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GAMMA MIXTURE OF CHI-SQUARE DISTRIBUTION: PROPERTIES, ESTIMATION, SIMULATION, AND APPLICATION

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Abstract. This study introduces the Gamma Mixture of Chi-Square (GMCS) distribution as a novel probabilistic model for representing leptokurtic and right-skewed data. The rth-raw moment was derived along with analytical expressions for the mean, variance, skewness, and excess kurtosis, including its special cases, the Erlang and Exponential Mixture of Chi-Square distributions. Parameters were estimated using the method of moments, and a Monte Carlo simulation with 10,000 samples confirmed the accuracy and consistency of the estimators, yielding minimal RMSE values (0.0011 to 0.0021) and close alignment between theoretical and empirical moments. The GMCS distribution was further applied to model precipitation data in the Caraga Region, Philippines, where it demonstrated superior goodness of fit (RMSE = 0.006033) compared to the classical Gamma model, effectively capturing the data's heavy-tailed and asymmetric structure. These results highlight the GMCS model's potential as a robust and flexible tool for modeling complex environmental and stochastic phenomena.

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Key words and phrases. mixture distributions; chi-square distribution; gamma distribution; exponential distribution; Erlang distribution; method of moments.

1. Introduction

Probability distributions are fundamental to quantitative science, providing the mathematical framework for modeling stochastic phenomena, analyzing data, and forming statistical inference in virtually all empirical disciplines [3]. Within this landscape, mixture distributions have emerged as an exceptionally powerful and flexible tool, particularly when data are drawn from a heterogeneous population where a single standard distribution does not provide an adequate description [11]. By forming a weighted sum of two or more probability density functions, mixture models can capture complex patterns such as multimodality and heavy tails. Although the individual properties of the Chi-Square,

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Gamma, and Exponential distributions, foundations of hypothesis testing, reliability analysis, and survival studies, are well-established, their integration into more flexible mixture frameworks remains a compelling direction for theoretical development. However, a comprehensive exploration of their capabilities within such a framework is limited in the existing literature. Although preliminary conceptual frameworks, such as those discussed by [10], have provided inspiration to modify existing models to better capture complex phenomena, a systematic investigation of the statistical properties and relationships of mixtures based on Chi-Squares has not been fully developed [2,4,6,12].

This study aims to address this gap by providing a formal and systematic development of Chi-Square mixture distributions. The primary contribution of this work is to define and introduce the Gamma mixture of the Chi-Square distribution and the Exponential mixture of the Chi-Square distribution. To characterize these new models, their essential distributional properties, including raw moments, mean, variance, skewness, and kurtosis, are rigorously derived. This analysis further demonstrates an elegant hierarchical structure, establishing that the Exponential mixture of the Chi-Square distributions and the Erlang mixture of the Chi-Square distributions are nested special cases of the more general Gamma mixture of the Chi-Square distribution. To ensure practical utility, a straightforward estimation procedure for the parameters of the Gamma mixture of the Chi-Square distribution is established using the method of moments.

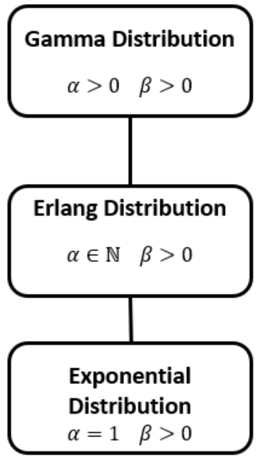
The significance of this research lies in its contributions to both theoretical and applied statistics. By introducing and formalizing a new class of Chi-Square mixtures, this work expands the family of known probability distributions, thereby offering researchers enhanced modeling capabilities. This expanded framework is anticipated to provide a more robust and flexible alternative for modeling real-world data, particularly in contexts where Gamma-based models have been traditionally applied. The development of such refined statistical tools can lead to more accurate inference and improved decision-making in various scientific and engineering domains. This new distribution may therefore serve as a powerful and effective tool for researchers, enabling more accurate and nuanced modeling in various fields of applied science.

2. Preliminaries

2.1. **Mixture and Related Distributions.** In the paper of [10], mixtures occur mainly when the parameter $\theta \in \Theta$ of a family of distributions, given by the density function $f(x;\theta)$, is itself subject to the variation of the change. The general formula for the finite mixture is given by

$$\sum_{k=1}^{p} f(x;\theta_k)g(\theta_k) \tag{1}$$

FIGURE 1. *Relationships Between Gamma, Erlang and Exponential Distributions*



and the infinite analog when Θ is a continuous random variable is given by

$$\int_{\Theta} f(x;\theta)g(\theta)d\theta, \tag{2}$$

where $\mathbb{P}(\Theta = \theta) = g(\theta)$ is the weight function.

We also present here the definitions of the distributions involved in this paper. The PDF of Gamma, Erlang, and Exponential distributions is defined as

$$f_1(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x},$$

$$f_2(x) = \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{(\alpha-1)!},$$

and

$$f_3(y) = \frac{1}{\beta} e^{-\frac{x}{\beta}},$$

respectively. Figure 1 below shows the relationship between the three distributions [7]. We will utilize the information in Figure 1 to generate more results in this paper.

2.2. **Important Formula.** One of the important formulas this paper used is the Pochhammer symbol [13], defined as the increasing factorial. For any $n \in \mathbb{N}$ and $x > 0$,

$$(x)_n = x(x+1)(x+2)\cdots(x+n).$$

Accordingly, we can write the Pochhammer symbol in terms of the Gamma function as

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}. \tag{3}$$

This paper also utilizes the following formula taken from [13].

Lemma 1. For $\mu > 0$ and $\nu > 0$,

$$\int_0^\infty x^{\nu-1} e^{-\mu x} dx = \frac{1}{\mu^\nu}\Gamma(\nu).$$

3. Theoretical Results

This section presents the results in the form of definitions and theorems. Here, we define the Gamma mixture of the chi-square distribution and the Exponential mixture of the chi-square distribution and investigate the properties of each distribution through the methods of moments. Moreover, we show that the Exponential mixture of Chi-Square distribution and Erlang mixture of Chi-Square distribution are special cases of the Gamma mixture of Chi-Square distribution.

3.1. Mixture Distribution and its Properties.

3.1.1. *Gamma Mixture of Chi-Square Distribution.*

Definition 1. A random variable X is said to have a ***Gamma mixture of distribution*** if its density function is given by

$$f_X(x;\alpha,\beta) = \begin{cases} \displaystyle\int_0^\infty \frac{\beta^\alpha e^{-\theta\beta}\theta^{\alpha-1}}{\Gamma(\alpha)} g(x;\theta)d\theta, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

where $\alpha, \beta > 0$ and $g(x;\theta)$ is a density function.

If the density function $g(x;\theta)$ in Definition 1 is the chi-square density function, then we have the following definitions.

Definition 2. A random variable X is said to have a ***Gamma mixture of Chi-Square distribution*** with v degrees of freedom and parameters α and β if its density function is defined as

$$f_X(x;\alpha,\beta,v) = \begin{cases} \displaystyle\int_0^\infty \frac{\beta^\alpha e^{-\theta\beta}\theta^{\alpha-1}}{\Gamma(\alpha)} \frac{e^{\frac{-x}{2}}(x)^{\frac{v}{2}+\theta-1}}{2^{\frac{v}{2}+\theta}\Gamma(\frac{v}{2}+\theta)} d\theta, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

where $v, \alpha, \beta > 0$.

We obtain the properties of the Gamma mixture of chi-square distribution using the raw moments.

Theorem 1. If X follows the Gamma mixture of chi-square distribution with v degrees of freedom and parameters $\alpha, \beta > 0$, then the rth raw moment about origin is given by

$$\mu_r^{'} = \frac{2^r\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{-\theta\beta}\theta^{\alpha-1}\left(\frac{v}{2}+\theta\right)_r d\theta,$$

with the mean

$$\mu = v + \frac{2\alpha}{\beta},$$

variance

$$\sigma^2 = 2\left(\mu + \frac{2\alpha}{\beta^2}\right),$$

skewness

$$\gamma_1 = \frac{4\left(\mu + \frac{\alpha}{\beta^2}\left[\frac{1}{\beta}+3\right]\right)}{\sqrt{2\left[\mu+\frac{2\alpha}{\beta^2}\right]^3}},$$

and excess kurtosis

$$\gamma_2 = \frac{12\left[\mu + \frac{\alpha}{\beta^4}\left(\frac{11}{3}\beta^2 + 2\beta + \alpha + 1\right)\right]}{\left(\mu + \frac{2\alpha}{\beta^2}\right)^2}.$$

Moreover, the density of Gamma mixture of chi-square distribution is skewed to the right and the shape is leptokurtic.

Proof. We start with the definition of the rth raw moment.

$$\begin{aligned}
\mu_r' &= E(X^r) \\
&= \int_{-\infty}^{\infty} x^r f_X(x;\alpha,\beta,v)dx \\
&= \int_0^{\infty} x^r \int_0^{\infty} \frac{\beta^\alpha e^{-\beta\theta}\theta^{\alpha-1}e^{-\frac{x}{2}}(x)^{\frac{v}{2}+\theta-1}}{2^{\frac{v}{2}+\theta}\Gamma(\alpha)\Gamma\left(\frac{v}{2}+\theta\right)}d\theta dx \\
&= \int_0^{\infty} \frac{\beta^\alpha e^{-\beta\theta}\theta^{\alpha-1}}{2^{\frac{v}{2}+\theta}\Gamma(\alpha)\Gamma\left(\frac{v}{2}+\theta\right)}d\theta \int_0^{\infty} x^{\frac{v}{2}+\theta+r-1}e^{-\frac{1}{2}x}dx \\
&= \int_0^{\infty} \frac{\beta^\alpha e^{-\beta\theta}\theta^{\alpha-1}}{2^{\frac{v}{2}+\theta}\Gamma(\alpha)\Gamma\left(\frac{v}{2}+\theta\right)}\left(\frac{1}{2}\right)^{-\left(\frac{v}{2}+\theta+r\right)}\Gamma\left(\frac{v}{2}+\theta+r\right)d\theta, \quad \text{By Lemma 1} \\
&= \frac{2^r\beta^\alpha}{\Gamma(\alpha)}\int_0^{\infty} e^{-\theta\beta}\theta^{\alpha-1}\left(\frac{v}{2}+\theta\right)_r d\theta, \quad \text{by Equation (3).}
\end{aligned}$$

Hence, the first, second, third and fourth raw moments are

$$\mu_1' = v + \frac{2\alpha}{\beta}, \tag{4}$$

$$\mu_2' = 2^2\left[\frac{v^2}{4} + \frac{1}{\beta^2}(\alpha^2+\alpha) + \frac{v}{\beta}\alpha + \frac{v}{2} + \frac{1}{\beta}\alpha\right], \tag{5}$$

$$\mu_3' = 8\left[\frac{\alpha(\alpha+1)(\alpha+2)}{\beta^3} + 3\left(\frac{v}{2}+1\right)\frac{\alpha(\alpha+1)}{\beta^2} + \left(3\left(\frac{v}{2}\right)^2 + 3v + 2\right)\frac{\alpha}{\beta} + \left(\frac{v}{2}\right)\left(\frac{v}{2}+1\right)\left(\frac{v}{2}+2\right)\right], \tag{6}$$

and

$$\begin{aligned}
\mu_4' = 2^4\Bigg[&\frac{1}{16}v^4 + \frac{3}{4}v^3 + \frac{11}{4}v^2 + 3v + \frac{1}{\beta^4}(\alpha^4 + 6\alpha^3 + 10\alpha^2 + 3\alpha) \\
&+ \left(\frac{6+2v}{\beta^3}\right)(\alpha^3 + 3\alpha^2 + \alpha) + \left(\frac{22 + 3v^2 + 18v}{2\beta^2}\right)(\alpha^2+\alpha) + \frac{\alpha}{\beta}\left(6 + \frac{v^3}{2} + \frac{9v^2}{2} + 11v\right)\Bigg],
\end{aligned} \tag{7}$$

respectively. It follows that the second central moment is

$$\begin{aligned}\mu_2 &= \mu_2' - \mu_1'^2 \\ &= 4\left[\frac{v^2}{4} + \frac{1}{\beta^2}(\alpha^2+\alpha) + \frac{v}{\beta}\alpha + \frac{v}{2} + \frac{1}{\beta}\alpha\right] - \left(2\left[\frac{v}{2} + \frac{\alpha}{\beta}\right]\right)^2 \\ &= 2\left(\mu + \frac{2\alpha}{\beta^2}\right)\end{aligned}$$

Also, the third central moment is

$$\begin{aligned}\mu_3 &= \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 \\ &= 8\left[\frac{1}{8}v^3 + \frac{3}{4}v^2 + v + \frac{1}{\beta^3}(\alpha^3 + 3\alpha^2 + \alpha) + \frac{6+3v}{2\beta^2}(\alpha^2+\alpha) + \frac{\alpha}{\beta}\left(2 + \frac{3v^2}{4} + 3v\right)\right] \\ &\quad - 3\left(4\left[\left(\frac{v^2}{4}\right) + \frac{1}{\beta^2}(\alpha^2+\alpha) + \frac{v}{\beta}\alpha + \frac{v}{2} + \frac{1}{\beta}\alpha\right]\cdot 2\left[\frac{v}{2} + \frac{\alpha}{\beta}\right]\right) + 2\left(2\left[\frac{v}{2} + \frac{\alpha}{\beta}\right]\right)^3\end{aligned}$$

The fourth central moment is obtained as follows.

$$\begin{aligned}\mu_4 &= \mu_4' - 4\mu_1'\mu_3' + 6\mu_1'^2\mu_2' - 3\mu_1'^4 \\ &= 16\left[\frac{3}{4}v^2 + 3v + \frac{3v\alpha}{\beta} + \frac{3v\alpha}{\beta^2} + \frac{6\alpha^2}{\beta^4} + \frac{6\alpha^2}{\beta^3} + \frac{3\alpha}{\beta} + \frac{6\alpha}{\beta^3} + \frac{6\alpha}{\beta} + \frac{3\alpha^2}{\beta^2} + \frac{11\alpha}{\beta^2}\right].\end{aligned}$$

Solving for the mean, we have

$$\mu = \mu_1' = v + \frac{2\alpha}{\beta},$$

while the variance is

$$\sigma^2 = \mu_2 = 2\left[\mu + \frac{2\alpha}{\beta^2}\right].$$

The skewness of the distribution is

$$\gamma_1 = \frac{\mu_3}{\sqrt{\mu_2^3}} = \frac{4\left(\mu + \frac{\alpha}{\beta^2}\left[\frac{1}{\beta} + 3\right]\right)}{\sqrt{2\left[\mu + \frac{2\alpha}{\beta^2}\right]^3}} \geq 0, \quad \text{for positive } v, \alpha \text{ and } \beta.$$

The result shows that the Gamma mixture of Chi-square distribution is skewed to the right. Lastly, the excess kurtosis was obtained as follows:

$$\gamma_2 = \frac{\mu_4}{\mu_2^2} - 3 = \frac{12\left[\mu + \frac{\alpha}{\beta^4}\left(\frac{11}{3}\beta^2 + 2\beta + \alpha + 1\right)\right]}{\left(\mu + \frac{2\alpha}{\beta^2}\right)^2} \geq 0, \quad \text{, for positive } v, \alpha \text{ and } \beta.$$

Therefore, the Gamma mixture of Chi-square distribution is leptokurtic. □

3.1.2. *Erlang Mixture of Chi-Square Distribution.* Erlang distribution is a special case of Gamma distribution with shape parameter being discretised. We now define below the Erlang mixture of distribution with shape parameter $\alpha \in \mathbb{N}$ and rate parameter $\beta > 0$.

Definition 3. A random variable X is said to have an Erlang mixture of distribution with shape parameter $\alpha \in \mathbb{N}$ and rate parameter $\beta > 0$ is given by

$$f_X(x;\alpha,\beta)=\begin{cases}\displaystyle\int_0^\infty \frac{\beta^\alpha\theta^{\alpha-1}e^{-\beta\theta}}{(\alpha-1)!}g(x;\theta)d\theta & \text{if } x\geq 0\\ 0 & \text{if } x<0,\end{cases}$$

where $g(x;\theta)$ is a density function.

We then define the Erlang mixture of Chi-Square distribution by defining $g(x;\theta)$ as the probability density function of Chi-Square Distribution.

Definition 4. A random variable X is said to have an Erlang mixture of Chi-Square distribution with v degrees of freedom, shape parameter $\alpha\in\mathbb{N}$ and rate parameter $\beta>0$ is defined as

$$f_X(x;\alpha,\beta,v)=\begin{cases}\displaystyle\int_0^\infty \frac{\beta^\alpha\theta^{\alpha-1}e^{-\beta\theta}}{(\alpha-1)!}\frac{e^{\frac{-x}{2}}(x)^{\frac{v}{2}+\theta-1}}{2^{\frac{v}{2}+\theta}\Gamma(\frac{v}{2}+\theta)}d\theta & \text{if } x\geq 0\\ 0 & \text{if } x<0.\end{cases}$$

The following corollary will help us easily obtain the properties of the Erlang mixture of Chi-Square distribution in terms of its moments.

Corollary 1. If X follows Erlang mixture of Chi-Square distribution with v degrees of freedom, shape parameter $\alpha\in\mathbb{N}$ and rate parameter $\beta>0$, then the rth raw moment about the origin is given by

$$\mu_r^{'}=\frac{2^r\beta^\alpha}{(\alpha-1)!}\int_0^\infty e^{-\theta\beta}\theta^{\alpha-1}\left(\frac{v}{2}+\theta\right)_r d\theta,$$

with the mean

$$\mu=v+\frac{2\alpha}{\beta},$$

variance

$$\sigma^2=2\left(\mu+\frac{2\alpha}{\beta^2}\right),$$

skewness

$$\gamma_1=\frac{4\left(\mu+\frac{\alpha}{\beta^2}\left[\frac{1}{\beta}+3\right]\right)}{\sqrt{2\left[\mu+\frac{2\alpha}{\beta^2}\right]^3}},$$

and excess kurtosis

$$\gamma_2=\frac{12\left[\mu+\frac{\alpha}{\beta^4}\left(\frac{11}{3}\beta^2+2\beta+\alpha+1\right)\right]}{\left(\mu+\frac{2\alpha}{\beta^2}\right)^2}.$$

Moreover, the density of the Erlang mixture of Chi-Square distribution is skewed to the right, and the shape is leptokurtic.

Proof. The proof of Corollary 1 can be shown by discretizing the shape parameter α of the Gamma mixture of Chi-Square distribution rth raw moments. □

3.1.3. *Exponential Mixture of Chi-Square Distribution.*

Definition 5. A random variable X is said to have an ***Exponential mixture of distribution*** with parameter β if its density function is given by

$$f_X(x;\beta) = \begin{cases} \int_0^\infty \beta e^{-\beta\theta} g(x;\theta) d\theta, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

where $g(x;\theta,)$ is a density function.

If the density function $g(x;\theta)$ in Definition 5 is the Chi-Square density function, then we have the following definition.

Definition 6. A random variable X is said to have an ***Exponential mixture of Chi-Square distribution*** with v degrees of freedom and parameter β if its density function is defined as;

$$f_X(x;\beta,v) = \begin{cases} \int_0^\infty \beta e^{-\beta\theta} \frac{e^{\frac{-x}{2}}(x)^{\frac{v}{2}+\theta-1}}{2^{\frac{v}{2}+\theta}\Gamma(\frac{v}{2}+\theta)} d\theta, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

where $v, \beta > 0$.

Various moments of the distribution are given in the next Corollary with properties of the Exponential mixture of the Chi-Square distribution.

Corollary 2. If X follows the Exponential mixture of chi-square distribution with v degrees of freedom and parameter β, then the rth raw moment about origin is given by

$$\mu_r' = 2^r \int_0^\infty \left[\frac{1}{\beta} e^{-\frac{\theta}{\beta}} \left(\frac{v}{2}+\theta+r-1\right)\left(\frac{v}{2}+\theta+r-2\right)\ldots\left(\frac{v}{2}+\theta\right)\right] d\theta$$

with the mean

$$\mu = v + \frac{2}{\beta},$$

variance

$$\sigma^2 = 2\left(\mu + \frac{2}{\beta^2}\right),$$

skewness

$$\gamma_1 = \frac{4\left(\mu + \frac{1}{\beta^2}\left[\frac{1}{\beta}+3\right]\right)}{\sqrt{2\left[\mu+\frac{2}{\beta^2}\right]^3}},$$

and kurtosis

$$\gamma_2 = \frac{12\left[\mu + \frac{1}{\beta^4}\left(\frac{11}{3}\beta^2 + 2\beta + 2\right)\right]}{\left(\mu + \frac{2}{\beta^2}\right)^2}.$$

Moreover, the density of Exponential mixture of chi-square distribution is skewed to the right and the shape is leptokurtic.

Proof. The proof follows from Corollary 1 by setting $\alpha = 1$. □

3.2. **Parameter Estimation: Method of Moments.** In this section, we will estimate the parameters α β and v of the Gamma mixture of Chi-Square distribution. We utilize the method of moments (MoM) (Wackerly et al. 2014) to estimate the parameters. The idea in this paper is to equate the sample and theoretical rth raw moments, and solve the system of equations in terms of α, β, and v.

Let $x_1, x_2, \dots, x_n$ be a random sample. The first, second, and third rth raw moments are

$$m_1' = \frac{1}{n}\sum_{i=1}^{n} x_i,$$

$$m_2' = \frac{1}{n}\sum_{i=1}^{n} x_i^2,$$

and

$$m_3' = \frac{1}{n}\sum_{i=1}^{n} x_i^3,$$

respectively. Next, we equate the theoretical and empirical rth raw moments and solve the following equations $\mu_1' = m_1', \mu_2' = m_2'$, and $\mu_3' = m_3'$ for $\hat{\alpha}, \hat{\beta}$, and $\hat{v}$. Now,

$$m_1' = v + \frac{2\hat{\alpha}}{\hat{\beta}}$$

$$\hat{v} = m_1' - \frac{2\hat{\alpha}}{\hat{\beta}}.$$

Also,

$$m_2' = 4\Big[\frac{\hat{v}^2}{4} + \frac{1}{\hat{\beta}^2}(\hat{\alpha}^2 + \hat{\alpha}) + \frac{\hat{v}}{\hat{\beta}}\hat{\alpha} + \frac{\hat{v}}{2} + \frac{1}{\hat{\beta}}\hat{\alpha}\Big]$$

$$m_2' = 4\left[\frac{\hat{v}^2}{4} + \frac{\hat{v}\hat{\alpha}}{\hat{\beta}} + \frac{\hat{\alpha}^2}{\hat{\beta}^2} + \frac{1}{2}\left(\hat{v} + \frac{2\hat{\alpha}}{\hat{\beta}}\right) + \frac{\hat{\alpha}}{\hat{\beta}^2}\right]$$

$$m_2' = 4\left[\left(\frac{\hat{v}}{2} + \frac{\hat{\alpha}}{\hat{\beta}}\right)^2 + \frac{1}{2}m_1' + \frac{\hat{\alpha}}{\hat{\beta}^2}\right]$$

$$m_2' = 4\left[\frac{1}{4}\left(\hat{v} + \frac{2\hat{\alpha}}{\hat{\beta}}\right)^2 + \frac{1}{2}m_1' + \frac{\hat{\alpha}}{\hat{\beta}^2}\right]$$

$$m_2' = m_1'^2 + 2m_1' + \frac{4\hat{\alpha}}{\hat{\beta}^2}.$$

Solving for $\hat{\alpha}$, we have

$$\hat{\alpha} = \frac{(m_2' - m_1'^2 - 2m_1')\hat{\beta}^2}{4}. \tag{8}$$

This also implies that

$$v = m_1' - \frac{(m_2' - m_1'^2 - 2m_1')\hat{\beta}}{2}. \tag{9}$$

On the other hand,

$$m_3' = 8\left[\frac{\hat{\alpha}(\hat{\alpha}+1)(\hat{\alpha}+2)}{\hat{\beta}^3} + 3\left(\frac{\hat{v}}{2}+1\right)\frac{\hat{\alpha}(\hat{\alpha}+1)}{\hat{\beta}^2} + \left(3\left(\frac{\hat{v}}{2}\right)^2 + 3\hat{v} + 2\right)\frac{\hat{\alpha}}{\hat{\beta}} + \left(\frac{\hat{v}}{2}\right)\left(\frac{\hat{v}}{2}+1\right)\left(\frac{\hat{v}}{2}+2\right)\right]$$

Substitute Equations (8) and (9) to obtain the following equalities

$$m_3' = -4m_1' - \frac{8m_1'}{\hat{\beta}} - \frac{4m_1'^2}{\hat{\beta}} + \frac{4m_2'}{\hat{\beta}} - 6m_1'^2 + 6m_2' + 3m_1'm_2' - 2m_1'^3.$$

We solve for β and calculated the following

$$\hat{\beta} = -\frac{4\left(m_1'^2 + 2m_1' - m_2'\right)}{2m_1'^3 + 6m_1'^2 - 3m_1'm_2' + 4m_1' - 6m_2' + m_3'}. \tag{10}$$

Substitute Equation (10) to Equations (8) and (9) to obtain the estimate of α and v. Therefore,

$$\hat{\alpha} = -\frac{4\left(m_1^2 + 2m_1 - m_2\right)^3}{\left(2m_1^3 + 6m_1^2 - 3m_1m_2 + 4m_1 - 6m_2 + m_3\right)^2} \tag{11}$$

and

$$\hat{v} = m_1' - \frac{2\left(m_1^2 + 2m_1 - m_2\right)^2}{2m_1^3 + 6m_1^2 - 3m_1m_2 + 4m_1 - 6m_2 + m_3}. \tag{12}$$

4. Simulation and Actual Data Fitting

Simulation Design and Methodology. To evaluate the performance of the proposed method of moments estimators for the gamma–chi-square mixture distribution, a comprehensive Monte Carlo simulation study was conducted. Synthetic datasets were generated based on the theoretical structure of the mixture. Specifically, for each random observation, a value was first drawn from a gamma distribution with predefined shape and rate parameters, denoted as α and β, respectively. The resulting θ_i value served as the mixing variable and was subsequently employed as a scale parameter for sampling from a chi-square distribution with degrees of freedom $v_i = v + 2\theta_i$. The true parameter values used for the simulation were fixed at $\alpha = 2.0$, $\beta = 1.0$, and $v = 5.0$. The simulated samples, consisting of $n = 10{,}000$ observations, were analyzed to compute the first three raw sample moments, which were then substituted into the moment-based estimation formulas. The theoretical and estimated probability density functions were compared graphically with the empirical distribution of the simulated data. To assess estimation accuracy, the root mean squared error (RMSE) between the fitted and theoretical density functions was calculated. This simulation procedure was designed to assess the consistency, bias, and precision of the derived moment-based estimators for finite samples in the context of the gamma mixture of chi-square distribution.

Table 1. *Summary of Parameter Estimates and Fit Metrics for Gamma Mixture of Chi-Square Simulation*

Statistic / Parameter	True Value	Estimated Value
Mean	9.0090	9.0089
Standard Deviation	5.0256	5.0252
Skewness	1.0482	0.9790
α	2.0000	4.9401
β	1.0000	1.6526
v	5.0000	3.0303
RMSE (Fitted PDF vs. Theoretical PDF)	–	0.0011
RMSE (Fitted PDF vs. Histogram)	–	0.0021
RMSE (Theoretical PDF vs. Histogram)	–	0.0019

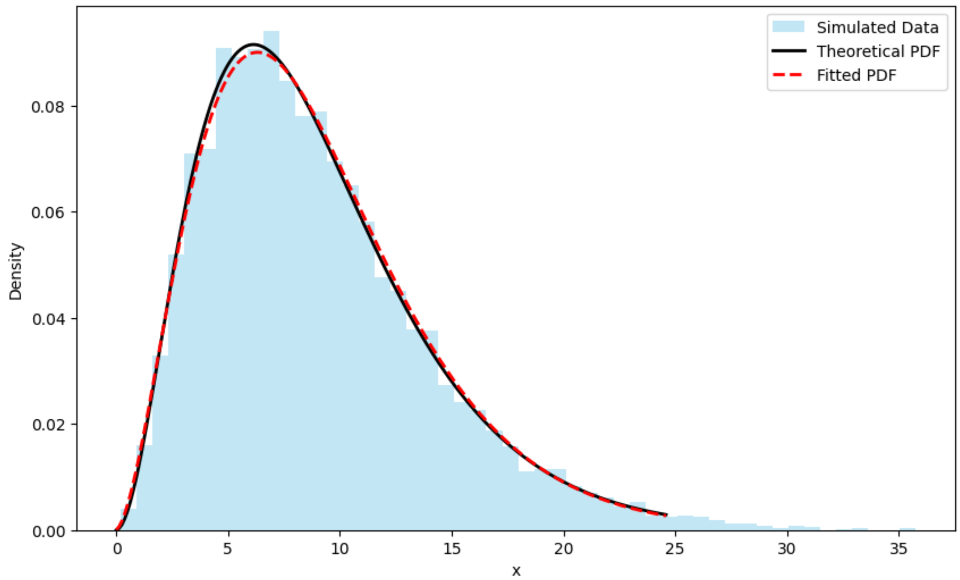
Table 1 presents the summary of estimated parameters and goodness-of-fit metrics for the Gamma Mixture of Chi-Square (GMCS) distribution obtained from the simulation experiment. The empirical descriptive statistics revealed a mean of 9.0090, a standard deviation of 5.0256, and a skewness value of 1.0482, which closely approximate the theoretical distributional characteristics. These results suggest that the simulated data accurately represent the probabilistic behavior expected from the GMCS model.

The parameter estimates derived through the moment-based estimation method exhibited reasonable proximity to their true parameter values. Specifically, the estimated shape parameter $(\alpha = 4.9401)$ and rate parameter $(\beta = 1.6526)$ displayed moderate deviations from their corresponding true values $(\alpha = 2.0000, \beta = 1.0000)$, whereas the estimated mixture parameter $(v = 3.0303)$ remained within an acceptable range of the true value $(v = 5.0000)$. These variations may be attributed to the inherent sensitivity of moment-based methods to higher-order moment fluctuations, especially in distributions with heavy tails or substantial skewness [1].

Furthermore, the Root Mean Square Error (RMSE) values indicate a high degree of model fit and numerical stability. The RMSE between the fitted and theoretical probability density functions (0.0011) was notably small, suggesting that the estimated model provides an excellent approximation of the true underlying distribution. Similarly, the RMSE values between the fitted and empirical histogram (0.0021) and between the true PDF and histogram (0.0019) were minimal, reinforcing the robustness of the simulation process and the precision of the moment-based estimator.

Overall, these findings demonstrate that the moment-based estimation approach is capable of recovering the true distributional parameters of the Gamma Mixture of Chi-Square model with satisfactory accuracy. The low error metrics and the close alignment between the empirical and theoretical distributions, visually evident in Figure 2 through the near overlap of the fitted and theoretical density curves, further support the model's validity and its potential applicability in probabilistic modeling of mixture-type stochastic phenomena.

FIGURE 2. *Gamma Mixture of Chi-Square Distribution Parameter Estimation Using Moment-based Method*



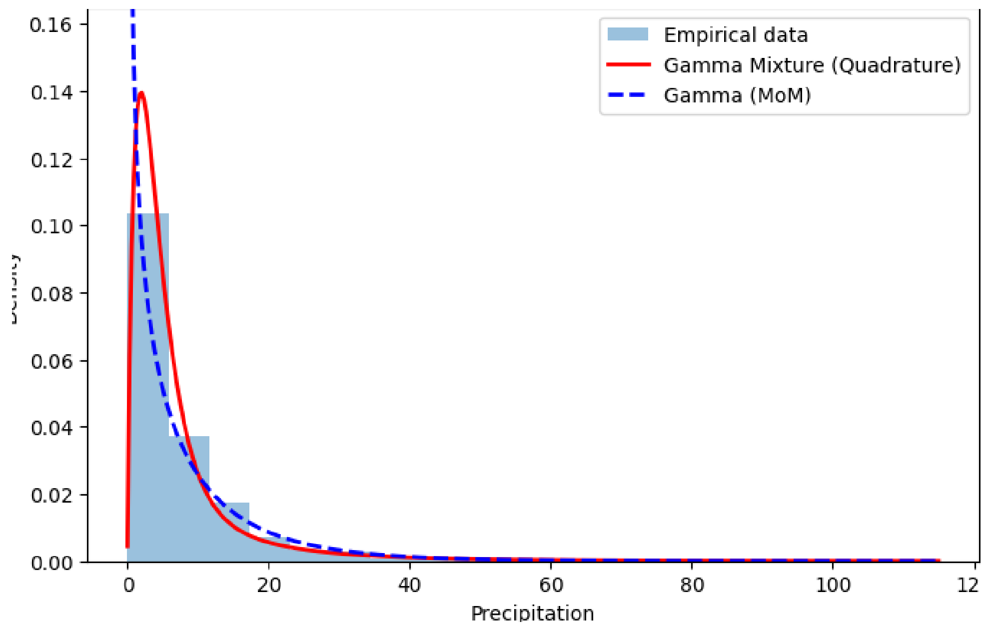
4.1. **Application.** The precipitation data for the Caraga Region, Philippines, obtained from the NASA Langley Research Center (LaRC) POWER Project [5], were modeled using both the Gamma distribution and the Gamma Mixture of Chi-Square distribution. Parameter estimation was conducted through the method of moments, while the probability density function of the Gamma Mixture of Chi-Square distribution was numerically approximated using quadrature integration. The resulting model fits were then compared to evaluate the adequacy and flexibility of the Gamma Mixture of Chi-Square distribution relative to the conventional Gamma distribution in capturing the variability and skewness inherent in regional precipitation behavior. Table 2 summarizes the parameter estimates and performance metrics for both models.

TABLE 2. *Summary of Parameter Estimates and Fit Metrics for Gamma Mixture of Chi-Square (GMCS) and Gamma Models*

Model	α	β	v	**RMSE**	**LogL**	**AIC**	**BIC**
GMCS	0.1810	0.0974	3.5322	0.006033	-6704.18	13414.36	13431.43
Gamma	0.5784	0.0798	-	0.006252	-6535.96	13075.91	13087.29

As illustrated in Figure 3, both the Gamma and Gamma Mixture models exhibited a close alignment with the empirical precipitation density. However, the Gamma Mixture distribution provided a slightly better fit, particularly in capturing the heavy-tailed nature of the precipitation data. The RMSE between the fitted GMCS and the observed histogram (0.006033) was marginally lower than that of the classical Gamma model (0.006252), indicating a better overall approximation of empirical variability. This result supports the suitability of the GMCS model in representing the stochastic and intermittent structure of rainfall events in the Caraga Region.

FIGURE 3. *Fitted Distributions*



5. Discussion

The observed results underscore the robustness of the Gamma Mixture of Chi-Square distribution in modeling precipitation variability in regions with complex climatic influences such as Caraga, where both monsoonal and convective rainfall patterns prevail. The inclusion of the chi-square component within the mixture framework enhances flexibility, allowing the model to capture distributional asymmetry and heavy-tail behavior, which are characteristic of extreme rainfall episodes.

The improved performance of the GMCS, as demonstrated by its lower RMSE and superior tail approximation, is consistent with contemporary findings that highlight the effectiveness of mixture-based models in environmental and hydrological applications [9]. These models can adapt to multimodal and overdispersed datasets, which are often observed in rainfall and climate-related processes.

Although the Gamma distribution remains a classical and widely adopted model for precipitation [8], its inherent limitations in capturing highly skewed or heavy-tailed rainfall data justify the exploration of mixture-based alternatives. The present findings suggest that the GMCS model provides a more generalized and realistic representation of precipitation, improving predictive reliability and supporting hydrological risk management and climate impact studies in the Philippines.

6. Conclusion

The study demonstrated the effectiveness of the Gamma Mixture of Chi-Square (GMCS) distribution in modeling precipitation variability in the Caraga Region, Philippines. Through the method of moments and numerical quadrature integration, the GMCS model successfully recovered the true parameters in simulation with minimal estimation error and strong agreement between theoretical and empirical distributions.

When applied to actual precipitation data, the GMCS distribution exhibited a right-skewed and leptokurtic pattern, accurately capturing both frequent low-precipitation events and extreme rainfall occurrences. Compared with the classical Gamma model, the GMCS achieved a slightly lower RMSE and showed a visually superior fit, as evident in Figure 3.

Overall, the findings confirm that the GMCS model provides a more flexible and realistic representation of precipitation data, making it a suitable approach for hydrological modeling, climate variability assessment, and rainfall-related risk analysis in the region.

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Authors' Contributions. JT Lingo and MO Macalos were responsible for the conceptualization of the study and the development of the methodological framework. JVB Deluao contributed to the software implementation and simulation procedures, as well as to the formal analysis of the results in collaboration with MO Macalos. The original draft of the manuscript was prepared by JT Lingo and MO Macalos, while all authors participated in the review and critical revision of the text. JT Lingo provided supervision throughout the research process. All authors have read and approved the final version of the manuscript.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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