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ADVANCED THEORETICAL FRAMEWORK OF TYPE-2 NEUTROSOPHIC TOPOLOGICAL SPACES: FORMAL DEFINITIONS, SET OPERATIONS, AND FUNDAMENTAL THEOREMS

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Abstract. This paper develops an advanced theoretical framework for Type2 Neutrosophic Topological Spaces (T2NTS) by introducing rigorous definitions, set operations, and fundamental theorems. The study begins with the formalization of Type2 neutrosophic sets, extending classical neutrosophic logic to capture a broader spectrum of uncertainty, indeterminacy, and inconsistency. Building upon these foundations, the research systematically explores topological notions such as Type2 neutrosophic interior, closure, exterior, and boundary, establishing their interrelations through a series of structural theorems and illustrative examples. The novelty of this work lies in demonstrating how Type2 neutrosophic topology generalizes conventional topological concepts, thereby offering a richer analytical tool for modeling systems characterized by higher-order vagueness and incomplete information. The proposed framework not only advances the mathematical foundations of neutrosophic theory but also opens new directions for its applications in decision-making, information systems, and medical diagnosis where uncertainty plays a critical role.

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1. Introduction

Classical topology, inspired by classical analysis, has steadily progressed and found applications across diverse research areas such as machine learning, data analysis, data mining, and quantum gravity [1–7]. Fundamentally, topology captures the notion of connectivity between spatial objects. Since the seminal work of Zadeh (1965) [8] on Fuzzy Sets, increasing attention has been devoted to

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the study of systems characterized by vagueness and imprecision. Fuzzy set theory provided a rigorous mathematical foundation for handling uncertainty, leading to significant applications in artificial intelligence, decision support systems, and control engineering. Building on this, Chang (1968) [9] introduced the concept of a fuzzy topological space, generalizing classical notions such as open sets, closed sets, continuity, and compactness within the fuzzy environment. Further developments on fuzzy topological spaces can be found in [10–14]. Later, Coker [15] extended these ideas by defining intuitionistic fuzzy topological spaces, and explored analogues of classical topological concepts under this framework. Additional studies on intuitionistic fuzzy topology are also available in [16–18]. In a parallel direction, Smarandache (1999) [19] introduced the concept of Neutrosophic Sets, which generalize fuzzy and intuitionistic fuzzy sets by incorporating three independent components: membership, indeterminacy, and non-membership. This advancement allowed for more realistic modeling of inconsistent, indeterminate, and incomplete information. Motivated by this, several researchers have investigated neutrosophic topological spaces [20–22]. Nevertheless, despite this growing interest, the integration of neutrosophic sets with topological structures remains relatively underdeveloped particularly at the Type2 level, which provides a more expressive framework for modeling higher-order uncertainty. This observation highlights the research gap: the lack of a comprehensive mathematical framework that unifies Type2 neutrosophic sets with topology, thereby extending classical and fuzzy topological concepts. The present paper addresses this gap by formulating and developing Type2 Neutrosophic Topological Spaces (T2NTS). Specifically, we introduce precise definitions, establish fundamental set operations, and prove key theorems related to topological notions such as interior, closure, exterior, and boundary within the Type2 neutrosophic setting. The scientific contribution of this work lies in showing that T2NTS is not merely an incremental extension of existing theories, but a robust analytical framework capable of representing complex systems dominated by uncertainty and incomplete information. Beyond strengthening the mathematical foundations of neutrosophic theory, this research also opens new directions for practical applications in medical diagnosis, decision-making systems, and uncertain data analysis, thus bridging the gap between abstract theory and real-world challenges.

2. Preliminaries

Definition 2.1. Let $\mathcal{G}$ be a non empty fixed set. A neutrosophic set (NS) on $\mathcal{G}$ is defined as

$$N = \{g, T_N(g), I_N(g), F_N(g) : g \in \mathcal{G}\}$$

where the function

$$T_N(g),\ I_N(g),\ F_N(g) : \mathcal{G} \to [0,1]$$

denote, respectively, the degree of membership, degree of indeterminacy, and degree of non-membership of the element $g \in \mathcal{G}$ in the set N. These degrees satisfy the condition

$$0 \leq T_N(g) + I_N(g) + F_N(g) \leq 3 \quad \text{for all } g \in \mathcal{G}.$$

Definition 2.2. Let $\mathcal{N}$ be a space of points with elements in $\mathcal{G}$. A Type-2 neutrosophic set (T2NS) $\mathcal{N}$ in $\mathcal{G}$ is characterized by truth membership function $T_{\mathcal{N}}(g)$, indeterminacy membership function $I_{\mathcal{N}}(g)$ and falsity membership function $F_{\mathcal{N}}(g)$, for each $g \in \mathcal{G}$. We have that

$$T_{\mathcal{N}}(g),\ I_{\mathcal{N}}(g),\ F_{\mathcal{N}}(g) \in [0,1],$$

and defined as

$$\mathcal{N} = \{x, [T^L_{\mathcal{N}}(g), T^U_{\mathcal{N}}(g)], [I^L_{\mathcal{N}}(g), I^U_{\mathcal{N}}(g)], [F^L_{\mathcal{N}}(g), F^U_{\mathcal{N}}(g)] : g \in \mathcal{G}\}.$$

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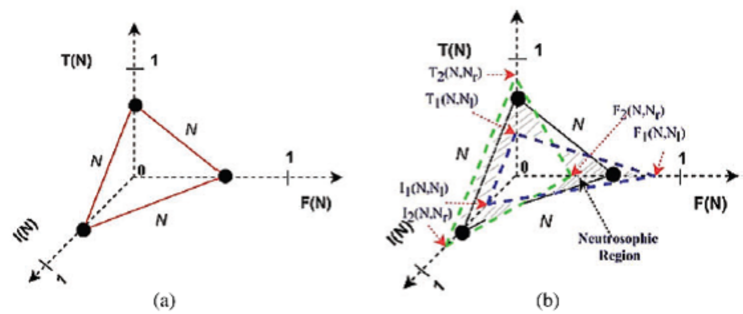

Definition 2.3. Let $\tilde{\tilde{\mathcal{Q}}}, \tilde{\tilde{\mathcal{R}}} \in T2NS$, Then,

i. **Inclusion:** $\tilde{\tilde{\mathcal{Q}}} \subseteq \tilde{\tilde{\mathcal{R}}}$ iff for every $g \in \mathcal{G}$ $\mathcal{T}_{\tilde{\tilde{\mathcal{Q}}}}{}^L(g) \leq \mathcal{T}_{\tilde{\tilde{\mathcal{R}}}}{}^L(g)$, $\mathcal{T}_{\tilde{\tilde{\mathcal{Q}}}}{}^U(g) \leq \mathcal{T}_{\tilde{\tilde{\mathcal{R}}}}{}^U(g)$, $\mathcal{I}_{\tilde{\tilde{\mathcal{Q}}}}{}^L(g) \geq \mathcal{I}_{\tilde{\tilde{\mathcal{R}}}}{}^L(g)$, $\mathcal{I}_{\tilde{\tilde{\mathcal{Q}}}}{}^U(g) \geq \mathcal{I}_{\tilde{\tilde{\mathcal{R}}}}{}^U(g)$, $\mathcal{F}_{\tilde{\tilde{\mathcal{Q}}}}{}^L(g) \geq \mathcal{F}_{\tilde{\tilde{\mathcal{R}}}}{}^L(g)$, $\mathcal{F}_{\tilde{\tilde{\mathcal{Q}}}}{}^U(g) \geq \mathcal{F}_{\tilde{\tilde{\mathcal{R}}}}{}^U(g)$.

ii. **Equality:** $\tilde{\tilde{\mathcal{Q}}} = \tilde{\tilde{\mathcal{R}}}$ iff for every $g \in \mathcal{G}$ $\mathcal{T}_{\tilde{\tilde{\mathcal{Q}}}}{}^L(g) = \mathcal{T}_{\tilde{\tilde{\mathcal{R}}}}{}^L(g)$, $\mathcal{T}_{\tilde{\tilde{\mathcal{Q}}}}{}^U(g) = \mathcal{T}_{\tilde{\tilde{\mathcal{R}}}}{}^U(g)$, $\mathcal{I}_{\tilde{\tilde{\mathcal{Q}}}}{}^L(g) = \mathcal{I}_{\tilde{\tilde{\mathcal{R}}}}{}^L(g)$, $\mathcal{I}_{\tilde{\tilde{\mathcal{Q}}}}{}^U(g) = \mathcal{I}_{\tilde{\tilde{\mathcal{R}}}}{}^U(g)$, $\mathcal{F}_{\tilde{\tilde{\mathcal{Q}}}}{}^L(g) = \mathcal{F}_{\tilde{\tilde{\mathcal{R}}}}{}^L(g)$, $\mathcal{F}_{\tilde{\tilde{\mathcal{Q}}}}{}^U(g) = \mathcal{F}_{\tilde{\tilde{\mathcal{R}}}}{}^U(g)$.

iii. **Complement:** The complement of $\tilde{\tilde{\mathcal{Q}}}$ is given by $\tilde{\tilde{\mathcal{Q}}}^c = \{g, [\mathcal{F}_{\tilde{\tilde{\mathcal{Q}}}}{}^L(g), \mathcal{F}_{\tilde{\tilde{\mathcal{Q}}}}{}^U(g)], [1 - \mathcal{I}_{\tilde{\tilde{\mathcal{Q}}}}{}^U(g), 1 - \mathcal{I}_{\tilde{\tilde{\mathcal{Q}}}}{}^L(g)], [\mathcal{T}_{\tilde{\tilde{\mathcal{Q}}}}{}^L(g), \mathcal{T}_{\tilde{\tilde{\mathcal{Q}}}}{}^U(g)] : g \in \mathcal{G}\}$.

iv. **Intersection:** $\tilde{\tilde{\mathcal{Q}}} \cap \tilde{\tilde{\mathcal{R}}} = \{g, [\min(\mathcal{T}_{\tilde{\tilde{\mathcal{Q}}}}{}^L(g), \mathcal{T}_{\tilde{\tilde{\mathcal{R}}}}{}^L(g)), \min(\mathcal{T}_{\tilde{\tilde{\mathcal{Q}}}}{}^U(g), \mathcal{T}_{\tilde{\tilde{\mathcal{R}}}}{}^U(g))], [\max(\mathcal{I}_{\tilde{\tilde{\mathcal{Q}}}}{}^L(g), \mathcal{I}_{\tilde{\tilde{\mathcal{R}}}}{}^L(x'))], [\max(\mathcal{I}_{\tilde{\tilde{\mathcal{Q}}}}{}^U(g), \mathcal{I}_{\tilde{\tilde{\mathcal{R}}}}{}^U(g))], [\max(\mathcal{F}_{\tilde{\tilde{\mathcal{Q}}}}{}^L(g), \mathcal{F}_{\tilde{\tilde{\mathcal{R}}}}{}^L(g))], [\max(\mathcal{F}_{\tilde{\tilde{\mathcal{Q}}}}{}^U(g), \mathcal{F}_{\tilde{\tilde{\mathcal{R}}}}{}^U(g))] : g \in \mathcal{G}\}$.

v. **Union:** $\tilde{\tilde{\mathcal{Q}}} \cup \tilde{\tilde{\mathcal{R}}} = \{g, [\max(\mathcal{T}_{\tilde{\tilde{\mathcal{Q}}}}{}^L(g), \mathcal{T}_{\tilde{\tilde{\mathcal{R}}}}{}^L(g)), \max(\mathcal{T}_{\tilde{\tilde{\mathcal{Q}}}}{}^U(g), \mathcal{T}_{\tilde{\tilde{\mathcal{R}}}}{}^U(x'))], [\min(\mathcal{I}_{\tilde{\tilde{\mathcal{Q}}}}{}^L(g), \mathcal{I}_{\tilde{\tilde{\mathcal{R}}}}{}^L(g))], [\min(\mathcal{I}_{\tilde{\tilde{\mathcal{Q}}}}{}^U(g), \mathcal{I}_{\tilde{\tilde{\mathcal{R}}}}{}^U(g))], [\min(\mathcal{F}_{\tilde{\tilde{\mathcal{Q}}}}{}^L(g), \mathcal{F}_{\tilde{\tilde{\mathcal{R}}}}{}^L(g))], [\min(\mathcal{F}_{\tilde{\tilde{\mathcal{Q}}}}{}^U(g), \mathcal{F}_{\tilde{\tilde{\mathcal{R}}}}{}^U(g))] : g \in \mathcal{G}\}$.

vi. **Universal set:**

$$1_{\mathcal{N}} = \{g : [1,1], [0,0], [0,0] : g \in \mathcal{G}\}.$$

vii. **Empty set:**

$$0_{\mathcal{N}} = \{g : [0,0], [1,1], [1,1] : g \in \mathcal{G}\}.$$

Example 2.4. Let $\mathcal{G} = \{g, h\}$ and $\tilde{\tilde{\mathcal{Q}}}, \tilde{\tilde{\mathcal{R}}}, \tilde{\tilde{\mathcal{S}}} \in T2NS$ such that

$$\tilde{\tilde{\mathcal{Q}}} = \{\{g, [0.3, 0.5], [0.5, 0.7], [0.3, 0.8]\}, \{h, [0.7, 0.8], [0.4, 0.7], [0.4, 0.7]\}\}$$

$$\tilde{\tilde{\mathcal{R}}} = \{\{g, [0.6, 0.7], [0.4, 0.6], [0.2, 0.7]\}, \{h, [0.8, 0.9], [0.3, 0.7], [0.3, 0.6]\}\}$$

$$\tilde{\tilde{\mathcal{S}}} = \{\{g, [0.7, 0.9], [0.3, 0.8], [0.1, 0.9]\}, \{h, [0.2, 0.6], [0.4, 0.6], [0.2, 0.9]\}\}$$

Then,

i. We have that $\tilde{\tilde{\mathcal{Q}}} \subseteq \tilde{\tilde{\mathcal{R}}}$

ii. $\tilde{\tilde{\mathcal{R}}} \cup \tilde{\tilde{\mathcal{S}}} = \{\{g, [0.7, 0.9], [0.3, 0.6], [0.1, 0.7]\}, \{h, [0.8, 0.9], [0.3, 0.6], [0.2, 0.6]\}\}$

iii. $\tilde{\tilde{\mathcal{Q}}} \cap \tilde{\tilde{\mathcal{S}}} = \{\{g, [0.3, 0.5], [0.5, 0.8], [0.3, 0.9]\}, \{h, [0.2, 0.6], [0.4, 0.7], [0.4, 0.9]\}\}$

iv. $\tilde{\tilde{\mathcal{S}}}^c = \{\{g, [0.1, 0.9], [0.7, 0.2], [0.7, 0.9]\}, \{h, [0.2, 0.9], [0.6, 0.4], [0.2, 0.6]\}\}$

Theorem 2.5. Let $\tilde{\tilde{\mathcal{Q}}}, \tilde{\tilde{\mathcal{R}}}, \tilde{\tilde{\mathcal{S}}} \in T2NS$, then the following hold.

i. $\tilde{\tilde{\mathcal{Q}}} \cup \tilde{\tilde{\mathcal{Q}}} = \tilde{\tilde{\mathcal{Q}}}$ and $\tilde{\tilde{\mathcal{Q}}} \cap \tilde{\tilde{\mathcal{Q}}} = \tilde{\tilde{\mathcal{Q}}}$

ii. $\tilde{\tilde{\mathcal{Q}}} \cup \tilde{\tilde{\mathcal{R}}} = \tilde{\tilde{\mathcal{R}}} \cup \tilde{\tilde{\mathcal{Q}}}$ and $\tilde{\tilde{\mathcal{Q}}} \cap \tilde{\tilde{\mathcal{R}}} = \tilde{\tilde{\mathcal{R}}} \cap \tilde{\tilde{\mathcal{Q}}}$

iii. $\tilde{\tilde{\mathcal{Q}}} \cap 0_{\mathcal{N}} = 0_{\mathcal{N}}$ and $\tilde{\tilde{\mathcal{Q}}} \cap 1_{\mathcal{N}} = \tilde{\tilde{\mathcal{Q}}}$

iv. $\tilde{\tilde{\mathcal{Q}}} \cup 0_{\mathcal{N}} = \tilde{\tilde{\mathcal{Q}}}$ and $\tilde{\tilde{\mathcal{Q}}} \cup 1_{\mathcal{N}} = 1_{\mathcal{N}}$

v. $\tilde{\tilde{\mathcal{Q}}} \cup (\tilde{\tilde{\mathcal{R}}} \cup \tilde{\tilde{\mathcal{S}}}) = (\tilde{\tilde{\mathcal{Q}}} \cup \tilde{\tilde{\mathcal{R}}}) \cup \tilde{\tilde{\mathcal{S}}}$ and $\tilde{\tilde{\mathcal{Q}}} \cap (\tilde{\tilde{\mathcal{R}}} \cap \tilde{\tilde{\mathcal{S}}}) = (\tilde{\tilde{\mathcal{Q}}} \cap \tilde{\tilde{\mathcal{R}}}) \cap \tilde{\tilde{\mathcal{S}}}$

v. $(\tilde{\tilde{\mathcal{Q}}}^c)^c = \tilde{\tilde{\mathcal{Q}}}$

Proof: It is clear

Theorem 2.6. Let $\tilde{\tilde{\mathcal{Q}}} \in$ T2NS, then the DeMorgan's law is valid.

i. $(\bigcup_{i\in I} \tilde{\tilde{\mathcal{Q}}}_i)^c = \bigcap_{i\in I} \tilde{\tilde{\mathcal{Q}}}_i^c$

ii. $(\bigcap_{i\in I} \tilde{\tilde{\mathcal{Q}}}_i)^c = \bigcup_{i\in I} \tilde{\tilde{\mathcal{Q}}}_i^c$

Proof:

i.

$$(\bigcup_{i\in I} \tilde{\tilde{\mathcal{Q}}}_i)^c = \{x' \mid [\max(T^L_{\tilde{\tilde{\mathcal{Q}}}_i}(g)), \max(T^U_{\tilde{\tilde{\mathcal{Q}}}_i}(g))], [\min(I^L_{\tilde{\tilde{\mathcal{Q}}}_i}(g)), \min(I^U_{\tilde{\tilde{\mathcal{Q}}}_i}(g))], [\min(F^L_{\tilde{\tilde{\mathcal{Q}}}_i}(g)), \min(F^U_{\tilde{\tilde{\mathcal{Q}}}_i}(g))]$$

$$= \{x' \mid \left[\min(F^L_{\tilde{\tilde{\mathcal{Q}}}_i}(g)), \min(F^U_{\tilde{\tilde{\mathcal{Q}}}_i}(g))\right], \left[\max(T^L_{\tilde{\tilde{\mathcal{Q}}}_i}(g)), \max(T^U_{\tilde{\tilde{\mathcal{Q}}}_i}(g))\right]$$

$$= \bigcap_{i\in I} \tilde{\tilde{\mathcal{Q}}}_i^c$$

ii. similar way to i

Theorem 2.7. Let $\tilde{\tilde{\mathcal{R}}} \in$ T2NS and $\{\tilde{\tilde{\mathcal{Q}}}_j : i \in I\} \subseteq$ T2NS. Then,

i. $\tilde{\tilde{\mathcal{R}}} \cap (\bigcup_{i\in I} \tilde{\tilde{\mathcal{Q}}}_i) = \bigcup_{i\in I} \tilde{\tilde{\mathcal{R}}} \cap \tilde{\tilde{\mathcal{Q}}}_i$

ii. $\tilde{\tilde{\mathcal{R}}} \cup (\bigcap_{i\in I} \tilde{\tilde{\mathcal{Q}}}_i) = \bigcap_{i\in I} \tilde{\tilde{\mathcal{R}}} \cup \tilde{\tilde{\mathcal{Q}}}_i$

Proof: It can be proved easily from definition 2.3

3. Type2 neutrosophic topological spaces

Definition 3.1. Let $\tilde{\tilde{\mathcal{T}}} \subseteq$ T2NS, then it is called **Type2 Neutrosophic topology** on $\mathcal{G}$ if

i. $0_{\mathcal{N}}, 1_{\mathcal{N}} \in \tilde{\tilde{\mathcal{T}}}$
ii. If $\tilde{\tilde{\mathcal{Q}}}$ and $\tilde{\tilde{\mathcal{R}}} \in \tilde{\tilde{\mathcal{T}}}$, then $\tilde{\tilde{\mathcal{Q}}} \cap \tilde{\tilde{\mathcal{R}}} \in \tilde{\tilde{\mathcal{T}}}$
iii. If $\tilde{\tilde{\mathcal{Q}}}_i \in \tilde{\tilde{\mathcal{T}}}$, then $\bigcup_{i\in I} \tilde{\tilde{\mathcal{Q}}}_i \in \tilde{\tilde{\mathcal{T}}}$

The pair $(\mathcal{G}, \tilde{\tilde{\mathcal{T}}})$ is called *Type2 Neutrosophic topological space* $(T2NTS)$ over $\mathcal{G}$ and the elements of $\tilde{\tilde{\mathcal{T}}}$ are said to be *Type2 Neutrosophic open set* $(T2NOS)$ in $\mathcal{G}$. If $\tilde{\tilde{\mathcal{Q}}}^c \in \tilde{\tilde{\mathcal{T}}}$, then $\tilde{\tilde{\mathcal{Q}}} \in T2NS$ is said to be *Type2 Neutrosophic closed set* $(T2NCS)$ in $\mathcal{G}$.

Theorem 3.2. Let $(\mathcal{G}, \tilde{\tilde{\mathcal{T}}})$ be T2NTS over $\mathcal{G}$. Then

(1) $0_{\mathcal{N}}, 1_{\mathcal{N}}$ are T2NCS over $\mathcal{G}$
(2) If $\tilde{\tilde{\mathcal{Q}}}_i \in$ T2NCS then $\bigcap_{i\in I} \tilde{\tilde{\mathcal{Q}}}_i \in$ T2NCS over $\mathcal{G}$
(3) If $\tilde{\tilde{\mathcal{Q}}}$ and $\tilde{\tilde{\mathcal{R}}} \in$ T2NCS then $\tilde{\tilde{\mathcal{Q}}} \cup \tilde{\tilde{\mathcal{R}}} \in$ T2NCS over $\mathcal{G}$

Proof: It is clear.

Example 3.3. Let $\mathcal{G} = \{q, r\}$ and $\tilde{\tilde{\mathcal{Q}}} \subseteq$ T2NS such that

$$\tilde{\tilde{\mathcal{Q}}} = \{(q, [0.2, 0.6], [0.3, 0.7], [0.4, 0.8]), (r, [0.1, 0.5], [0.2, 0.6], [0.3, 0.7])\}$$

Then, $\tilde{\tilde{\mathcal{T}}} = \{0_{\mathcal{N}}, 1_{\mathcal{N}}, \tilde{\tilde{\mathcal{Q}}}\}$ is T2NTS over $\mathcal{G}$.

Theorem 3.4. Let $(\mathcal{G}, \tilde{\tilde{\mathcal{T}}}_1)$ and $(\mathcal{G}, \tilde{\tilde{\mathcal{T}}}_2)$ be two T2NTS over$\mathcal{G}$, then $(X, \tilde{\tilde{\mathcal{T}}}_1 \cap \tilde{\tilde{\mathcal{T}}}_2)$ is T2NTS over $\mathcal{G}$. **Proof:** Let $(\mathcal{G}, \tilde{\tilde{\mathcal{T}}}_1)$ and $(\mathcal{G}, \tilde{\tilde{\mathcal{T}}}_2)$ be two T2NTS over $\mathcal{G}$. It can be seen clearly that $0_{\mathcal{N}}, 1_{\mathcal{N}} \in \tilde{\tilde{\mathcal{T}}}_1 \cap \tilde{\tilde{\mathcal{T}}}_2$. If $\tilde{\tilde{\mathcal{R}}} \in \tilde{\tilde{\mathcal{T}}}_1$ and $\tilde{\tilde{\mathcal{Q}}}, \tilde{\tilde{\mathcal{R}}} \in \tilde{\tilde{\mathcal{T}}}_2$, then $\tilde{\tilde{\mathcal{Q}}} \cap \tilde{\tilde{\mathcal{R}}} \in \tilde{\tilde{\mathcal{T}}}_1 \cap \tilde{\tilde{\mathcal{T}}}_2$. Let $\tilde{\tilde{\mathcal{Q}}}_i : i \in I\} \subseteq \tilde{\tilde{\mathcal{T}}}_1 \cap \tilde{\tilde{\mathcal{T}}}_2$, then $\tilde{\tilde{\mathcal{Q}}}_i \in \tilde{\tilde{\mathcal{T}}}_1 \cap \tilde{\tilde{\mathcal{T}}}_2$ for all $i \in I$. Thus $\tilde{\tilde{\mathcal{Q}}}_i \in \tilde{\tilde{\mathcal{T}}}_1$ and $\tilde{\tilde{\mathcal{Q}}}_i \in \tilde{\tilde{\mathcal{T}}}_2$ for all $i \in I$, so we have $\bigcup_{i\in I} \tilde{\tilde{\mathcal{Q}}}_i \in \tilde{\tilde{\mathcal{T}}}_1 \cap \tilde{\tilde{\mathcal{T}}}_2$.

Corollary 3.5. Let $\{(\mathcal{G}, \tilde{\tilde{\mathcal{T}}}_i) : i \in I\}$ be a family of T2NTS over $\mathcal{G}$. Then $(\mathcal{G}, \bigcap_{i\in I} \tilde{\tilde{\mathcal{T}}}_i)$ is T2NTS over $\mathcal{G}$.

Proof: Similar to Theorem 3.4.

Remark 3.6. If we get the union operation instead of the intersection in the theorem 3.4 the claim may not be correct. The following example shows this.

Example 3.7. Let $\mathcal{G} = \{q, r\}$ and $\tilde{\tilde{\mathcal{Q}}}, \tilde{\tilde{\mathcal{R}}} \subseteq$ T2NS such that

$$\tilde{\tilde{\mathcal{Q}}} = \{(q, [0.2, 0.6], [0.3, 0.7], [0.4, 0.9]), (r, [0.1, 0.6], [0.3, 0.6], [0.3, 0.8])\}$$

$$\tilde{\tilde{\mathcal{R}}} = \{(q, [0.2, 0.3], [0.3, 0.9], [0.5, 0.8]), (r, [0.2, 0.5], [0.2, 0.7], [0.4, 0.7])\}$$

Then $\tilde{\tilde{\mathcal{T}}}_1 = \{0_N, 1_N, \tilde{\tilde{\mathcal{Q}}}\}$ and $\tilde{\tilde{\mathcal{T}}}_2 = \{0_{\mathcal{N}}, 1_{\mathcal{N}}, \tilde{\tilde{\mathcal{R}}}\}$ be two T2NTS. But $\tilde{\tilde{\mathcal{T}}}_1 \cup \tilde{\tilde{\mathcal{T}}}_2 = \{0_N, 1_N, \tilde{\tilde{\mathcal{Q}}}, \tilde{\tilde{\mathcal{R}}}\}$ is not T2NTS because $\tilde{\tilde{\mathcal{Q}}} \cup \tilde{\tilde{\mathcal{R}}} = \{(q, [0.2, 0.6], [0.3, 0.7], [0.4, 0.8]), (r, [0.2, 0.6], [0.2, 0.6], [0.3, 0.7])\} \notin \tilde{\tilde{\mathcal{T}}}_1 \cup \tilde{\tilde{\mathcal{T}}}_2$. So $\tilde{\tilde{\mathcal{T}}}_1 \cup \tilde{\tilde{\mathcal{T}}}_2$ is not T2NTS over $\mathcal{G}$.

Definition 3.8. Let $(\mathcal{G}, \tilde{\tilde{\mathcal{T}}})$ be T2NTS over $\mathcal{G}$ and $\tilde{\tilde{\mathcal{Q}}} \in$ T2NS. Then the Type2 Neutrosophic interior of $\tilde{\tilde{\mathcal{Q}}}$

$$T2Int(\tilde{\tilde{\mathcal{Q}}}) = \bigcup\{\tilde{\tilde{\mathcal{R}}} : \tilde{\tilde{\mathcal{R}}} \in T2NOS \text{ in } \mathcal{G} \text{ such that } \tilde{\tilde{\mathcal{R}}} \subseteq \tilde{\tilde{\mathcal{Q}}}\}.$$

Theorem 3.9. Let $(\mathcal{G}, \tilde{\tilde{\mathcal{T}}})$ be T2NTS over $\mathcal{G}$ and $\tilde{\tilde{\mathcal{Q}}}, \tilde{\tilde{\mathcal{R}}} \in$ T2NS. Then

i. T2Int$(0_{\mathcal{N}}) = 0_{\mathcal{N}}$ and T2Int$(1_{\mathcal{N}}) = 1_{\mathcal{N}}$
ii. T2Int$(\tilde{\tilde{\mathcal{Q}}}) \subseteq \tilde{\tilde{\mathcal{Q}}}$
iii. $\tilde{\tilde{\mathcal{Q}}}$ is T2NOS iff T2Int$(\tilde{\tilde{\mathcal{Q}}}) = \tilde{\tilde{\mathcal{Q}}}$
iv. T2Int(T2Int$(\tilde{\tilde{\mathcal{Q}}})$) = T2Int$(\tilde{\tilde{\mathcal{Q}}})$
v. $\tilde{\tilde{\mathcal{Q}}} \subseteq \tilde{\tilde{\mathcal{R}}} \implies$ T2Int$(\tilde{\tilde{\mathcal{Q}}}) \subseteq$ T2Int$(\tilde{\tilde{\mathcal{R}}})$
vi. $T2Int(\tilde{\tilde{\mathcal{Q}}}) \cup T2Int(\tilde{\tilde{\mathcal{R}}}) \subseteq T2Int(\tilde{\tilde{\mathcal{Q}}} \cup \tilde{\tilde{\mathcal{R}}})$.
vii. $T2Int(\tilde{\tilde{\mathcal{Q}}} \cap \tilde{\tilde{\mathcal{R}}}) = T2Int(\tilde{\tilde{\mathcal{Q}}}) \cap T2Int(\tilde{\tilde{\mathcal{R}}})$.

Proof: i and ii are clear

iii $\Rightarrow$ If $\tilde{\tilde{\mathcal{Q}}}$ is T2NOS over $\mathcal{G}$ then $\tilde{\tilde{\mathcal{Q}}}$ itself T2NOS over X which contains $\tilde{\tilde{\mathcal{Q}}}$. So $\tilde{\tilde{\mathcal{Q}}}$ is the largest T2NOS contained in $\tilde{\tilde{\mathcal{Q}}}$, and $T2Int(\tilde{\tilde{\mathcal{Q}}}) = \tilde{\tilde{\mathcal{Q}}}$.
$\Leftarrow$ let $T2Int(\tilde{\tilde{\mathcal{Q}}}) = \tilde{\tilde{\mathcal{Q}}}$, then $\tilde{\tilde{\mathcal{Q}}} \in T2NOS$.

iv. Let $T2Int(\tilde{\tilde{\mathcal{Q}}}) = \tilde{\tilde{\mathcal{R}}}$. Then $T2Int(\tilde{\tilde{\mathcal{R}}}) = \tilde{\tilde{\mathcal{R}}}$, from iii and then $T2Int(T2Int(\tilde{\tilde{\mathcal{Q}}})) = T2Int(\tilde{\tilde{\mathcal{Q}}})$.

v. Suppose that $\tilde{\tilde{\mathcal{Q}}} \subseteq \tilde{\tilde{\mathcal{R}}}$. As $T2Int(\tilde{\tilde{\mathcal{Q}}}) \subseteq \tilde{\tilde{\mathcal{Q}}} \subseteq \tilde{\tilde{\mathcal{R}}}$, $T2Int(\tilde{\tilde{\mathcal{Q}}})$ is T2NOS of $\tilde{\tilde{\mathcal{R}}}$, so by definition 3.8 we have that $T2Int(\tilde{\tilde{\mathcal{Q}}}) \subseteq T2Int(\tilde{\tilde{\mathcal{R}}})$.

vi. It is clear that $\tilde{\tilde{\mathcal{Q}}} \subseteq \tilde{\tilde{\mathcal{Q}}} \cup \tilde{\tilde{\mathcal{R}}}$ and $\tilde{\tilde{\mathcal{R}}} \subseteq \tilde{\tilde{\mathcal{Q}}} \cup \tilde{R}$, Thus

$$T2Int(\tilde{\tilde{\mathcal{Q}}}) \subseteq T2Int(\tilde{\tilde{\mathcal{Q}}} \cup \tilde{\tilde{\mathcal{R}}}) \quad \text{and} \quad T2Int(\tilde{\tilde{\mathcal{R}}}) \subseteq T2Int(\tilde{\tilde{\mathcal{Q}}} \cup \tilde{\tilde{\mathcal{R}}}),$$

so we have that $T2Int\tilde{\tilde{\mathcal{Q}}}) \cup T2Int(\tilde{\tilde{\mathcal{R}}}) \subseteq T2Int(\tilde{\tilde{\mathcal{Q}}} \cup \tilde{\tilde{\mathcal{R}}})$.

viii. It is known that $T2Int(\tilde{\tilde{\mathcal{Q}}} \cap \tilde{\tilde{\mathcal{R}}}) \subseteq T2Int(\tilde{\tilde{\mathcal{Q}}})$ and $T2Int(\tilde{\tilde{\mathcal{Q}}} \cap \tilde{\tilde{\mathcal{R}}}) \subseteq T2Int(\tilde{\tilde{\mathcal{R}}})$, so that

$$T2Int(\tilde{\tilde{\mathcal{Q}}} \cap \tilde{\tilde{\mathcal{R}}}) \subseteq T2Int(\tilde{\tilde{\mathcal{Q}}}) \cap T2Int(\tilde{\tilde{\mathcal{R}}}).$$

Also from $T2Int(\tilde{\tilde{\mathcal{Q}}}) \subseteq \tilde{\tilde{\mathcal{Q}}}$ and $T2Int(\tilde{\tilde{\mathcal{R}}}) \subseteq \tilde{\tilde{\mathcal{R}}}$ we have $T2Int(\tilde{\tilde{\mathcal{Q}}}) \cap T2Int(\tilde{\tilde{\mathcal{R}}}) \subseteq \tilde{\tilde{\mathcal{Q}}} \cap \tilde{\tilde{\mathcal{R}}}$, these implies that

$$T2Int(\tilde{\tilde{\mathcal{Q}}} \cap \tilde{\tilde{\mathcal{R}}}) = T2Int(\tilde{\tilde{\mathcal{Q}}}) \cap T2Int(\tilde{\tilde{\mathcal{R}}}).$$

Example 3.10. Let $\mathcal{G} = \{q, r\}$ and $\tilde{\tilde{\mathcal{Q}}}, \tilde{\tilde{\mathcal{R}}}, \tilde{\tilde{\mathcal{S}}} \in T2NS$ such that

$$\tilde{\tilde{\mathcal{Q}}} = \{(q, [0.4, 0.6], [0.3, 0.7], [0.4, 0.8]), (r, [0.2, 0.5], [0.3, 0.9], [0.6, 0.7])\},$$

$$\tilde{\tilde{\mathcal{R}}} = \{(q, [0.3, 0.7], [0.2, 0.5], [0.2, 0.7]), (r, [0.3, 0.4], [0.1, 0.6], [0.3, 0.8])\},$$

$$\tilde{\tilde{\mathcal{S}}} = \{(q, [0.7, 0.9], [0.3, 0.4], [0.3, 0.6]), (r, [0.1, 0.5], [0.2, 0.5], [0.2, 0.7])\}.$$

Then, $\tilde{\tilde{\mathcal{T}}} = \{0_N, 1_N, \tilde{\tilde{\mathcal{Q}}}\}$ is T2NTS over X. Therefore $T2Int(\tilde{\tilde{\mathcal{R}}}) = 0_N$ and $T2Int(\tilde{\tilde{\mathcal{S}}}) = 0_N$.

And

$$\tilde{\tilde{\mathcal{R}}} \cup \tilde{\tilde{\mathcal{S}}} = \{(q, [0.7, 0.9], [0.2, 0.4], [0.2, 0.6]), (r, [0.3, 0.5], [0.1, 0.5], [0.2, 0.7])\}.$$

$T2Int(\tilde{\tilde{\mathcal{R}}} \cup \tilde{\tilde{\mathcal{S}}}) = \tilde{\tilde{\mathcal{Q}}}$, So $T2Int(\tilde{\tilde{\mathcal{R}}}) \cup T2Int(\tilde{\tilde{\mathcal{S}}}) \neq T2Int(\tilde{\tilde{\mathcal{R}}} \cup \tilde{\tilde{\mathcal{S}}})$.

Definition 3.11. Let $(\mathcal{G}, \tilde{\tilde{\mathcal{T}}})$ be T2NTS over $\mathcal{G}$ and $\tilde{\tilde{\mathcal{Q}}} \in T2NS$. Then the Type2 Neutrosophic closure of $\tilde{\tilde{\mathcal{Q}}}$ is

$$T2Cl(\tilde{\tilde{\mathcal{R}}}) = \bigcap\{\tilde{\tilde{\mathcal{R}}} : \tilde{\tilde{\mathcal{R}}} \text{ is T2NCS in } \mathcal{G} \text{ such that } \tilde{\tilde{\mathcal{Q}}} \subseteq \tilde{\tilde{\mathcal{R}}}\}.$$

Theorem 3.12. Let $(\mathcal{G}, \tilde{\tilde{\mathcal{T}}})$ be T2NTS over $\mathcal{G}$ and $\tilde{\tilde{\mathcal{Q}}}, \tilde{\tilde{\mathcal{R}}} \in T2NS$. Then

i. $T2Cl(0_{\mathcal{N}}) = 0_{\mathcal{N}}$ and $T2Cl(1_{\mathcal{N}}) = 1_{\mathcal{N}}$.
ii. $\tilde{\tilde{\mathcal{Q}}} \subseteq T2Cl(\tilde{\tilde{\mathcal{Q}}})$.
iii. $\tilde{\tilde{\mathcal{Q}}}$ is T2NCS iff $T2Cl(\tilde{\tilde{\mathcal{Q}}}) = \tilde{\tilde{\mathcal{Q}}}$.
iv. $T2Cl(T2Cl(\tilde{\tilde{\mathcal{Q}}})) = T2Cl(\tilde{\tilde{\mathcal{Q}}})$.
v. $\tilde{\tilde{\mathcal{Q}}} \subseteq \tilde{\tilde{\mathcal{R}}}$ implies $T2Cl(\tilde{\tilde{\mathcal{Q}}}) \subseteq T2Cl(\tilde{\tilde{\mathcal{R}}})$.
vi. $T2Cl(\tilde{\tilde{\mathcal{Q}}} \cup \tilde{\tilde{\mathcal{R}}}) = T2Cl(\tilde{\tilde{\mathcal{Q}}}) \cup T2Cl(\tilde{\tilde{\mathcal{R}}})$.
vii. $T2Cl(\tilde{\tilde{\mathcal{Q}}} \cap \tilde{\tilde{\mathcal{R}}}) \subseteq T2Cl(\tilde{\tilde{\mathcal{Q}}}) \cap T2Cl(\tilde{\tilde{\mathcal{R}}})$.

Proof :

i., ii., vi and vii are similar to **theorem 3.9**

iii. if is a T2NCS over $\mathcal{G}$, then $\tilde{\tilde{\mathcal{Q}}}$ is itself a T2NCS over $\mathcal{G}$ which contains $\tilde{\tilde{\mathcal{R}}}$, therefore $\tilde{\tilde{\mathcal{Q}}}$ is the smallest T2NCS containing $\tilde{\tilde{\mathcal{Q}}}$ and T2Cl$(\tilde{\tilde{\mathcal{Q}}}) = \tilde{\tilde{\mathcal{Q}}}$.

Conversely suppose that$T2Cl(\tilde{\tilde{\mathcal{Q}}}) = \tilde{\tilde{\mathcal{Q}}}$ as $\tilde{\tilde{\mathcal{Q}}}$ T2NCS, so $\tilde{\tilde{\mathcal{Q}}}$ T2NCS over $\mathcal{G}$.

iv. is a T2NCS so by iii. then we have $T2Cl(\tilde{\tilde{\mathcal{Q}}}) = \tilde{\tilde{\mathcal{Q}}}$.

v. Let $\tilde{\tilde{\mathcal{Q}}} \subseteq \tilde{\tilde{\mathcal{R}}}$ then every T2NCS of $\tilde{\tilde{\mathcal{R}}}$ will also contain $\tilde{\tilde{\mathcal{Q}}}$. This means that every T2NCS of $\tilde{\tilde{\mathcal{R}}}$ is also a T2NCS of $\tilde{\tilde{\mathcal{Q}}}$. Hence the intersection of T2NCS of $\tilde{\tilde{\mathcal{Q}}}$ is contained in the intersection of T2NCS of $\tilde{\tilde{\mathcal{R}}}$. Thus $T2Cl(\tilde{\tilde{\mathcal{Q}}}) \subseteq T2Cl(\tilde{\tilde{\mathcal{R}}})$.

Remark 3.13. The following example shown that the equality in vii is not hold.

Example 3.14. Let $\mathcal{G} = \{q, r\}$ and $\tilde{\tilde{\mathcal{Q}}}, \tilde{\tilde{\mathcal{R}}} \in$ T2NS such that
$\tilde{\tilde{\mathcal{Q}}} = \{\{q, [0.4, 0.6]\}, [0.5, 0.4], [0.4, 0.8]\}, \tilde{\tilde{\mathcal{R}}} = \{\{q, [0.3, 0.7]\}, [0.2, 0.5], [0.2, 0.7]\}$,
then $\tilde{\tilde{\mathcal{T}}} = \{0_N, 1_N, \tilde{\tilde{\mathcal{Q}}}, \tilde{\tilde{\mathcal{R}}}, \tilde{\tilde{\mathcal{Q}}} \cup \tilde{\tilde{\mathcal{R}}}, \tilde{\tilde{\mathcal{Q}}} \cap \tilde{\tilde{\mathcal{R}}}\}$ is T2NTS over $\mathcal{G}$.
where $\tilde{\tilde{\mathcal{Q}}} \cup \tilde{\tilde{\mathcal{R}}}$ = {(q, [0.4, 0.7], [0.2, 0.4], [0.2, 0.7]), (r, [0.3, 0.5], [0.1, 0.3], [0.3, 0.7]) }.
$\tilde{\tilde{\mathcal{Q}}} \cap \tilde{\tilde{\mathcal{R}}}$ = {(q, [0.3, 0.6], [0.5, 0.5], [0.4, 0.8]), (r, [0.2, 0.4], [0.5, 0.5], [0.6, 0.8]) }.

Moreover T2NCS over $\mathcal{G}$ is $\{0_{\mathcal{N}}, 1_{\mathcal{N}}, \tilde{\tilde{\mathcal{Q}}}^c, \tilde{\tilde{\mathcal{R}}}^c, (\tilde{\tilde{\mathcal{Q}}} \cup \tilde{\tilde{\mathcal{R}}})^c, (\tilde{\tilde{\mathcal{Q}}} \cap \tilde{\tilde{\mathcal{R}}})^c\}$, where

$$\tilde{\tilde{\mathcal{Q}}}^c = \{(q, [0.4, 0.8], [0.5, 0.6], [0.4, 0.6]), (r, [0.6, 0.7], [0.5, 0.7], [0.2, 0.5])\}$$

$$\tilde{\tilde{\mathcal{R}}}^c = \{(q, [0.2, 0.7], [0.8, 0.5], [0.3, 0.7]), (r, [0.3, 0.8], [0.9, 0.5], [0.3, 0.4])\}$$

$$(\tilde{\tilde{\mathcal{Q}}} \cup \tilde{\tilde{\mathcal{R}}})^c = \{(q, [0.2, 0.7], [0.8, 0.6], [0.4, 0.7]), (r, [0.3, 0.7], [0.9, 0.7], [0.3, 0.5])\}$$

$$(\tilde{\tilde{\mathcal{Q}}} \cap \tilde{\tilde{\mathcal{R}}})^c = \{(q, [0.4, 0.8], [0.5, 0.5], [0.3, 0.6]), (r, [0.6, 0.8], [0.5, 0.5], [0.2, 0.4])\}$$

$T2Cl(\tilde{\tilde{\mathcal{Q}}}) = 1_{\mathcal{N}}$, $T2Cl(\tilde{\tilde{\mathcal{R}}}) = 1_{\mathcal{N}}$, $T2Cl(\tilde{\tilde{\mathcal{Q}}} \cap \tilde{\tilde{\mathcal{R}}}) = (\tilde{\tilde{\mathcal{Q}}} \cap \tilde{\tilde{\mathcal{R}}})^c$
Hence $T2Cl(\tilde{\tilde{\mathcal{Q}}} \cap \tilde{\tilde{\mathcal{R}}}) \subseteq T2Cl(\tilde{\tilde{\mathcal{Q}}}) \cap T2Cl(\tilde{\tilde{\mathcal{R}}})$.

Theorem 3.15. Let $(\mathcal{G}, \tilde{\tilde{\mathcal{T}}})$ be T2NTS over $\mathcal{G}$ and $\tilde{\tilde{\mathcal{Q}}}, \tilde{\tilde{\mathcal{R}}} \in T2NS$. Then

i. T2Int$(\tilde{\tilde{\mathcal{Q}}}^c) = (T2Cl(\tilde{\tilde{\mathcal{Q}}}))^c$
ii. T2Cl$(\tilde{\tilde{\mathcal{Q}}}^c) = (T2Int(\tilde{\tilde{\mathcal{Q}}}))^c$

Proof: Let $\tilde{\tilde{\mathcal{Q}}}, \tilde{\tilde{\mathcal{R}}} \in T2NS$. Then

i. It is known that $T2Cl(\tilde{\tilde{\mathcal{Q}}}) = \bigcap_{\tilde{\tilde{\mathcal{R}}}^c \in \tilde{\tilde{\mathcal{T}}}, \tilde{\tilde{\mathcal{Q}}} \subseteq \tilde{\tilde{\mathcal{R}}}} \tilde{\tilde{\mathcal{R}}}$, therefore we have that $(T2Cl(\tilde{\tilde{\mathcal{Q}}}))^c = \bigcup_{\tilde{\tilde{\mathcal{R}}}^c \in \tilde{\tilde{\mathcal{T}}}, \tilde{\tilde{\mathcal{R}}}^c \subseteq \tilde{\tilde{\mathcal{Q}}}^c} \tilde{\tilde{\mathcal{R}}}^c$
Right side of above equality is T2Int$(\tilde{\tilde{\mathcal{Q}}}^c)$, thus T2Int$(\tilde{\tilde{\mathcal{Q}}}^c) = (T2Cl(\tilde{\tilde{\mathcal{Q}}}))^c$
ii. Replace $\tilde{\tilde{\mathcal{Q}}}^c$ by $\tilde{\tilde{\mathcal{Q}}}$ in i. then $(T2Cl(\tilde{\tilde{\mathcal{Q}}}^c))^c = T2Int((\tilde{\tilde{\mathcal{Q}}}^c)^c) = T2Int(\tilde{\tilde{\mathcal{Q}}})$. So $T2Cl(\tilde{\tilde{\mathcal{Q}}}^c) = (T2Int(\tilde{\tilde{\mathcal{Q}}}))^c$.

Definition 3.16. Let $(\mathcal{G}, \tilde{\tilde{\mathcal{T}}})$ be T2NTS over $\mathcal{G}$ and $\tilde{\tilde{\mathcal{Q}}}, \in T2NS$. Then the Type2 Neutrosophic exterior of $\tilde{\tilde{\mathcal{Q}}}$ denoted by $T2ext(\tilde{\tilde{\mathcal{Q}}})$ and is defined as $T2ext(\tilde{\tilde{\mathcal{Q}}}) = T2Int(\tilde{\tilde{\mathcal{Q}}}^c)$.

Theorem 3.17. Let $(\mathcal{G}, \tilde{\tilde{\mathcal{T}}})$ be T2NTS over $\mathcal{G}$ and $\tilde{\tilde{\mathcal{Q}}}, \tilde{\tilde{\mathcal{R}}} \in T2NS$. Then

i. T2ext$(\tilde{\tilde{\mathcal{Q}}} \cup \tilde{\tilde{\mathcal{R}}}) = T2ext(\tilde{\tilde{\mathcal{Q}}}) \cap T2ext(\tilde{\tilde{\mathcal{R}}})$
ii. T2ext$(\tilde{\tilde{\mathcal{Q}}}) \cup T2ext(\tilde{\tilde{\mathcal{R}}}) \subseteq T2ext(\tilde{\tilde{\mathcal{Q}}} \cap \tilde{\tilde{\mathcal{R}}})$

Proof: Let $\tilde{\tilde{\mathcal{Q}}}, \tilde{\tilde{\mathcal{R}}} \in T2NS$. Then

i. T2ext$(\tilde{\tilde{\mathcal{Q}}} \cup \tilde{\tilde{\mathcal{R}}}) = T2Int(\tilde{\tilde{\mathcal{Q}}} \cup \tilde{\tilde{\mathcal{R}}})^c$ = T2Int$(\tilde{\tilde{\mathcal{Q}}})^c \cap T2Int(\tilde{\tilde{\mathcal{R}}})^c$ = T2ext$(\tilde{\tilde{\mathcal{Q}}}) \cap T2ext(\tilde{\tilde{\mathcal{R}}})$.
ii. It is similar to i.

Definition 3.18. Let $(\mathcal{G}, \tilde{\tilde{\mathcal{T}}})$ be T2NTS over $\mathcal{G}$ and $\tilde{\tilde{\mathcal{Q}}}, \in T2NS$. Then the Type2 Neutrosophic boundary of $\tilde{\tilde{\mathcal{Q}}}$ denoted by $T2Bou(\tilde{\tilde{\mathcal{Q}}})$ and is defined as

$$T2Bou(\tilde{\tilde{\mathcal{Q}}}) = T2Cl(\tilde{\tilde{\mathcal{Q}}}) \cap T2Cl(\tilde{\tilde{\mathcal{Q}}}^c).$$

Theorem 3.19. Let $(\mathcal{G}, \tilde{\tilde{\mathcal{T}}})$ be T2NTS over $\mathcal{G}$ and $\tilde{\tilde{\mathcal{Q}}}, \in T2NS$. Then

(i) (T2Bou$(\tilde{\tilde{\mathcal{Q}}}))^c = T2ext(\tilde{\tilde{\mathcal{Q}}}) \cup T2Int(\tilde{\tilde{\mathcal{R}}})$
(ii) (T2Cl$(\tilde{\tilde{\mathcal{Q}}}))^c = T2Int(\tilde{\tilde{\mathcal{Q}}}) \cup T2Bou(\tilde{\tilde{\mathcal{R}}})$

Proof: Let $\tilde{\tilde{\mathcal{Q}}} \in T2NS$. Then

(i) By theorem 3.15 (i), we have

$$(T2Bou(\tilde{\tilde{\mathcal{Q}}}))^c = (T2Cl(\tilde{\tilde{\mathcal{Q}}}) \cap T2Bou(\tilde{\tilde{\mathcal{Q}}}^c))^c$$

$$= T2Cl(\tilde{\tilde{\mathcal{Q}}})^c \cup T2Bou(\tilde{\tilde{\mathcal{Q}}}^c)^c$$

$$= T2Cl(\tilde{\tilde{\mathcal{Q}}}^c) \cup (T2Int(\tilde{\tilde{\mathcal{Q}}}^c))^c = T2ext(\tilde{\tilde{\mathcal{Q}}}) \cup T2Int(\tilde{\tilde{\mathcal{Q}}}).$$

(ii) By theorem 3.15 (i), we have
$T2Int(\tilde{\tilde{\mathcal{Q}}}) \cup T2Bou(\tilde{\tilde{\mathcal{Q}}})$
$= T2Int(\tilde{\tilde{\mathcal{Q}}}) \cup (T2Cl(\tilde{\tilde{\mathcal{Q}}}) \cap T2Bou(\tilde{\tilde{\mathcal{Q}}}))$
$= T2Int(\tilde{\tilde{\mathcal{Q}}}) \cup T2Cl(\tilde{\tilde{\mathcal{Q}}}) \cap T2Int(\tilde{\tilde{\mathcal{Q}}}) \cup T2Bou(\tilde{\tilde{\mathcal{Q}}})$
$= T2Cl(\tilde{\tilde{\mathcal{Q}}}) \cap T2Int(\tilde{\tilde{\mathcal{Q}}}) \cup (T2Int(\tilde{\tilde{\mathcal{Q}}}))^c$
$= T2Cl(\tilde{\tilde{\mathcal{Q}}}) \cap 1_{\mathcal{N}} = T2Cl(\tilde{\tilde{\mathcal{Q}}})$.

Theorem 3.20. Let $(\mathcal{G}, \tilde{\tilde{\mathcal{T}}})$ be T2NTS over $\mathcal{G}$ and $\tilde{\tilde{\mathcal{Q}}} \in T2NS$. Then

(i) $\tilde{\tilde{\mathcal{Q}}}$ is T2NOS over $\mathcal{G}$ iff $\tilde{\tilde{\mathcal{Q}}} \cap T2Bou(\tilde{\tilde{\mathcal{Q}}}) = 0_{\mathcal{N}}$.

(ii) $\tilde{\tilde{\mathcal{Q}}}$ is T2NCS over $\mathcal{G}$ iff $T2Bou(\tilde{\tilde{\mathcal{Q}}}) \subseteq \tilde{\tilde{\mathcal{Q}}}$.

Proof: Let $\tilde{\tilde{\mathcal{Q}}} \in T2NS$. Then

(i) Assume that $\tilde{\tilde{\mathcal{Q}}}$ is T2NOS over $\mathcal{G}$, then $T2Int(\tilde{\tilde{\mathcal{Q}}}) = \tilde{\tilde{\mathcal{Q}}}$, then by theorem above

$$\tilde{\tilde{\mathcal{Q}}} \cap T2Bou(\tilde{\tilde{\mathcal{Q}}}) = T2Int(\tilde{\tilde{\mathcal{Q}}}) \cap T2Bou(\tilde{\tilde{\mathcal{Q}}}) = T2Cl(\tilde{\tilde{\mathcal{Q}}}) \cap (T2Int(\tilde{\tilde{\mathcal{Q}}}))^c.$$

So

$$T2Bou(\tilde{\tilde{\mathcal{Q}}}) \cap T2Int(\tilde{\tilde{\mathcal{Q}}}) = T2Cl(\tilde{\tilde{\mathcal{Q}}}) \cap (T2Int(\tilde{\tilde{\mathcal{Q}}}))^c \cap T2Int(\tilde{\tilde{\mathcal{Q}}})$$

$$= T2Cl(\tilde{\tilde{\mathcal{Q}}}) \cap \tilde{\tilde{\mathcal{Q}}}^c \cap \tilde{\tilde{\mathcal{Q}}} = 0_{\mathcal{N}}.$$

Conversely, let $\tilde{\tilde{\mathcal{Q}}} \cap T2Bou(\tilde{\tilde{\mathcal{Q}}}) = 0_{\mathcal{N}}$. Then $\tilde{\tilde{\mathcal{Q}}} \cap T2Cl(\tilde{\tilde{\mathcal{Q}}}) \cap T2Bou(\tilde{\tilde{\mathcal{Q}}}^c) = 0_{\mathcal{N}}$. Or $\tilde{\tilde{\mathcal{Q}}} \cap T2Bou(\tilde{\tilde{\mathcal{Q}}}^c) = 0_{\mathcal{N}}$ or $T2Cl(\tilde{\tilde{\mathcal{Q}}}) \subseteq \tilde{\tilde{\mathcal{Q}}}^c$ which implies $\tilde{\tilde{\mathcal{Q}}}^c$ is a (T2NS) and so $\tilde{\tilde{\mathcal{Q}}}$ is (T2NOS).

(ii) Let $\tilde{\tilde{\mathcal{Q}}}$ be a (T2NCS), then $T2Cl(\tilde{\tilde{\mathcal{Q}}}) = \tilde{\tilde{\mathcal{Q}}}$,
since $T2Bou(\tilde{\tilde{\mathcal{Q}}}) = T2Cl(\tilde{\tilde{\mathcal{Q}}}) \cap T2Bou(\tilde{\tilde{\mathcal{Q}}}^c) \subseteq T2Cl(\tilde{\tilde{\mathcal{Q}}}) = \tilde{\tilde{\mathcal{Q}}}$,
therefore $T2Bou(\tilde{\tilde{\mathcal{Q}}}) \subseteq \tilde{\tilde{\mathcal{Q}}}$.
Conversely, $T2Bou(\tilde{\tilde{\mathcal{Q}}}) \subseteq \tilde{\tilde{\mathcal{Q}}}$, then $T2Bou(\tilde{\tilde{\mathcal{Q}}}) \cap \tilde{\tilde{\mathcal{Q}}}^c = 0_{\mathcal{N}}$,
since $T2Bou(\tilde{\tilde{\mathcal{Q}}}) = T2Bou(\tilde{\tilde{\mathcal{Q}}}) \cap \tilde{\tilde{\mathcal{Q}}} \cup \tilde{\tilde{\mathcal{Q}}}^c$, so $T2Bou(\tilde{\tilde{\mathcal{Q}}}) \cap \tilde{\tilde{\mathcal{Q}}}^c = 0_{\mathcal{N}}$
by (i). $\tilde{\tilde{\mathcal{Q}}}$ is a (T2NOS) and so $\tilde{\tilde{\mathcal{Q}}}$ is a (T2NCS).

Theorem 3.21. Let $(\mathcal{G}, \tilde{\tilde{\mathcal{T}}})$ be T2NTS over $\mathcal{G}$ and $\tilde{\tilde{\mathcal{Q}}} \in T2NS$. Then

(i) $T2Bou(\tilde{\tilde{\mathcal{Q}}}) \cap T2Int(\tilde{\tilde{\mathcal{Q}}}) = 0_N$.

(ii) $T2Bou(T2Int(\tilde{\tilde{\mathcal{Q}}})) \subseteq T2Bou(\tilde{\tilde{\mathcal{Q}}})$.

Proof: Let $\tilde{\tilde{\mathcal{Q}}} \in T2NS$. Then

(i) From theorem above it is clear.

(ii) $T2Bou(T2Int(\tilde{\tilde{\mathcal{Q}}})) = T2Cl(T2Int(\tilde{\tilde{\mathcal{Q}}})) \cap T2Cl(T2Int(\tilde{\tilde{\mathcal{Q}}}))^c$
$= T2Cl(T2Int(\tilde{\tilde{\mathcal{Q}}})) \cap T2Bou(\tilde{\tilde{\mathcal{Q}}}) \subseteq T2Cl(\tilde{\tilde{\mathcal{Q}}}) \cap T2Bou(\tilde{\tilde{\mathcal{Q}}}) = T2Bou(\tilde{\tilde{\mathcal{Q}}})$.

Definition 3.22. Let $(\mathcal{G}, \tilde{\tilde{\mathcal{T}}})$ be T2NTS and $\mathcal{H}$ be (T2NS) of $\mathcal{G}$, then the Type2 neutrosophic relative topology (T2NRT) on $\mathcal{H}$ is defined by

$$\tilde{\tilde{\mathcal{T}}}_{\mathcal{H}} = \{\tilde{\tilde{\mathcal{Q}}} \cap \mathcal{H} : \tilde{\tilde{\mathcal{Q}}} \in \tilde{\tilde{\mathcal{T}}}\}$$

where

$$\mathcal{H}(h) = \begin{cases} (\{1,0,0\},[0,0,0],[0,0,0]) & h \in \mathcal{H} \\ ([0,0,0],[1,1,1],[1,1,1]) & \text{otherwise} \end{cases}$$

Thus $(\mathcal{G}, \tilde{\tilde{\mathcal{T}}}_H)$ is called a Type2 neutrosophic subspace (T2NSS) of $(\mathcal{G}, \tilde{\tilde{\mathcal{T}}})$.

Example 3.23. Let $\mathcal{G} = \{q, r, s\},\ \mathcal{H} = \{q, r\} \subseteq \mathcal{G}$ and $\tilde{\tilde{\mathcal{Q}}}, \tilde{\tilde{\mathcal{R}}} \in (T2NS)$, such that

$$\tilde{\tilde{\mathcal{Q}}} = \{(q,[0.4,0.6],[0.2,0.4],[0.4,0.8]),\ (r,[0.2,0.5],[0.1,0.3],[0.6,0.7]),\ (s,[0.2,0.5],[0.2,0.3],[0.6,0.7])\}$$

$$\tilde{\tilde{\mathcal{R}}} = \{(q,[0.3,0.7],[0.2,0.5],[0.2,0.7]),\ (r,[0.3,0.4],[0.2,0.5],[0.3,0.8]),\ (s,[0.4,0.5],[0.3,0.6],[0.6,0.7])\}$$

Then $\tilde{\tilde{\mathcal{T}}} = \{0_{\mathcal{N}}, 1_{\mathcal{N}}, \tilde{\tilde{\mathcal{Q}}}, \tilde{\tilde{\mathcal{R}}}, \tilde{\tilde{\mathcal{Q}}} \cap \tilde{\tilde{\mathcal{R}}}, \tilde{\tilde{\mathcal{Q}}} \cup \tilde{\tilde{\mathcal{R}}}\}$ is a (T2NTS) on $\mathcal{G}$, therefore
$\tilde{\tilde{\mathcal{T}}}_{\mathcal{H}} = \{0_{\mathcal{N}}, 1_{\mathcal{N}}, \tilde{\tilde{\mathcal{O}}}, \tilde{\tilde{\mathcal{L}}}, \tilde{\tilde{\mathcal{K}}}, \tilde{\tilde{\mathcal{M}}}\}$ is (T2NRT) on $\mathcal{H}$, such that
$\tilde{\tilde{\mathcal{O}}} = \mathcal{H} \cap \tilde{\tilde{\mathcal{Q}}}, \quad \tilde{\tilde{\mathcal{L}}} = \mathcal{H} \cap \tilde{\tilde{\mathcal{R}}}, \quad \tilde{\tilde{\mathcal{K}}} = \mathcal{H} \cap (\tilde{\tilde{\mathcal{Q}}} \cap \tilde{\tilde{\mathcal{R}}}), \quad \tilde{\tilde{\mathcal{M}}} = \mathcal{H} \cap (\tilde{\tilde{\mathcal{Q}}} \cup \tilde{\tilde{\mathcal{R}}})$.

4. Conclusions

This study establishes a solid theoretical foundation for Type2 Neutrosophic Topological Spaces (T2NTS), offering a framework that is both mathematically rigorous and conceptually rich. The findings demonstrate that T2NTS is not merely an extension of classical topology, but rather a more powerful and flexible analytical tool for representing systems characterized by high levels of uncertainty, inconsistency, and incomplete information.

By introducing and analyzing the fundamental structures of interior, closure, exterior, and boundary in the Type2 neutrosophic context, the research highlights their strong interrelations and mathematical robustness. More importantly, the study shows the potential of T2NTS to go beyond theoretical advancements, serving as a practical foundation for real-world applications in areas such as medical diagnosis, decision-support systems, and uncertain data analysis.

In conclusion, this work positions Type2 neutrosophic topology as a promising frontier in modern mathematics, opening new avenues for interdisciplinary research and encouraging further exploration of its applications in artificial intelligence, computational sciences, and applied decision-making.

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