)$, for all $x \in D(A)$ if and only if A is the infinitesimal generator of a c_0-semigroup $\{Tt\}_{t \geq 0}$.*

When extending the domain of the semigroup parameter to regions over $\mathbb{C}$ at which the non-negative real axis is included, it is essential that in order to preserve the semigroup structure to consider domains at which the complex parameter varies with respect to angles around the positive real axis.

Definition 1.2. [13] *Let $\Delta = \{z \in \mathbb{C} : \varphi_1 < |\arg z| < \varphi_2,\ \varphi_1 < 0 < \varphi_2\}$ be a sector in $\mathbb{C}$ such that for $z \in \Delta$ we have Tz is a bounded linear operator. Then a c_0-semigroup $\{Tz\}_{z \in \Delta}$ is said to be a strongly continuous analytic semigroup if*

$$(i)\ z \to Tz \text{ is analytic in } \Delta,$$
$$(ii)\ T0 = I \text{ and } \lim_{\substack{z \to 0 \\ z \in \Sigma}} Tzx = x \text{ for very } x \in X,$$
$$(iii)\ T(z_1 + z_2) = Tz_1 Tz_2 \text{ for } z_1, z_2 \in \Delta.$$

2. Fractional Powers of Operators

The study of fractional powers of operators in Banach spaces is one of the main ingredients in spectral theory. For $\alpha \in \mathbb{R}$, we can associate a closed linear operator $A : D(A) \subseteq X \to X$ where X is a non-tirvial complex Banach space, another operator A^{α}, namely the fractional power of A with exponent α. Here, we emphasize that for A^{α} to exist, then the following assumption (6.1 page 69 of [13]) must be satisfied

$$\{z \in \mathbb{C} : 0 < |\arg(z)| < \pi\} \subset \rho(A),$$

where $\rho(A)$ is the resolvent set of the operator A.

With X being a complex Banach space, there were extensive studies related to this area during the second half of the twentieth century [1]. Moreover, utilizing the function $f(z) = z^{-\alpha}$, in which an operator A can be plugged in, leads to the field of the fractional calculus of the so called sectorial operators. During the 1960s, the semigroups for both fractional powers of a bounded operator and fractional powers for the negatives of infinitesimal generators were studied in [3], [4], [5]. Later, a new extension of this concept to closed linear operators A in a Banach space X, such that the negative real axis is contained in the resolvent set of A, was carried out in [8]. Recently, Cafarelli and Silvestre [9] presented a new approach to describe fractional powers of operators using the Laplacian. A very detailed study about fractional powers of operators in Banach spaces is available in [14].

Definition 2.1. [14] *Let X be a non-tirvial complex Banach space (X is defined over $\mathbb{C}$) and $A : D(A) \subseteq X \to X$ be a closed and densely defined linear operator with a non-empty resolvent set $\rho(A)$. Then A is called a sectorial operator of angle $\varphi \in \left(0, \frac{\pi}{2}\right)$ over a sector $\Delta_{\varphi,a}$ if there exist constants $a \in \mathbb{R}$ and $N \geq 1$ such that*

$$\Delta_{\varphi,a} := \{\lambda \in \mathbb{C} : 0 < \varphi \leq |\arg(\lambda - a)| \leq \pi\} \subset \rho(A),$$

and

$$\left\|(\lambda I - A)^{-1}\right\|_{L(X,X)} \leq \tfrac{N}{|\lambda - a|}, \text{ for all } \lambda \in \Delta_{\varphi,a} \backslash \{a\}.$$

Remark 1. *In definition 3, if $a = 0$, then A is called a positive sectorial operator , and the sector in this case is denoted by Δ_{φ}.*

Theorem 2.1. [13] *Let $A : D(A) \subseteq X \to X$ be a linear operator. Then the following are equivalent:*

(***i***) *A is the infinitesimal generator of an analytic semigroup,*

(***ii***) *$-A$ is a sectorial operator in X.*

Definition 2.2. [15] *Let $A : D(A) \subseteq X \to X$ be a closed linear positive sectorial operator such that*

$$\Delta_{\varphi} := \{\lambda \in \mathbb{C} : 0 < \varphi \leq |\arg(\lambda)| \leq \pi\} \cup \delta \subset \rho(A),$$

where δ is a neighborhood of zero, and the resolvent operator $(\lambda I - A)^{-1}$ is subjected to the bound

$$\left\|(\lambda I - A)^{-1}\right\|_{L(X,X)} \leq \frac{N(\varphi)}{1 + |\lambda|}, \text{ for all } \lambda \in \Delta_{\varphi},$$

where $N > 0$ depends on the value of φ. Then $A^{-\alpha}$ where $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha > 0$ can be defined by the following contour integral

$$A^{-\alpha} = \frac{1}{2\pi i}\int_{\Gamma} z^{-\alpha}\left(A - zI\right)^{-1} dz, \tag{2.1}$$

where Γ is any piecewise smooth curve running throughout the resolvent set of A from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$ such that $\varphi < \theta < \pi$, avoids the negative real axis and the origin where $z^{-\alpha}$ is taken to be positive for real positive values of z.

It is easy to check that for $\alpha = n \in \mathbb{N}$, the contour of integration Γ can be deformed to a small one around the origin. Therefore, one can apply the residue theorem to show that the integral equals A^{-n}.

3. Main Results

In this section, we provide the solution of the following second order homogeneous abstract Cauchy problem

$$Au''(t) + Bu'(t) + Cu(t) = 0, \tag{3.1}$$

where A, B, and C are linear operators with domains containing the range of $u(t)$ with resolvents, respectively, $\rho(A)$, $\rho(B)$, and $\rho(C)$ lying in the first quadrant such that $\rho(T) \subseteq E = \{(x, y) : x > 0, y > 0\}$ for $T = A$, B, and C.

Before we start solving (3.1), we begin our main result with the following definition.

Definition 3.1. *Let X and Y be any two Banach spaces. Then a closed operator $T : X \to Y$ is said to be a nice operator if*

$$\rho(T) \subseteq E = \{(x, y) : x > 0, y > 0\}.$$

Clearly, definition 5 enables the fractional power of T given in (2.1) to exist. Consequently, in (3.1), A^{α}, B^{α}, and C^{α} are defined for all $\alpha > 0$.

For things to be easy, we assume that A, B, and C are of exponential type [16]. This implies that the operator $T(t)x = e^{tA}x$ exists and is well defined for all $x \in Dom(A)$ in the sense:

(i) T is densely defined,
(ii) $T(t)x = e^{tA}x = \sum_{i=m}^{n} \frac{t^n}{n!}\left\|T^n x\right\| < \infty$, for all $x \in Dom(T)$,
(iii) If $x \in Dom(T)$, then $C\left(x, T\right) = \left\{x,\ Tx,\ T^2 x, \cdots\right\} \subseteq Dom(T)$,

where T represents A, B, and C. Clearly, every bounded linear operator is of exponential type.

Now, we come to the main result.

Theorem 3.1. *Let $u : [0, \infty] \to X$, where X is a normed space. Consider the equation*

$$Au''(t) + Bu'(t) + Cu(t) = 0. \tag{3.2}$$

Assume

$$\begin{aligned} &(\boldsymbol{i}) \quad A,\ B,\ \text{and } C \text{ are nice operators of exponential type,} \\ &(\boldsymbol{ii}) \quad B^2-4AC \text{ is a nice operators of exponential type,} \\ &(\boldsymbol{iii}) \quad A^{-1} \text{ exists and satisfies } (2.1)\,. \end{aligned} \tag{3.3}$$

Then (3.2) *has the following solution*

$$u(t)=e^{\left(\frac{-B+(B^2-4AC)^{\frac{1}{2}}}{2A}t\right)}x_\circ+e^{\left(\frac{-B-(B^2-4AC)^{\frac{1}{2}}}{2A}t\right)}x_\circ, \tag{3.4}$$

where the initial conditions are assumed to be

$$u(0)=2x_\circ \text{ and } u'(0)=-2\frac{Bx_\circ}{2A}, \tag{3.5}$$

and by the operator $\frac{1}{2A}$ *we mean* $\frac{1}{2}A^{-1}$.

Proof. We are going to use the novel idea that we treat A, B, and C as constants in (3.2). This implies that the characteristic equation of (3.2) is

$$Ar^2+Br+C=0, \tag{3.6}$$

which has two solutions

$$r_1=\frac{-B+\sqrt{B^2-4AC}}{2A} \text{ and } r_1=\frac{-B-\sqrt{B^2-4AC}}{2A}. \tag{3.7}$$

From assumptions (i) and (ii) in (3.3), $\sqrt{B^2-4AC}$ exists, well defined, and of exponential type. Therefore, the general solution of (3.2) is as follows:

$$u(t)=c_1e^{\left(\frac{-B+(B^2-4AC)^{\frac{1}{2}}}{2A}t\right)}x_\circ+c_2e^{\left(\frac{-B-(B^2-4AC)^{\frac{1}{2}}}{2A}t\right)}x_\circ, \tag{3.8}$$

But, from the initial conditions(3.5), we obtain

$$\begin{aligned} &u(0)=c_1x_\circ+c_2x_\circ=2x_\circ, \\ &\text{and} \\ &u'(0)=c_1\tfrac{-B+\sqrt{B^2-4AC}}{2A}x_\circ+c_2\tfrac{-B-\sqrt{B^2-4AC}}{2A}x_\circ=-2\tfrac{Bx_\circ}{2A}. \end{aligned} \tag{3.9}$$

Hence,

$$(2-c_2)\,\frac{-B+\sqrt{B^2-4AC}}{2A}x_\circ+c_2\frac{-B-\sqrt{B^2-4AC}}{2A}x_\circ=-2\frac{Bx_\circ}{2A}. \tag{3.10}$$

Equivalently,

$$\frac{2\sqrt{B^2-4AC}}{2A}x_\circ=2c_2\frac{\sqrt{B^2-4AC}}{2A}x_\circ. \tag{3.11}$$

Thus, $c_2=1$ and consequently, $c_1=1$. So,

$$u(t)=e^{\left(\frac{-B+(B^2-4AC)^{\frac{1}{2}}}{2A}t\right)}x_\circ+e^{\left(\frac{-B-(B^2-4AC)^{\frac{1}{2}}}{2A}t\right)}x_\circ. \tag{3.12}$$

Now, to make sure that (3.12) is a solution to (3.2), it is enough to prove that both

$$u_1(t) = e^{\left(\frac{-B+(B^2-4AC)^{\frac{1}{2}}}{2A}t\right)}x_\circ \text{ and } u_2(t) = e^{\left(\frac{-B-(B^2-4AC)^{\frac{1}{2}}}{2A}t\right)}x_\circ,$$

are solutions to (3.2). For $u_1(t)$, we have

$$\begin{aligned} u_1''(t) &= \left(\tfrac{-B+\sqrt{B^2-4AC}}{2A}\right)^2 e^{\left(\frac{-B+\sqrt{B^2-4AC}}{2A}t\right)}x_\circ, \\ &\text{and} \\ u_1'(t) &= \tfrac{-B+\sqrt{B^2-4AC}}{2A}\, e^{\left(\frac{-B+\sqrt{B^2-4AC}}{2A}t\right)}x_\circ. \end{aligned} \tag{3.13}$$

Therefore, by substituting (3.13) into the left hand side of (3.2), we have

$$\left[A\left(\frac{-B+\sqrt{B^2-4AC}}{2A}\right)^2 + B\left(\frac{-B+\sqrt{B^2-4AC}}{2A}\right) + C\right] e^{\left(\frac{-B+\sqrt{B^2-4AC}}{2A}t\right)}x_\circ. \tag{3.14}$$

But,

$$\left[A\left(\frac{-B+\sqrt{B^2-4AC}}{2A}\right)^2 + B\left(\frac{-B+\sqrt{B^2-4AC}}{2A}\right) + C\right] \tag{3.15}$$

$$\begin{aligned} &= \frac{A}{4A^2}\left(B^2+B^2-4AC-2B\sqrt{B^2-4AC}\right) \\ &\quad -\frac{B^2}{4A^2}+\frac{B}{2A}\sqrt{B^2-4AC}+C \\ &= 2\frac{B^2}{4A}-4\frac{AC}{4A}-2\frac{B}{4A}\sqrt{B^2-4AC}-\frac{B^2}{4A^2}+\frac{B}{2A}\sqrt{B^2-4AC}+C \\ &= 0. \end{aligned}$$

Hence, $u_1(t)$ is a solution to (3.2). Similarly, one can prove that $u_2(t)$ is also a solution to (3.2). This ends the proof.

4. Conclusions

This paper has successfully introduced an extraordinary analytical approach for solving second order homogeneous abstract Cauchy problem, namely, $Au''(t)+Bu'(t)+Cu(t)=0$ subjected to certain initial conditions. The method is based on the concept of fractional powers of operators in Banach spaces. The originality of treating the treating A, B, and C as constants together with the assumption that the operators are nice and of exponential type enable the use of classical method for solving a second order homogeneous differential equation.

Authors' Contributions. All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] C.M. Carracedo, M.S. Alix, The Theory of Fractional Powers of Operators, Elsevier, 2001. https://doi.org/10.1016/s0304-0208(00)x8025-5.

[2] E. Hille, R.S. Phillips, Functional Analysis and Semi-Groups, American Mathematical Society, (1948).

[3] E. Hille, Notes on Linear Transformations. II. Analyticity of Semi-Groups, Ann. Math. 40 (1939), 1–47. https://doi.org/10.2307/1968541.

[4] S. Bochner, Diffusion Equation and Stochastic Processes, Proc. Natl. Acad. Sci. U.S.A. 35 (1949), 368–370. https://doi.org/10.1073/pnas.35.7.368.

[5] R.S. Phillips, On the Generation of Semigroups of Linear Operators, Pac. J. Math. 2 (1952), 343–369. https://doi.org/10.2140/pjm.1952.2.343.

[6] K. YOSIDA, On the Differentiability and the Representation of One-Parameter Semi-Group of Linear Operators., J. Math. Soc. Jpn. 1 (1948), 15–21. https://doi.org/10.2969/jmsj/00110015.

[7] K. Yosida, Fractional Powers of Infinitesimal Generators and the Analyticity of the Semi-Groups Generated by Them, Proc. Jpn. Acad. Ser. Math. Sci. 36 (1960), 86–89. https://doi.org/10.3792/pja/1195524080.

[8] A.V. Balakrishnan, An Operational Calculus for Infinitesimal Generators of Semigroups, Trans. Am. Math. Soc. 91 (1959), 330–353. https://doi.org/10.2307/1993125.

[9] L. Caffarelli, L. Silvestre, An Extension Problem Related to the Fractional Laplacian, Commun. Partial. Differ. Equ. 32 (2007), 1245–1260. https://doi.org/10.1080/03605300600987306.

[10] W. Feller, THE GENERAL DIFFUSION OPERATOR AND POSITIVITY PRESERVING Semi-Groups IN ONE DIMENSION, Sel. Pap. II (2015), 273–292. https://doi.org/10.1007/978-3-319-16856-2_16.

[11] A. Favini, A. Yagi, Degenerate Differential Equations in Banach Spaces, CRC Press, 1998. https://doi.org/10.1201/9781482276022.

[12] G. Barbatis, E. Davies, J. Erdos, Course on Operator Theory, Lecture Notes, King's College London, 2003.

[13] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer, (1983).

[14] M. Haase, The Functional Calculus for Sectorial Operators, Birkhäuser, Basel, 2006. https://doi.org/10.1007/3-7643-7698-8.

[15] A. Ashyralyev, A Note on Fractional Derivatives and Fractional Powers of Operators, J. Math. Anal. Appl. 357 (2009), 232–236. https://doi.org/10.1016/j.jmaa.2009.04.012.

[16] R. Khalil, M. Horani, R. Lahcene, Operators of Exponential Type and the Abstract Cauchy Problem, J. Semigroup Theory Appl. 2020 (2020), 4. https://doi.org/10.28919/jsta/4421.

[17] I.M. Batiha, S.A. Njadat, R.M. Batyha, A. Zraiqat, A. Dababneh, et al., Design Fractional-Order PID Controllers for Single-Joint Robot Arm Model, Int. J. Adv. Soft Comput. Appl. 14 (2022), 97–114. https://doi.org/10.15849/ijasca.220720.07.

[18] I.M. Batiha, J. Oudetallah, A. Ouannas, A.A. Al-Nana, I.H. Jebril, Tuning the Fractional-order PID-Controller for Blood Glucose Level of Diabetic Patients, Int. J. Adv. Soft Comput. Appl. 13 (2021), 1–10.