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CLASS OF ALGEBRAS WHOSE TENSOR PRODUCT WITH A BERNSTEIN ALGEBRA IS A BERNSTEIN ALGEBRA

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Abstract. In this paper, we study the structure of a class of algebras satisfying the polynomial identity $(x^2)^3 = 2\omega(x)^2(x^2)^2 - \omega(x)^4x^2$. We show that the tensor product of an algebra in this class with an evolution Bernstein algebra remains an evolution Bernstein algebra. Using Peirce decomposition, we establish the connections between this class of algebras and other classes of non-associative algebras, in particular evolution algebras and principal train algebras.

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1. Introduction

In 2006, J. P. Tian introduced a new class of algebras called evolution algebras (see [15]). Theses algebras are definite using a notion of natural basis. Following Tian, several authors have studied this class of nonassociative commutative algebras (see [6], [7]). In [8], Conseibo and al studied evolution algebras that are Bernstein, ie satisfying the identity $(x^2)^2 - \omega(x)^2x^2 = 0$. We can also cite Emile and al ([12]) who studied evolution algebras that are almost Jordan algebras. In this paper, we study the structure of a class of finite dimensional evolution algebras that satisfy the polynomial identity $(x^2)^3 = 2\omega(x)^2(x^2)^2 - \omega(x)^4x^2$. This polynomial identity is a special case of the identity introduced by Basso ([2]) which designates the class of cubic algebras of exponent 2. The relevance of this study lies in the fact that the tensor product of an evolution algebra satisfying the polynomial identity $(x^2)^3 = 2\omega(x)^2(x^2)^2 - \omega(x)^4x^2$ under certain conditions with an evolution Bernstein algebra remains an evolution Bernstein algebra. Next, we describe the structure of this class of algebras and highlight the link with principal train algebras of rank ≤ 4.

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2. Preliminaries

Let $\mathbb{K}$ be a commutative field and A a commutative and not necessarily associative $\mathbb{K}$-algebra.

Definition 2.1. *We say that A is baric if there exists a non-zero algebra morphism $\omega : A \longrightarrow \mathbb{K}$ called the weight function of the algebra A. The weight of any element x of A is the scalar $\omega(x)$.*

Definition 2.2. *A baric algebra (A, ω) is said to be a principal train algebra of rank $r \geq 2$ if there exists γ_i in $\mathbb{K}$ $(i \in \{1; \cdots ; r-1\})$ such that $x^r + \gamma_1 \omega(x) x^{r-1} + \cdots + \gamma_{r-1}\omega(x)^{r-1}x = 0$, for all x in A and r the smallest integer satisfying this equality.*

Definition 2.3. *We say that a n-dimensional $\mathbb{K}$-algebra A is an evolution algebra if it admits a basis $B = \{e_1, \cdots , e_n\}$ such that for all $i, j \in \{1, \cdots , n\}$:*

$$e_i^2 := e_i e_i = \sum_{k=1}^{n} \omega_{ik} e_k \text{ and } e_i e_j = 0 \text{ if } i \neq j, \tag{1}$$

B is called a natural basis of A and the scalars ω_{ik} are the structural constants of A relative to the basis B. The matrix $M_B = (\omega_{ik})_{1 \leqslant i,k \leqslant n}$ is called the structural constants matrix of A relative to B.

Definition 2.4. *The tensor product of two vector spaces V and W over a field $\mathbb{K}$ is a vector space defined by the set*

$$V \otimes W = \{\sum_{i=1}^{n} \lambda_i v_i \otimes w_i |\, \lambda_i \in \mathbb{K}, v_i \in V, w_i \in W, n \in \mathbb{N}^*\},$$

where $v_i \otimes w_i$ for all $1 \leq i \leq n$ is a bilinear map defined by

$$\begin{array}{rcl} v \otimes w : V^* \times W^* & \to & \mathbb{K} \\ (f, g) & \mapsto & f(v)g(w) \end{array},$$

where V^* and W^* denote respectively the dual vector spaces of V and W. Furthermore, if $B_1 = \{e_i\}_{1 \leq i \leq n}$ and $B_2 = \{f_j\}_{1 \leq j \leq m}$ are bases of the vector spaces V and W, respectively, then $B_1 \otimes B_2 := \{e_i \otimes f_j\}_{1 \leqslant i \leqslant n; 1 \leqslant j \leqslant m}$, is a basis for the tensor product $V \otimes W$.

Definition 2.5. *Let A_1 and A_2 be two $\mathbb{K}$-algebras with bases $B_1 = \{e_i\}_{1 \leqslant i \leqslant n}$ and $B_2 = \{f_j\}_{1 \leqslant j \leqslant m}$ respectively. We define the multiplication table of the tensor product $A_1 \otimes A_2$ in the basis $B_1 \otimes B_2$ as follows:*

$$(e_i \otimes f_j) \cdot (e_k \otimes f_r) = e_i e_k \otimes f_j f_r.$$

In the rest of this paper, the field $\mathbb{K}$ has characteristic different from 2. In [5], the authors show that the tensor of two evolution algebras is an evolution algebra.

Theorem 2.1. [5, *Theorem 3.2*] *If A_1 and A_2 are evolution algebras, then $A_1 \otimes A_2$ is also an evolution algebra. Furthermore, if $B_1 = \{e_i\}_{1 \leqslant i \leqslant n}$ and $B_2 = \{f_j\}_{1 \leqslant j \leqslant m}$ are natural bases of A_1 and A_2, respectively, then $\{e_i \otimes f_j\}_{1 \leqslant i \leqslant n; 1 \leqslant j \leqslant m}$ is a natural basis for $A_1 \otimes A_2$.*

Theorem 2.2. *[8, Theorem 4.2] A n-dimensional baric evolution algebra (A, ω) with a natural basis $\{e_i\}_{1\leqslant i\leqslant n}$ is Bernstein if and only if the following conditions are satisfied:*

i) $(e_1^2)^2 = e_1^2$;

ii) $e_i^2 e_j^2 = 0$ *for* $2 \leqslant i, j \leqslant n$;

iii) $e_1^2 e_i^2 = \frac{1}{2} e_i^2$ *for* $2 \leqslant i \leqslant n$.

3. Tensor product of evolution algebras satisfying some conditions

Proposition 3.1. *Let (A_1, ω) and (A_2, ω) be baric evolution algebras with natural bases $B = \{e_i; 1 \leqslant i \leqslant n\}$ and $B' = \{e'_j; 1 \leqslant j \leqslant m\}$, respectively. Then, the tensor product $A_1 \otimes A_2$ is an evolution Bernstein algebra with the natural basis $B \otimes B' = \{e_i \otimes e'_j, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}$ if and only if:*

i) $(e_1^2)^2 \otimes ({e'_1}^2)^2 = {e_1}^2 \otimes {e'_1}^2$;

ii) ${e_i}^2 e_r^2 \otimes e_j'^2 e_s'^2 = 0$ *for* $2 \leqslant i, r \leqslant n$ *and* $2 \leqslant j, s \leqslant m$;

iii) $e_1^2 e_i^2 \otimes e_1'^2 e_j'^2 = \frac{1}{2} e_i^2 \otimes e_j'^2$ *for* $2 \leqslant i \leqslant n$ *and* $2 \leqslant j \leqslant m$.

Proof. This result is a consequence of Theorems 2.1 and 2.2. □

Remark 3.1. *If A_1 and A_2 are evolution Bernstein algebras, then $A_1 \otimes A_2$ is not necessarily an evolution Bernstein algebra.*

Proof. Let $B_1 = \{e_i\}_{1\leqslant i\leqslant n}$ and $B_2 = \{f_j\}_{1\leqslant j\leqslant m}$ be natural bases of A_1 and A_2, respectively, then $\{e_i \otimes f_j\}_{1\leqslant i\leqslant n; 1\leqslant j\leqslant m}$ is a natural basis for $A_1 \otimes A_2$ according to Theorem 2.1. We have, using Theorem 2.2: $(e_1^2)^2 \otimes ({f_1}^2)^2 = (e_1^2) \otimes ({f_1}^2)$, ${e_i}^2 e_r^2 \otimes f_j^2 f_s^2 = 0 \times 0 = 0$ for $2 \leqslant i, r \leqslant n$; and $2 \leqslant j, s \leqslant m$; and $e_1^2 e_i^2 \otimes f_1^2 f_j^2 = (\frac{1}{2} e_i^2) \otimes (\frac{1}{2} f_j^2) \neq \frac{1}{2} e_i^2 \otimes f_j^2$ for $2 \leqslant i \leqslant n$; and $2 \leqslant j \leqslant m$. □

We give an example of an evolution algebra whose tensor product with a Bernstein evolution algebra is a Bernstein evolution algebra.

Example 3.1. *Let A_1 be a $\mathbb{K}$-baric evolution algebra of dimension 3 with natural basis $B = \{e_1, e_2, e_3\}$ whose multiplication table is given by: $e_1^2 = e_1 + e_2 + e_3$, $e_2^2 = \frac{1}{2}(e_2 - e_3)$ and $e_3^2 = -e_2^2$. Consider the algebra morphism $\omega : A_1 \longrightarrow \mathbb{K}$ such that $\omega(e'_1) = 1$ and $\omega(e'_2) = \omega(e'_3) = 0$. This algebra is Bernstein. Indeed, let $y = ae'_1 + be'_2 + ce'_3$. Then we have,*

$y^2 y^2 = a^4 e_1'^2 + a^2(b^2 - c^2) e_2'^2$ and $\omega(y)^2 y^2 = a^4 e_1'^2 + a^2(b^2 - c^2) e_2'^2$. We therefore have $y^2 y^2 = \omega(y)^2 y^2$.
Let A_2 be a baric evolution algebra of dimension 3, with basis $B_1 = \{e'_1, e'_2, e'_3\}$ and multiplication table given by: $e_1'^2 = e'_1 + e'_2 + e'_3$, $e_2'^2 = \frac{1}{4}(e'_2 - e'_3)$ and $e_3'^2 = -e_2'^2$. The tensor product $A_1 \otimes A_2$ is defined in the natural basis $B = \{e'_i \otimes e_j\}_{1\leqslant i,j\leqslant 3}$. We verify that $A_1 \otimes A_2$ is an evolution Bernstein algebra. Indeed, let $u_{ij} = e'_i \otimes e_j$, then we have:
$u_{11}^2 = e_1'^2 \otimes e_1^2 = (e'_1 + e'_2 + e'_3) \otimes (e_1 + e_2 + e_3)$, $(u_{11}^2)^2 = u_{11}^2 + u_{12}^2 + u_{13}^2 = u_{11}^2$ because $u_{12}^2 = -u_{13}^2$.
We also have, $u_{12}^2 u_{13}^2 = (e_1'^2 \otimes e_2^2)(e_1'^2 \otimes e_3^2) = e_1'^2 e_1'^2 \otimes e_2^2 e_3^2$. Or $e_2^2 e_3^2 = 0$, according to Theorem 2.2, so

$u_{12}^2 u_{13}^2 = 0$. Similarly, we can show that $u_{ij}^2 u_{kl}^2 = 0$, with $1 \leqslant i, j, k, l \leqslant 3$ and $u_{ij}^2 u_{kl}^2 \neq (u_{11}^2)^2$.
$u_{11}^2 u_{12}^2 = (e_1'^2 \otimes e_1^2)(e_1'^2 \otimes e_2^2) = e_1'^2 e_1'^2 \otimes e_1^2 e_2^2$ and by Theorem 2.2, $e_1^2 e_2^2 = \frac{1}{2} e_2^2$, so $u_{11}^2 u_{12}^2 = \frac{1}{2} u_{12}^2$. Similarly, we can verify that $u_{11}^2 u_{kl}^2 = \frac{1}{2} u_{kl}^2$ with $1 \leqslant k, l \leqslant 3$ and $u_{kl} \neq u_{11}$. The tensor product $A_1 \otimes A_2$ is therefore an evolution Bernstein algebra.

More generally we have the following result.

Theorem 3.1. *Let T_1 be an evolution Bernstein algebra with a natural basis $B = \{e_i; 1 \leqslant i \leqslant n\}$ and T_2 an evolution algebra with a natural basis $B' = \{e_j'; 1 \leqslant j \leqslant m\}$ satisfying the following identities:*

$$(e_1'^2)^2 = e_1'^2; e_j'^2 e_r'^2 = 0 \text{ for } 2 \leqslant j, r \leqslant m \text{ and } e_1'^2 e_j'^2 = e_j'^2 \text{ for } 2 \leqslant j \leqslant m. \tag{2}$$

Then, $T_1 \otimes T_2$ is an evolution Bernstein algebra of dimension $n \times m$.

Proof. By using Proposition 3.1 we prove that $T_1 \otimes T_2$ is an evolution and Bernstein algebra of dimension $n \times m$. □

We can observe that the algebra T_2 belongs to a class of baric evolution algebra satisfying a polynomial identity.

Remark 3.2. *Let consider A_2 be a baric evolution $\mathbb{K}$-algebra of dimension 3 with natural basis $B = \{e_1, e_2, e_3\}$ whose multiplication table is given by: $e_1^2 = e_1 + e_2 + e_3$, $e_2^2 = \frac{1}{4}(e_2 - e_3)$ and $e_3^2 = -e_2^2$. Consider the algebra morphism $\omega : A_2 \longrightarrow \mathbb{K}$ such that $\omega(e_1) = 1$ and $\omega(e_2) = \omega(e_3) = 0$. We show that A_2 satisfies the identities (2) and the polynomial identity*

$$(x^2)^3 = 2\omega(x)^2 (x^2)^2 - \omega(x)^4 x^2. \tag{3}$$

Let $x = ae_1 + be_2 + ce_3$, then we have $x^2 = a^2 e_1^2 + (b^2 - c^2) e_2^2$, $(x^2)^2 = a^4 e_1^2 + 2a^2(b^2 - c^2) e_2^2$ and $(x^2)^3 = a^6 e_1^2 + 3a^4(b^2 - c^2) e_2^2$.
By direct calculation, we obtain $2\omega(x)^2(x^2)^2 - \omega(x)^4 x^2 = a^6 e_1^2 + 3a^4(b^2 - c^2) e_2^2$ and since $(x^2)^3 = a^6 e_1^2 + 3a^4(b^2 - c^2) e_2^2$, therefore, $(x^2)^3 = 2\omega(x)^2(x^2)^2 - \omega(x)^4 x^2$. However, A is not Bernstein, because $(x^2)^2 - \omega(x)^2 x^2 = a^2(b^2 - c^2) e_2^2 \neq 0$ for a certain element x in A. Hence, A is not Bernstein.

Beyond the example indicated in the previous remark, we establish a more general result.

Theorem 3.2. *Any evolution algebra satisfying relation (2) also satisfies polynomial identity (3).*

Proof. Let A be an evolution algebra with a natural basis $B = \{e_i; 1 \leqslant i \leqslant n\}$ satisfying (2). Consider the algebra morphism $\omega : A \longrightarrow \mathbb{K}$ such that $\omega(e_1) = 1$ and $\omega(e_i) = 0$, for any $i \geq 2$. We show that (A, ω) satisfy (3). Since the set H of elements of weight 1 is dense in A according to Zariski Toplogy, the proof is done for $x \in H$. Let $x = e_1 + \sum_{j=2}^{n} x_j e_j \in A$. We have: $x^2 = e_1^2 + \sum_{j=2}^{n} x_j^2 e_j^2$. This gives $\omega(x)^4 x^2 = e_1^2 + \sum_{j=2}^{n} x_j^2 e_j^2$.

Also, $(x^2)^2 = e_1^2e_1^2 + 2\sum_{j=2}^{n} x_j^2(e_1^2e_j^2) + \sum_{j,k=2}^{n} x_j^2x_k^2(e_j^2e_k^2)$, so $2(x^2)^2 = 2e_1^2e_1^2 + 4\sum_{j=2}^{n} x_j^2(e_1^2e_j^2) + 2\sum_{j,k=2}^{n} x_j^2x_k^2(e_j^2e_k^2)$. Since, according to relation (2), $(e_1^2)^2 = e_1^2$, $e_1^2e_j^2 = e_j^2$ and $e_j^2e_k^2 = 0$, we have $2(x^2)^2 = 2e_1^2 + 4\sum_{j=2}^{n} x_j^2e_j^2$ then, $2(x^2)^2 - x^2 = e_1^2 + 3\sum_{j=2}^{n} x_j^2e_j^2$.

$(x^2)^3 = (e_1^2e_1^2)e_1^2 + 2\sum_{j=2}^{n} x_j^2(e_1^2e_j^2)e_1^2 + \sum_{j,k=2}^{n} x_j^2x_k^2(e_j^2e_k^2)e_1^2 + \sum_{j=2}^{n} x_j^2(e_1^2e_1^2)e_j^2 + 2\sum_{j,k=2}^{n} x_j^2x_k^2(e_1^2e_j^2)e_k^2 + \sum_{i,j,k=2}^{n} x_i^2x_j^2x_k^2(e_i^2e_j^2)e_k^2$. According to relation (2), $(x^2)^3 = e_1^2 + 3\sum_{j=2}^{n} x_j^2e_j^2$. Therefore, A satisfies $(x^2)^3 = 2(x^2)^2 - x^2$ and therefore, according to Zariski Topology, the polynomial identity $(x^2)^3 = 2\omega(x)^2(x^2)^2 - \omega(x)^4x^2$. □

Remark 3.3. *In [2], the authors show that any algebra satisfying the identity $(x^2)^3 = 2\omega(x)^2(x^2)^2 - \omega(x)^4x^2$ admits non-zero idempotents.*

Remark 3.4. *Is it clear that the class of algebras satisfying identity (3) contains the class of principal train algebra of rank ≤ 3 and equation $x^3 - 2\omega(x)x^+\omega(x)^2x^2 = 0$ and the class of of Bernstein algebras. Indeed, we have $(x^2)^3 - 2\omega(x)^2(x^2)^2 + \omega(x)^4x^2 = x^2[(x^2)^2 - \omega(x)^2x^2] - \omega(x)^2[(x^2)^2 - \omega(x)^2x^2] = 0$ because $(x^2)^2 - \omega(x)^2x^2 = 0$. Therefore, the identity (3) strictly contains the Bernstein algebras, as illustrated in Example 3.1.*

In the following section, we study the structure of evolution algebras satisfying the identity (3).

4. Structure of evolution algebras satisfying identity (3)

Proposition 4.1. *Let (A, ω) be a baric $\mathbb{K}$-algebra satisfying the identity $(x^2)^3 = 2\omega(x)^2(x^2)^2 - \omega(x)^4x^2$. Then, for all x, y and z in A, we have:*

i) $(xy)(x^2)^2 + 2x^2((xy)x^2) = 4\omega(x)^2((xy)x^2) + 2\omega(xy)(x^2)^2 - \omega(x)^4(xy) - 2\omega(x^3y)x^2$;

ii) $4(xy)((xz)x^2) + (yz)(x^2)^2 + 4(xz)((xy)x^2) + 4x^2((xy)(xz)) + 2x^2((yz)x^2) = 8\omega(x)^2((xy)(xz)) + 4\omega(x)^2((yz)x^2) + 8\omega(xz)((xy)x^2) + 8\omega(xy)((xz)x^2) + 2\omega(yz)(x^2)^2 - \omega(x)^4(yz) - 4\omega(x^3y)(xz) - 4\omega(x^3z)(xy) - 6\omega((x^2z)y)x^2$;

iii) $8(xy)((xy)x^2) + y^2(x^2)^2 + 4(xy)^2x^2 + 4(x^2y^2)x^2 = 8\omega(x)^2(xy)^2 + 4\omega(x)^2(x^2y^2) + 16\omega(xy)((xy)x^2) + 2\omega(y)^2(x^2)^2 - \omega(x)^4y^2 - 8\omega(x^3y)(xy) - 6\omega((x^2y)y)x^2$;

iv) $8(xy)((xt)(xz)) + 4(xy)((zt)x^2) + 4(yt)((xz)x^2) + 4(yz)((xt)x^2) + 8(xz)((xy)(xt)) + 4(xz)((yt)x^2) + 4(zt)((xy)x^2) + 4x^2((xy)(zt)) + 4x^2((yt)(xz)) + 8(xt)((xy)(xz)) + 4x^2((yz)(xt)) + 4(xt)((yz)x^2) = 8\omega(x)^2((xy)(zt)) + 8\omega(x)^2((xz)(yt)) + 16\omega(xt)((xy)(xz)) + 8\omega(x)^2((xt)(yz)) + 8\omega(xt)((yz)x^2) + 16\omega(xz)((xt)(xy)) + 8\omega(xz)((yt)x^2) + 8\omega(zt)((xy)x^2) + 16\omega(xy)((xz)(xt)) +$

$8\omega(yt)((xz)x^2) + 8\omega(yz)((xt)x^2) - 4\omega(x^3t)(yz) - 12\omega((x^2t)z)(xy) - 4\omega(x^3z)(yt) - 12\omega((x^2t)y)(xz) - 4\omega(x^3y)(zt) - 12\omega(((xt)z)y)x^2 - 12\omega((x^2z)y)(xt)$.

Proof. The different assertions are obtained by a partial linearization of the identity (3). □

In [2], the authors prove the following results.

Theorem 4.1. *Let A be an algebra satisfying identity (3). Then, for any non-zero idempotent e, A admits the Peirce decomposition given by: $A = \mathbb{K}e \oplus A_0 \oplus A_{\frac{1}{2}} \oplus A_1$ where $A_\alpha = \{x \in \ker\omega / ex = \alpha x\}$ with $\alpha \in \{0; 1; \frac{1}{2}\}$.*

Theorem 4.2. *Let A be an algebra satisfying identity (3) and admitting a non-zero idempotent e, whose Peirce decomposition is given by $A = \mathbb{K}e \oplus A_0 \oplus A_{\frac{1}{2}} \oplus A_1$. Then the following relations hold:*

i) $A_0^2 \subset A_{\frac{1}{2}} \oplus A_1$;
ii) $A_{\frac{1}{2}}^2 \subset A_0 \oplus A_1$;
iii) $A_1^2 = 0$;
iv) $A_0 A_{\frac{1}{2}} \subset A_{\frac{1}{2}} \oplus A_1$;
v) $A_0 A_1 \subset A_{\frac{1}{2}} \oplus A_1$;
vi) $A_1 A_{\frac{1}{2}} \subset A_0 \oplus A_{\frac{1}{2}}$.

Now assume that (A, ω) is a n-dimensional evolution algebra satisfying the identity (3). In [8], the authors show that a baric evolution algebra (A, ω) of dimension n and natural basis $B = \{e_1, ..., e_n\}$ has the multiplication table given by

$$e_1^2 = e_1 + \sum_{k=2}^{n} a_{1k} e_k \text{ and } e_j^2 = \sum_{k=2}^{n} a_{jk} e_k \tag{4}$$

the other products being zero with $2 \leqslant j \leqslant n$, $\omega(e_1) = 1$ and $\omega(e_j) = 0$.

Theorem 4.3. *Let (A, ω) a finite-dimensional baric evolution algebra with natural basis $B = \{e_1, ..., e_n\}$ and multiplication table given in (4). Then the algebra A is an algebra that satisfy the identity $(x^2)^3 = 2\omega(x)^2(x^2)^2 - \omega(x)^4 x^2$ if and only if the following assertions hold:*

i) $(e_1^2 e_1^2) e_1^2 = 2 e_1^2 e_1^2 - e_1^2$;
ii) $2(e_1^2 e_j^2) e_1^2 + (e_1^2 e_1^2) e_j^2 = 4 e_1^2 e_j^2 - e_j^2$, *for* $2 \leqslant j \leqslant n$;
iii) $2(e_1^2 e_j^2) e_k^2 + (e_j^2 e_k^2) e_1^2 = 2 e_j^2 e_k^2$, *for* $2 \leqslant j, k \leqslant n$;
iv) $(e_i^2 e_j^2) e_k^2 = 0$, *for* $2 \leqslant i, j, k \leqslant n$.

Proof. Suppose that A satisfies the identity $(x^2)^3 = 2\omega(x)^2(x^2)^2 - \omega(x)^4 x^2$.
By setting $x = e_1$ in the identity $(x^2)^3 = 2\omega(x)^2(x^2)^2 - \omega(x)^4 x^2$, we obtain the assertion $i)$.
For $ii)$, let $x = e_1$ and $y = e_j$ in $iii)$ of Proposition 4.1, we have $2e_j^2(e_1^2 e_1^2) + 4(e_1^2 e_j^2) e_1^2 = 8 e_1^2 e_j^2 - 2 e_j^2$, from which $2(e_1^2 e_j^2) e_1^2 + (e_1^2 e_1^2) e_j^2 = 4 e_1^2 e_j^2 - e_j^2$, for $2 \leqslant j \leqslant n$.
The assertion $iii)$ is obtained by setting $x = e_k$, $y = z = e_1$ and $t = e_j$ in $iv)$ of Proposition 4.1. Which

gives $8e_1^2(e_k^2(e_ke_j)) + 8e_k^2(e_1^2(e_ke_j)) + 8(e_ke_j)(e_1^2e_k^2) = 16e_k^2(e_ke_j)$. For $j \neq k$, the equation cancels and for $j = k$, we have: $8e_1^2(e_k^2e_j^2) + 8e_k^2(e_1^2e_j^2) + 8e_k^2(e_1^2e_j^2) = 16e_k^2e_j^2$. Which finally gives
$2(e_1^2e_j^2)e_k^2 + (e_j^2e_k^2)e_1^2 = 2e_j^2e_k^2$; for $2 \leqslant j, k \leqslant n$.

For $iv)$ we set $x = e_i$, $y = z = e_j$ and $t = e_k$ in $iv)$ of Proposition 4.1, and we assume that $i = j = k$.

Conversely, let us assume that relations $i)$ to $iv)$ hold. Let us set $x = x_1e_1 + \sum_{j=2}^{n} x_je_j$ with $\omega(x) = x_1$.

We have: $x^2 = x_1^2e_1^2 + \sum_{j=2}^{n} x_j^2e_j^2$. Which gives $\omega(x)^4x^2 = x_1^6e_1^2 + \sum_{j=2}^{n} x_1^4x_j^2e_j^2$. So, $(x^2)^2 = x_1^4e_1^2e_1^2 + 2\sum_{j=2}^{n} x_1^2x_j^2(e_1^2e_j^2) + \sum_{j,k=2}^{n} x_j^2x_k^2(e_j^2e_k^2)$, which gives

$2\omega(x)^2(x^2)^2 = 2x_1^6e_1^2e_1^2 + 4\sum_{j=2}^{n} x_1^4x_j^2(e_1^2e_j^2) + 2\sum_{j,k=2}^{n} x_1^2x_j^2x_k^2(e_j^2e_k^2)$.

$(x^2)^3 = x_1^6(e_1^2e_1^2)e_1^2 + 2\sum_{j=2}^{n} x_1^4x_j^2(e_1^2e_j^2)e_1^2 + \sum_{j,k=2}^{n} x_1^2x_j^2x_k^2(e_j^2e_k^2)e_1^2 + \sum_{j=2}^{n} x_1^4x_j^2(e_1^2e_1^2)e_j^2 + 2\sum_{j,k=2}^{n} x_1^2x_j^2x_k^2(e_1^2e_j^2)e_k^2 + \sum_{i,j,k=2}^{n} x_i^2x_j^2x_k^2(e_i^2e_j^2)e_k^2$.

$(x^2)^3 = x_1^6(e_1^2e_1^2)e_1^2 + \sum_{j=2}^{n} x_1^4x_j^2[2(e_1^2e_j^2)e_1^2 + (e_1^2e_1^2)e_j^2] + \sum_{j,k=2}^{n} x_1^2x_j^2x_k^2[2(e_1^2e_j^2)e_k^2 + (e_j^2e_k^2)e_1^2] + \sum_{i,j,k=2}^{n} x_i^2x_j^2x_k^2(e_i^2e_j^2)e_k^2$. Since relations $i)$ to $iv)$ are satisfied, then

$(x^2)^3 = x_1^6(2e_1^2e_1^2 - e_1^2) + \sum_{j=2}^{n} x_1^4x_j^2[4(e_1^2e_j^2) - e_j^2] + 2\sum_{j,k=2}^{n} x_1^2x_j^2x_k^2e_j^2e_k^2$

$(x^2)^3 = 2x_1^6e_1^2e_1^2 + 4\sum_{j=2}^{n} x_1^4x_j^2e_1^2e_j^2 + 2\sum_{j,k=2}^{n} x_1^2x_j^2x_k^2e_j^2e_k^2 - x_1^6e_1^2 - \sum_{j=2}^{n} x_1^4x_j^2e_j^2$. Therefore, $(x^2)^3 = 2\omega(x)^2(x^2)^2 - \omega(x)^4x^2$. □

Proposition 4.2. *Let $A = \mathbb{K}e \oplus ker\omega$ be a baric evolution algebra satisfying $(x^2)^3 = 2\omega(x)^2(x^2)^2 - \omega(x)^4x^2$, and ω be a weight function with $e_1^2 = e_1 + z$ where $z \in ker\omega$ and $e_1ker\omega = 0$. Then, we have the following relations:*

i) $z^3 = z^2$;
ii) $z^2z^2 = 0$;
iii) $y^2z^2 + 2(y^2z)z - 3y^2z + y^2 = 0$, *for any $y \in ker\omega$;*
iv) $2(y^2z)y^2 + (y^2y^2)z - 2y^2y^2 = 0$, *for any $y \in ker\omega$.*

Proof. Let $e_1^2 = e_1 + z$ where $z \in ker\omega$ and $e_1ker\omega = 0$. For $i)$, we have $e_1^2e_1^2 = e_1^2 + z^2$; $(e_1^2e_1^2)e_1^2 = e_1^2e_1^2 + e_1^2z^2 = 2e_1^2e_1^2 - e_1^2$. Which gives $e_1^2z^2 = e_1^2e_1^2 - e_1^2$; This gives us $z^3 = e_1^2 + z^2 - e_1^2$. Therefore, $z^3 = z^2$.

For $ii)$, we set $x = e_1$ and $y = z$ in $c)$ of Proposition 4.1 and we have

$2z^2(e_1^2e_1^2)+4(e_1^2z^2)e_1^2=8(e_1^2z^2)-2z^2$, which is equivalent to $2z^2e_1^2+2z^2z^2+4e_1^2z^2=6z^2$, so $z^2z^2=0$.
For $iii)$ and $iv)$, we set $x=\alpha e_1+y$ with $y\in ker\omega$. We therefore have
$x^2=\alpha^2e_1^2+y^2$; $(x^2)^2=\alpha^4e_1^2e_1^2+2\alpha^2e_1^2y^2+(y^2)^2$; $(x^2)^3=\alpha^6(e_1^2e_1^2)e_1^2+\alpha^4(e_1^2e_1^2)y^2+2\alpha^4(e_1^2y^2)e_1^2+2\alpha^2(e_1^2y^2)y^2+\alpha^2(y^2)^2e_1^2+(y^2)^3$. By identifying the coefficients of α^4 and α^2, we obtain

$$(e_1^2e_1^2)y^2+2(e_1^2y^2)e_1^2=4e_1^2y^2-y^2 \tag{5}$$

and

$$2(e_1^2y^2)y^2+(y^2)^2e_1^2=2(y^2)^2 \tag{6}$$

Replacing $e_1^2=e_1+z$ in the relation (5), we obtain $y^2z^2+2(y^2z)z-3y^2z+y^2=0$, which is $iii)$.
Doing the same in the relation (6), we obtain $iv)$. □

Proposition 4.3. *Let (A,ω) be a baric evolution algebra satisfying the identity $(x^2)^3=2\omega(x)^2(x^2)^2-\omega(x)^4x^2$. Then, $(ker\omega)^2$ is nil with nil-index at most 3.*

Proof. Let $x=\sum_{j=2}^{n}x_je_j^2\in(ker\omega)^2$. Then, we have: $x^2=\sum_{j,k=2}^{n}x_jx_ke_j^2e_k^2$ and
$x^3=\sum_{i,j,k=2}^{n}x_kx_jx_k(e_j^2e_k^2)e_i^2$, by $iv)$ of theorem 4.3, $(e_j^2e_k^2)e_i^2=0$, hence $x^3=0$, consequently, $(ker\omega)^2$ is nil with nil-index at most 3. □

The following theorem gives the conditions for the tensor product of a Bernstein algebra by an algebra satisfying $(x^2)^3=2\omega(x)^2(x^2)^2-\omega(x)^4x^2$ to be Bernstein.

Theorem 4.4. *Let (A_1,ω) be a Bernstein evolution algebra and (A_2,ω) be a baric evolution algebra satisfying $(x^2)^3=2\omega(x)^2(x^2)^2-\omega(x)^4x^2$. If A_2 is an associative algebra and e_1^2 is a non-zero idempotent of A_2, then the tensor product $A_1\otimes A_2$ is a Bernstein evolution algebra.*

Proof. Suppose A_2 is an associative algebra that satisfy the identity $(x^2)^3=2\omega(x)^2(x^2)^2-\omega(x)^4x^2$ and e_1^2 is a non-zero idempotent of A_2, we have:

$$e_1^2e_1^2=e_1^2. \tag{7}$$

Taking the identities obtained in theorem 4.3, the relation $ii)$ gives us
$2(e_1^2e_j^2)e_1^2+(e_1^2e_1^2)e_j^2=4e_1^2e_j^2-e_j^2$, for $2\leqslant j\leqslant n$. A_2 being associative and $e_1^2e_1^2=e_1^2$, we obtain

$$2(e_1^2e_j^2)e_1^2+e_1^2e_j^2=4e_1^2e_j^2-e_j^2$$

$$2(e_1^2e_1^2)e_j^2+e_1^2e_j^2=4e_1^2e_j^2-e_j^2,$$

$$2e_1^2e_j^2+e_1^2e_j^2=4e_1^2e_j^2-e_j^2,$$

$$3e_1^2e_j^2=4e_1^2e_j^2-e_j^2$$

, thus

$$e_1^2 e_j^2 = e_j^2, \text{ for } 2 \leqslant j \leqslant n. \tag{8}$$

The identity $iii)$ of theorem 4.3 gives us

$$2(e_1^2 e_j^2)e_k^2 + (e_j^2 e_k^2)e_1^2 = 2e_j^2 e_k^2$$

Since according to the relation (8), $e_1^2 e_j^2 = e_j^2$, we obtain

$$2e_j^2 e_k^2 + e_j^2 e_k^2 = 2e_j^2 e_k^2$$

therefore,

$$e_j^2 e_k^2 = 0, \text{ for } 2 \leqslant j, k \leqslant n. \tag{9}$$

Since the relations (7), (8) and (9) satisfy the relation (2) of the Theorem 3.1, then the tensor product $A_1 \otimes A_2$ is a Bernstein evolution algebra. □

Proposition 4.4. *Let A be a baric associative evolution algebra with a natural basis $B = \{e_1, ..., e_n\}$ and a multiplication table given in (4) satisfying the identity $(x^2)^3 = 2\omega(x)^2(x^2)^2 - \omega(x)^4 x^2$. Then the following relations hold:*

i) $A_1(e_1^2) = \{x \in ker\omega | e_1^2 x = x\} = (ker\omega)^2$;

ii) $A_0(e_1^2) = \{x \in ker\omega | e_1^2 x = 0\} =< e_i - a_{1i}e_1^2 | 2 \leqslant i \leqslant n >$;

iii) $A_{\frac{1}{2}}(e_1^2) = \{x \in ker\omega | e_1^2 x = \frac{1}{2}x\} = 0$.

Proof. i) Let A be a baric associative evolution algebra with a natural basis $B = \{e_1, ..., e_n\}$. By theorem 4.4, we have e_1^2 an idempotent and $e_1^2 e_j^2 = e_j^2$, then $e_j^2 \in A_1(e_1^2)$, for all $j \in \{2, ..., n\}$, so $(ker\omega)^2 \subseteq A_1(e_1^2)$.. Let us now show that $A_1(e_1^2) \subseteq (ker\omega)^2$.

Let $x = \sum_{i=2}^{n} x_i e_i$ in A_1, then

$$x = e_1^2 x = (e_1 + \sum_{i=2}^{n} a_{1i} e_i)(\sum_{i=2}^{n} x_i e_i) = \sum_{i=2}^{n} x_i (a_{1i} e_i^2) \in (ker\omega)^2.$$

Therefore, $A_1(e_1^2) \subseteq (ker\omega)^2$, hence $A_1(e_1^2) = (ker\omega)^2$.

ii) For $i = \{2, ..., n\}$, $e_1^2(e_i - a_{1i}e_1^2) = 0$, then $< e_i - a_{1i}e_1^2 | 2 \leqslant i \leqslant n > \subset A_0(e_1^2)$.

Soit $x = \sum_{i=2}^{n} x_i e_i \in A_0(e_1^2)$, we have:

$0 = e_1^2 x = \sum_{i=2}^{n} x_i (a_{1i} e_i^2)$. So,

$$x = \sum_{i=2}^{n} x_i e_i - \sum_{i=2}^{n} x_i (a_{1i} e_i^2),$$

Thus
$$x = \sum_{i=2}^{n} x_i(e_i - a_{1i}e_1^2)$$
and we therefore have $A_0(e_1^2) \subset < e_i - a_{1i}e_1^2 | 2 \leqslant i \leqslant n >$. We therefore deduce that $A_0(e_1^2) = < e_i - a_{1i}e_1^2 | 2 \leqslant i \leqslant n >$.

iii) Let $x \in A_{\frac{1}{2}}$, as $e_k^2 \in (ker\omega)^2 = A_1(e_1^2)$, then we have
$$x = 2e_1^2 x = 2\sum_{i=2}^{n} x_i(a_{1i}e_i^2) \in (ker\omega)^2.$$
Therefore, $x \in A_1(e_1^2)$. This implies that $\frac{1}{2}x = 0$ and therefore $x = 0$ and we obtain $A_{\frac{1}{2}}(e_1^2) = \{x \in ker\omega | e_1^2 x = \frac{1}{2}x\} = 0$.

□

5. Algebras satisfying the identity (3) that are principal train algebras of rank 4

In [2], the authors studied the structure of cubic algebras of exponent 2 that are principal train algebras of rank ≤ 3. In this paper we characterize the structure of algebras satisfying 3 that are principal train algebras of rank 4. In [4] and [11], the authors give the Peirce decomposition of a principal train algebra as well as a characterization of principal train algebras of rank 4. These two results will be useful to us in making the connection with principal train algebras of rank 4.

Proposition 5.1. *Let $A = \mathbb{K}e \oplus A_0 \oplus A_{\frac{1}{2}} \oplus A_1$ be a Peirce decomposition an algebra satisfying identity (3). If A is a principal train algebra of rank 4, then its train equation satisfies the following equations:*

i) $x^4 - (1+\gamma)\omega(x)x^3 + \gamma\omega(x)^2x^2 = 0, \gamma \in \{0, 1, \frac{1}{2}\}$;
ii) $x^4 - \frac{5}{2}\omega(x)x^3 + 2\omega(x)^2x^2 - \frac{1}{2}\omega(x)^3x = 0$;
iii) $x^4 - 3\omega(x)x^3 + 3\omega(x)^2x^2 - \omega(x)^3x = 0$.

Proof. Let $A = \mathbb{K}e \oplus A_0 \oplus A_{\frac{1}{2}} \oplus A_1$ be an algebra satisfying the identity $(x^2)^3 = 2\omega(x)^2(x^2)^2 - \omega(x)^4x^2$. Suppose that A is a train of rank 4, then its train equation is of the form
$x^4 - (1+\gamma+\beta)\omega(x)x^3 + \gamma\omega(x)^2x^2 + \beta\omega(x)^3x = 0$, with $\gamma, \beta \in \mathbb{K}$. Then its minimal train polynomial is given by $P(x) = X^4 - (1+\gamma_1+\gamma_2)X^3 + (\gamma_1+\gamma_2+\gamma_1\gamma_2)X^2 - \gamma_1\gamma_2X = X(X-1)(X-\gamma_1)(X-\gamma_2)$; with $\gamma = \gamma_1+\gamma_2+\gamma_1\gamma_2$ and $\beta = -\gamma_1\gamma_2$. This gives
$$x^4 - (1+\gamma_1+\gamma_2)\omega(x)x^3 + (\gamma_1+\gamma_2+\gamma_1\gamma_2)\omega(x)^2x^2 - \gamma_1\gamma_\omega(x)^3x = 0. \tag{10}$$
This gives us several scenarios to examine.

$1st$ **case**: $\gamma_1 \neq \gamma_2$; $\gamma_1 \neq \frac{1}{2}$ and $\gamma_2 \neq \frac{1}{2}$.

Using Theorem 5 of [11] and Theorem 4.1, we obtain that A admits relatively an idempotent e, the following Peirce decomposition: $A = \mathbb{K}e \oplus A_{\frac{1}{2}} \oplus A_{\gamma_1} \oplus A_{\gamma_2} = \mathbb{K}e \oplus A_{\frac{1}{2}} \oplus A_0 \oplus A_1$, so $A_{\gamma_1} \oplus A_{\gamma_2} = A_0 \oplus A_1$. By identification, we have $\gamma_1, \gamma_2 \in \{0, 1\}$. We can assume without loss of generality that $\gamma_1 = 0$ and

$\gamma_2 = 1$, so relation (10) becomes $x^4 - 2\omega(x)x^3 + \omega(x)^2x^2 = 0$.

$2nd$ **case**: $\gamma_1 \neq \gamma_2$; $\gamma_1 = \frac{1}{2}$

Considering Theorem 1 of [4] and Theorem 4.1, we have that $A = \mathbb{K}e \oplus A_{\frac{1}{2}} \oplus A_{\gamma_2}$, with $\gamma_2 \in \{0,1\}$. The train equation of A is therefore one of the following forms:

For $\gamma_1 = \frac{1}{2}$ and $\gamma_2 = 0$; we have equation (10), which becomes $x^4 - \frac{3}{2}\omega(x)x^3 + \frac{1}{2}\omega(x)^2x^2 = 0$.

For $\gamma_1 = \frac{1}{2}$ and $\gamma_2 = 1$; we have: $x^4 - \frac{5}{2}\omega(x)x^3 + 2\omega(x)^2x^2 - \frac{1}{2}\omega(x)^3x = 0$.

$3rd$ **case**: $\gamma_1 = \gamma_2 \neq \frac{1}{2}$.

Considering Theorem 1 of [4] and since A admits non-zero idempotents, then the Peirce decomposition of A relative to an idempotent e is $A = \mathbb{K}e \oplus A_{\frac{1}{2}} \oplus V$, where $V = \ker\omega \cap \ker(L_e - \lambda_1 i_d)^2$. If $V = 0$, for $x = e + x_{\frac{1}{2}} \in A$, we have $x^2 = e + x_{\frac{1}{2}}$, and therefore $x^2 = \omega(x)x$. This is impossible because A is a train algebra of rank 4, so $V \neq 0$. We then obtain the following situations :

- $\gamma_1 = \gamma_2 = 0$, then equation (10) becomes $x^4 - \omega(x)x^3 = 0$.
- $\gamma_1 = \gamma_2 = 1$, then we have $x^4 - 3\omega(x)x^3 + 3\omega(x)^2x^2 - \omega(x)^3x = 0$.

4^{th} **case**: $\gamma_1 = \gamma_2 = \frac{1}{2}$

The train equation of A becomes $x^4 - 2\omega(x)x^3 + \frac{5}{4}\omega(x)^2x^2 - \frac{1}{4}\omega(x)^3x = 0$. □

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