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INTUITIONISTIC NEUTROSOPHIC $\hat{Z}$-IDEALS AND THEIR PROPERTIES IN $\hat{Z}$-ALGEBRAS

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Abstract. In this study, we introduce and investigate intuitionistic neutrosophic $\hat{Z}$-ideals as a novel extension within the framework of $\hat{Z}$-algebras, aimed at effectively modeling ambiguity, indeterminacy, and uncertainty. By incorporating three membership functions — truth, indeterminacy, and falsity — this structure generalizes classical, fuzzy, intuitionistic fuzzy, and neutrosophic $\hat{Z}$-ideals. We rigorously explore the algebraic properties of these ideals, including their stability under intersection, closure under homomorphism mappings, and Cartesian products. Several theorems are presented to characterize their behavior, accompanied by illustrative examples. The proposed framework enhances the expressive power of $\hat{Z}$-algebras and provides a foundation for further generalizations in uncertain algebraic systems.

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1. Introduction

The introduction of fuzzy sets by Zadeh in 1965 [23] marked a significant shift in the mathematical treatment of uncertainty. Unlike classical sets, where an element is either a member or not, fuzzy sets accommodate partial membership, enabling a nuanced representation of imprecise or ambiguous information. Building on this, Atanassov later proposed intuitionistic fuzzy sets (IFS) [1], which characterize each element through both a membership and a non-membership degree, while leaving a degree of indeterminacy implicit. This dual-character framework enhanced the applicability of fuzzy theory, especially in contexts involving incomplete or conflicting information. These foundational ideas have since been extended to a variety of algebraic structures, broadening the reach of fuzzy mathematics.

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During the same period, the development of generalized algebraic frameworks prompted researchers to explore new structures capable of modeling uncertainty. Chandramouleeswaran et al. [3] introduced the notion of $\hat{Z}$-algebras, which extend classical algebraic systems and opened avenues for integrating fuzzy and neutrosophic logic. Subsequent studies, such as those by Hemavathi et al. [4], focused on interval-valued intuitionistic fuzzy β-subalgebras. At the same time, Sowmiya and Jeyalakshmi [16,18] formulated fuzzy $\hat{Z}$-ideals and examined their structural and homomorphic behaviors. Building upon these efforts, Shanmugapriya and Hemavathi [13] proposed a new framework that embeds interval-valued neutrosophic sets into $\hat{Z}$-algebras, emphasizing enhanced granularity in representing ambiguity, indeterminacy, and falsity. Further advancing this direction, Shanmugapriya et al. [14] conducted an in-depth study on the structural dynamics of neutrosophic $\hat{Z}$-ideals, demonstrating their stability and transformations within $\hat{Z}$-algebraic systems. Their investigation highlighted how neutrosophic logic enriches the algebraic behavior of ideals, reinforcing the flexibility and depth of the $\hat{Z}$-algebraic framework in modeling generalized uncertainty. Collectively, these foundational studies set the stage for the present investigation into intuitionistic neutrosophic $\hat{Z}$-ideals, which unifies and extends these prior approaches by incorporating graded truth, indeterminacy, and falsity into a cohesive ideal-theoretic model.

Smarandache [15] made a pivotal advancement by introducing neutrosophic sets, which extend classical and fuzzy frameworks through the explicit inclusion of indeterminacy. This component captures uncertainty between truth and falsity. This three-part structure—truth, indeterminacy, and falsity—offers a powerful tool for modeling inconsistent or incomplete information. Building upon this, Wang et al. [22] developed interval neutrosophic sets, while Bhowmik and Pal [2] proposed intuitionistic neutrosophic sets, combining features of both intuitionistic fuzzy and neutrosophic systems. These generalizations significantly enhance the capacity to represent vagueness, hesitation, and ambiguity in mathematical modeling.

These foundational principles have been extensively applied to various algebraic structures. For instance, Jun et al. [5] studied BCI-implicative ideals within the neutrosophic framework, while Metawee and Iampan [17] extended the theory to UP-algebras. Muhiuddin [6,8,9] made substantial contributions by developing neutrosophic $\mathcal{N}$-structures across semigroups, semirings, and BCI/BCK-algebras. Additionally, Muralikrishna [10,11] introduced neutrosophic β-ideals and cubic β-subalgebras, and Nagaiah et al. [12] offered a broader perspective on the general theory of neutrosophic algebras. More recently, Suayngam et al. [21] investigated the structure of IUP-algebras through the lens of intuitionistic neutrosophic set theory, offering refined algebraic insights and demonstrating the adaptability of neutrosophic logic to emerging algebraic frameworks. Their results illustrate how intuitionistic neutrosophic sets can capture nuanced algebraic behavior in extended settings beyond classical neutrosophic systems.

These developments naturally culminate in the concept of intuitionistic neutrosophic $\hat{Z}$-ideals, which blend the structural flexibility of intuitionistic fuzzy sets with the descriptive power of neutrosophic logic. This framework generalizes fuzzy $\hat{Z}$-ideals by incorporating the dimensions of uncertainty, hesitation, and indeterminacy in a unified manner. The present work aims to formally define these ideals, investigate their core algebraic properties, and explore their relationships with existing ideal structures in $\hat{Z}$-algebras. Beyond their theoretical significance, such ideals offer a robust foundation for modeling incomplete, inconsistent, or ambiguous information in uncertain algebraic systems.

2. Preliminaries

This section presents the foundational concepts required for the development of intuitionistic neutrosophic $\hat{Z}$-ideals. We begin with the basic definitions of fuzzy sets, intuitionistic fuzzy sets, and neutrosophic sets, followed by the algebraic structure of $\hat{Z}$-algebras and their generalizations. Key properties and illustrative examples are also provided to support subsequent discussions.

Throughout this paper, we denote the universal sets by $\mathfrak{M}$ and $\mathfrak{Y}$. Elements of $\mathfrak{M}$ are represented by γ, while elements of $\mathfrak{Y}$ are denoted by ω.

Definition 2.1. [23] *A fuzzy set ζ from the universal set $\mathfrak{M}$ is defined as a mapping*

$$\mu_\zeta : \mathfrak{M} \to [0, 1],$$

for each element $\gamma \in \mathfrak{M}$, where $\mu_\zeta(\gamma)$ is known as the membership value of γ.

Definition 2.2. [1] *A non-empty set $\mathfrak{M}$ is called an Intuitionistic fuzzy set (IFS), defined as*

$$\zeta = \{\langle \gamma, \mu_\zeta(\gamma), \vartheta_\zeta(\gamma)\rangle \mid \gamma \in \mathfrak{M}\},$$

where $\mu_\zeta : \mathfrak{M} \to [0, 1]$ represents the membership function and $\vartheta_\zeta : \mathfrak{M} \to [0, 1]$ represents the non-membership function. These functions satisfy the condition

$$0 \le \mu_\zeta(\gamma) + \vartheta_\zeta(\gamma) \le 1, \quad \forall \gamma \in \mathfrak{M}.$$

Definition 2.3. [15] *A neutrosophic set ζ in $\mathfrak{M}$ is defined as*

$$\zeta = \{\langle \gamma, \mu_{\zeta_T}(\gamma), \mu_{\zeta_I}(\gamma), \mu_{\zeta_F}(\gamma)\rangle \mid \gamma \in \mathfrak{M}\},$$

where μ_{ζ_T}, μ_{ζ_I} and μ_{ζ_F} are fuzzy sets in $\mathfrak{M}$, denoting respectively:

- μ_{ζ_T} *- truth-membership function,*
- μ_{ζ_I} *- indeterminacy-membership function,*
- μ_{ζ_F} *- falsity-membership function.*

Definition 2.4. [16] *Let μ_ζ and μ_ξ be two fuzzy sets of $\mathfrak{M}$. Then their intersection $\mu_\zeta \cap \mu_\xi$ is defined as*

$$(\mu_\zeta \cap \mu_\xi)(\gamma) = \min\{\mu_\zeta(\gamma), \mu_\xi(\gamma)\}, \quad \forall \gamma \in \mathfrak{M}.$$

Definition 2.5. [8] *The structure*

$$\zeta = \{\langle \gamma, \mu_{\zeta_{(T,I,F)}}(\gamma), \vartheta_{\zeta_{(T,I,F)}}(\gamma)\rangle \mid \gamma \in \mathfrak{M}\}$$

is said to be an Intuitionistic Neutrosophic set in $\mathfrak{M}$, where $\mu_{\zeta_{(T,I,F)}} : \mathfrak{M} \to [0,1]$ and $\vartheta_{\zeta_{(T,I,F)}} : \mathfrak{M} \to [0,1]$. Here,

- *$\mu_{\zeta_{(T,I,F)}}$ denotes the true, indeterminate, and false membership functions,*
- *$\vartheta_{\zeta_{(T,I,F)}}$ denotes the true, indeterminate, and false non-membership functions.*

Having established the foundations of fuzzy, intuitionistic fuzzy, and neutrosophic sets, we now turn to the algebraic structure within which these set-theoretic concepts are applied. The notion of a $\hat{Z}$-algebra, introduced by Chandramouleeswaran et al. [3], serves as the underlying framework for the development of $\hat{Z}$-ideals. This structure generalizes classical algebraic systems by incorporating a binary operation with specific absorptive and commutative properties. The formal definition is given below.

Definition 2.6. [3] *Suppose $\mathfrak{M}$ is a non-empty set with a binary operation $*$ and a constant element 0. Then $(\mathfrak{M}, *, 0)$ is called a $\hat{Z}$-algebra if the following conditions hold:*

(i) $\gamma * 0 = 0$

(ii) $0 * \gamma = \gamma$

(iii) $\gamma * \gamma = \gamma$

(iv) $\gamma * \omega = \omega * \gamma$ *when* $\gamma \neq 0$ *and* $\omega \neq 0$, *for all* $\gamma, \omega \in \mathfrak{M}$.

Example 2.1. *Consider the $\hat{Z}$-algebra $(\mathfrak{M}, *, 0)$ defined on the set $\mathfrak{M} = \{0, \upsilon_1, \upsilon_2, \upsilon_3\}$ with the following Cayley table:*

$*$	0	υ_1	υ_2	υ_3
0	0	υ_1	υ_2	υ_3
υ_1	0	υ_1	υ_1	υ_3
υ_2	0	υ_1	υ_2	υ_2
υ_3	0	υ_3	υ_2	υ_3

Definition 2.7. [3] *If $\mathfrak{M}$ is a non-empty subset of a $\hat{Z}$-algebra, then $\mathfrak{M}$ is called a $\hat{Z}$-subalgebra of $\mathfrak{M}$ if*

$$\gamma * \omega \in \mathfrak{M}, \quad \forall \gamma, \omega \in \mathfrak{M}.$$

Definition 2.8. [20] *Let $\mathfrak{M}$ be a $\hat{Z}$-algebra and $I \subseteq \mathfrak{M}$. Then I is called a $\hat{Z}$-ideal of $\mathfrak{M}$ if the following conditions hold for all $\gamma, \omega \in \mathfrak{M}$:*

(i) $0 \in I$

(ii) *if* $\gamma * \omega \in I$ *and* $\omega \in I$, *then* $\gamma \in I$.

Definition 2.9. [20] *Let $(\mathfrak{M}, *, 0)$ be a $\hat{Z}$-algebra. A fuzzy set ζ on $\mathfrak{M}$, defined by the membership function μ_ζ, is called a fuzzy $\hat{Z}$-ideal of $\mathfrak{M}$ if for every $\gamma, \omega \in \mathfrak{M}$ the following properties hold:*

(*i*) $\mu_\zeta(0) \geq \mu_\zeta(\gamma)$

(*ii*) $\mu_\zeta(\gamma) \geq \min\{\mu_\zeta(\gamma * \omega), \mu_\zeta(\omega)\}$.

Definition 2.10. [11] *Let $(\mathfrak{M}, *, 0)$ be a $\hat{Z}$-algebra. A neutrosophic set*

$$\xi = \{\langle \gamma, \mu_{\zeta_T}(\gamma), \mu_{\zeta_I}(\gamma), \mu_{\zeta_F}(\gamma) \rangle \mid \gamma \in \mathfrak{M}\}$$

is called a neutrosophic $\hat{Z}$-subalgebra of $\mathfrak{M}$ if for all $\gamma, \omega \in \mathfrak{M}$ the following conditions hold:

(*i*) $\mu_{\zeta_T}(\gamma * \omega) \geq \min\{\mu_{\zeta_T}(\gamma), \mu_{\zeta_T}(\omega)\}$

(*ii*) $\mu_{\zeta_I}(\gamma * \omega) \geq \min\{\mu_{\zeta_I}(\gamma), \mu_{\zeta_I}(\omega)\}$

(*iii*) $\mu_{\zeta_F}(\gamma * \omega) \leq \max\{\mu_{\zeta_F}(\gamma), \mu_{\zeta_F}(\omega)\}$.

Definition 2.11. [19] *Let U be a subset of the universal set $\mathfrak{M}$. The supremum and infimum property of an intuitionistic fuzzy set ζ is defined as*

$$\mu_\zeta(\gamma_o) = \sup_{\gamma \in U} \mu_\zeta(\gamma) \ \text{ and } \ \vartheta_\zeta(\gamma_o) = \inf_{\gamma \in U} \vartheta_\zeta(\gamma),$$

if there exists $\gamma_o \in U$.

Definition 2.12. [19] *Let U be a subset of the universal set $\mathfrak{M}$. A neutrosophic set ζ in $\mathfrak{M}$ is said to satisfy the sup-sup-inf property if there exists an element $\gamma_o \in U$ such that*

$$\mu_{\zeta_T}(\gamma_o) = \sup_{\gamma \in U} \mu_{\zeta_T}(\gamma), \quad \mu_{\zeta_I}(\gamma_o) = \sup_{\gamma \in U} \mu_{\zeta_I}(\gamma), \ \text{ and } \ \mu_{\zeta_F}(\gamma_o) = \inf_{\gamma \in U} \mu_{\zeta_F}(\gamma).$$

Definition 2.13. [19] *Let $\zeta = \{\langle \gamma, \mu_{\zeta_T}(\gamma), \mu_{\zeta_I}(\gamma), \mu_{\zeta_F}(\gamma) \rangle \mid \gamma \in \mathfrak{M}\}$ be a neutrosophic set in a $\hat{Z}$-algebra, and let $f_\varrho : \mathfrak{M} \to \mathfrak{Y}$ be a mapping. Then the image of ξ under f_ϱ, denoted as*

$$f_\varrho(\zeta) = \{\langle \omega, f_{\varrho_{\sup}}(\mu_{\zeta_T})(\omega), f_{\varrho_{\sup}}(\mu_{\zeta_I})(\omega), f_{\varrho_{\inf}}(\mu_{\zeta_F})(\omega) \rangle \mid \omega \in \mathfrak{Y}\},$$

is defined as

$$f_{\varrho_{\sup}}(\mu_{\zeta_T})(\omega) = \begin{cases} \sup_{\gamma \in f_\varrho^{-1}(\omega)} \mu_{\zeta_T}(\gamma), & \text{if } f_\varrho^{-1}(\omega) \neq \emptyset, \\ 1, & \text{otherwise,} \end{cases}$$

$$f_{\varrho_{\sup}}(\mu_{\zeta_I})(\omega) = \begin{cases} \sup_{\gamma \in f_\varrho^{-1}(\omega)} \mu_{\zeta_I}(\gamma), & \text{if } f_\varrho^{-1}(\omega) \neq \emptyset, \\ 1, & \text{otherwise,} \end{cases}$$

$$f_{\varrho_{\inf}}(\mu_{\zeta_F})(\omega) = \begin{cases} \inf_{\gamma \in f_\varrho^{-1}(\omega)} \mu_{\zeta_F}(\gamma), & \text{if } f_\varrho^{-1}(\omega) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.14. [19] *If $f_\varrho : \mathfrak{M} \to \mathfrak{Y}$ is a function, and let $\mu_{\zeta_{(T_1,I_1,F_1)}}$ and $\mu_{\zeta_{(T_2,I_2,F_2)}}$ be two neutrosophic sets in $\mathfrak{M}$ and $\mathfrak{Y}$, respectively, then the inverse image of $\zeta_{(T,I,F)}$ under f_ϱ is defined as*

$$f_\varrho^{-1}(\mu_{\zeta_{(T_2,I_2,F_2)}}) = \{\langle \gamma, f_\varrho^{-1}(\mu_{\zeta_{T_2}}(\gamma)), f_\varrho^{-1}(\mu_{\zeta_{I_2}}(\gamma)), f_\varrho^{-1}(\mu_{\zeta_{F_2}}(\gamma))\rangle \mid \gamma \in \mathfrak{M}\},$$

such that

$$f_\varrho^{-1}(\mu_{\zeta_{T_2}})(\gamma) = \mu_{\zeta_{T_2}}(f_\varrho(\gamma)),$$
$$f_\varrho^{-1}(\mu_{\zeta_{I_2}})(\gamma) = \mu_{\zeta_{I_2}}(f_\varrho(\gamma)),$$
$$f_\varrho^{-1}(\mu_{\zeta_{F_2}})(\gamma) = \mu_{\zeta_{F_2}}(f_\varrho(\gamma)).$$

Definition 2.15. [19] *Let h be a $\hat{Z}$-endomorphism of neutrosophic $\hat{Z}$-algebras, and let*

$$\zeta = \{\langle \gamma, \mu_{\zeta_{(T,I,F)}}(\gamma), \vartheta_{\zeta_{(T,I,F)}}(\gamma)\rangle \mid \gamma \in \mathfrak{M}\}$$

be a neutrosophic set in $\mathfrak{M}$. Then the fuzzy set ζ^h in $\mathfrak{M}$ is symbolized as

$$\xi_{\zeta^h}(\gamma) = \xi_\zeta(h(\gamma)), \quad \forall \gamma \in \mathfrak{M}.$$

Definition 2.16. [19] *Let $\zeta = \{\langle \gamma, \mu_{\zeta_{(T,I,F)}}(\gamma), \vartheta_{\zeta_{(T,I,F)}}(\gamma)\rangle \mid \gamma \in \mathfrak{M}\}$ and $\xi = \{\langle \omega, \mu_{\xi_{(T,I,F)}}(\omega), \vartheta_{\xi_{(T,I,F)}}(\omega)\rangle \mid \omega \in \mathfrak{Y}\}$ be two neutrosophic sets of $\mathfrak{M}$ and $\mathfrak{Y}$, respectively. The Cartesian product of ζ and ξ is defined as*

$$\zeta \times \xi = \{\langle (\gamma, \omega), \mu_{\zeta_T \times \xi_T}(\gamma, \omega), \mu_{\zeta_I \times \xi_I}(\gamma, \omega), \mu_{\zeta_F \times \xi_F}(\gamma, \omega)\rangle \mid \gamma \in \mathfrak{M}, \omega \in \mathfrak{Y}\},$$

where

$$\mu_{\zeta_T \times \xi_T}(\gamma, \omega) = \min\{\mu_{\zeta_T}(\gamma), \mu_{\xi_T}(\omega)\},$$
$$\mu_{\zeta_I \times \xi_I}(\gamma, \omega) = \min\{\mu_{\zeta_I}(\gamma), \mu_{\xi_I}(\omega)\},$$
$$\mu_{\zeta_F \times \xi_F}(\gamma, \omega) = \max\{\mu_{\zeta_F}(\gamma), \mu_{\xi_F}(\omega)\}.$$

3. Intuitionistic neutrosophic $\hat{Z}$-ideals

In this section, we introduce and analyze the structure of intuitionistic neutrosophic $\hat{Z}$-ideals within the framework of $\hat{Z}$-algebras. Building upon the previously defined fuzzy, intuitionistic fuzzy, and neutrosophic ideals, this extension incorporates multiple degrees of membership to model uncertainty more effectively. We investigate the fundamental algebraic properties of these ideals, examine their behavior under standard operations such as intersection and homomorphism, and discuss their structural significance in abstract algebraic analysis.

Definition 3.1. *Let $(\mathfrak{M}, *, 0)$ be a $\hat{Z}$-algebra. An intuitionistic neutrosophic set*

$$\zeta = \{\langle \gamma, \mu_{\zeta_{(T,I,F)}}(\gamma), \vartheta_{\zeta_{(T,I,F)}}(\gamma)\rangle \mid \gamma \in \mathfrak{M}\}$$

in $\mathfrak{M}$ is considered as an intuitionistic neutrosophic $\hat{Z}$-ideal of $\mathfrak{M}$ if the following conditions hold for all $\gamma, \omega \in \mathfrak{M}$,

(i) $\mu_{\zeta_{(T,I)}}(0) \geq \mu_{\zeta_{(T,I)}}(\gamma)$ *and* $\vartheta_{\zeta_{(T,I)}}(0) \leq \vartheta_{\zeta_{(T,I)}}(\gamma)$

(ii) $\mu_{\zeta_F}(0) \leq \mu_{\zeta_F}(\gamma)$ *and* $\vartheta_{\zeta_F}(0) \geq \vartheta_{\zeta_F}(\gamma)$

(iii) $\mu_{\zeta_{(T,I)}}(\gamma) \geq \min\{\mu_{\zeta_{(T,I)}}(\gamma * \omega), \mu_{\zeta_{(T,I)}}(\omega)\}$ *and* $\vartheta_{\zeta_{(T,I)}}(\gamma) \leq \max\{\vartheta_{\zeta_{(T,I)}}(\gamma * \omega), \vartheta_{\zeta_{(T,I)}}(\omega)\}$

(iv) $\mu_{\zeta_F}(\gamma) \leq \max\{\mu_{\zeta_F}(\gamma * \omega), \mu_{\zeta_F}(\omega)\}$ *and* $\vartheta_{\zeta_F}(\gamma) \geq \min\{\vartheta_{\zeta_F}(\gamma * \omega), \vartheta_{\zeta_F}(\omega)\}$.

Example 3.1. *Illustrate with Example 2.1, we define*

$$\mu_{\zeta_{(T,I)}}(\gamma) = \begin{cases} 0.9, & \text{if } \gamma = 0, \\ 0.7, & \text{if } \gamma = \upsilon_1, \upsilon_3, \\ 0.5, & \text{if } \gamma = \upsilon_2. \end{cases} \qquad \vartheta_{\zeta_{(T,I)}}(\gamma) = \begin{cases} 0.1, & \text{if } \gamma = 0, \\ 0.2, & \text{if } \gamma = \upsilon_1, \upsilon_3, \\ 0.5, & \text{if } \gamma = \upsilon_2. \end{cases}$$

$$\mu_{\zeta_{(F)}}(\gamma) = \begin{cases} 0.6, & \text{if } \gamma = 0, \\ 0.7, & \text{if } \gamma = \upsilon_1, \upsilon_3 \\ 0.8, & \text{if } \gamma = \upsilon_2 \end{cases} \qquad \vartheta_{\zeta_{(F)}}(\gamma) = \begin{cases} 0.4, & \text{if } \gamma = 0, \\ 0.3, & \text{if } \gamma = \upsilon_1, \upsilon_3 \\ 0.2, & \text{if } \gamma = \upsilon_2. \end{cases}$$

Then $\zeta = (\mu_{\zeta_{(T,I,F)}}, \vartheta_{\zeta_{(T,I,F)}})$ is an intuitionistic neutrosophic $\hat{Z}$-ideal of $\mathfrak{M}$.

Theorem 3.1. *The intersection of any two intuitionistic neutrosophic $\hat{Z}$-ideals of $\mathfrak{M}$ is again an intuitionistic neutrosophic $\hat{Z}$-ideal of $\mathfrak{M}$.*

Proof. Let $\gamma, \omega \in \mathfrak{M}$. Then

$$\begin{aligned} \mu_{\zeta_{(T,I)} \cap \xi_{(T,I)}}(0) &= \min\{\mu_{\zeta_{(T,I)}}(0), \mu_{\xi_{(T,I)}}(0)\} \\ &\geq \min\{\mu_{\zeta_{(T,I)}}(\gamma), \mu_{\xi_{(T,I)}}(\gamma)\} \\ &= \mu_{\zeta_{(T,I)} \cap \xi_{(T,I)}}(\gamma), \\ \vartheta_{\zeta_{(T,I)} \cap \xi_{(T,I)}}(0) &= \max\{\vartheta_{\zeta_{(T,I)}}(0), \vartheta_{\xi_{(T,I)}}(0)\} \\ &\leq \max\{\vartheta_{\zeta_{(T,I)}}(\gamma), \vartheta_{\xi_{(T,I)}}(\gamma)\} \\ &= \vartheta_{\zeta_{(T,I)} \cap \xi_{(T,I)}}(\gamma), \end{aligned}$$

$$\begin{aligned} \mu_{\zeta_F \cap \xi_F}(0) &= \max\{\mu_{\zeta_F}(0), \mu_{\xi_F}(0)\} \\ &\leq \max\{\mu_{\zeta_F}(\gamma), \mu_{\xi_F}(\gamma)\} \\ &= \mu_{\zeta_F \cap \xi_F}(\gamma), \\ \vartheta_{\zeta_F \cap \xi_F}(0) &= \min\{\vartheta_{\zeta_F}(0), \vartheta_{\xi_F}(0)\} \\ &\geq \min\{\vartheta_{\zeta_F}(\gamma), \vartheta_{\xi_F}(\gamma)\} \\ &= \vartheta_{\zeta_F \cap \xi_F}(\gamma), \end{aligned}$$

$$\begin{aligned}
&\mu_{\zeta_{(T,I)}\cap\xi_{(T,I)}}(\gamma)\\
&=\min\{\mu_{\zeta_{(T,I)}}(\gamma),\mu_{\xi_{(T,I)}}(\gamma)\}\\
&\geq\min\{\min\{\mu_{\zeta_{(T,I)}}(\gamma*\omega),\mu_{\zeta_{(T,I)}}(\omega)\},\min\{\mu_{\xi_{(T,I)}}(\gamma*\omega),\mu_{\xi_{(T,I)}}(\omega)\}\}\\
&=\min\{\min\{\mu_{\zeta_{(T,I)}}(\gamma*\omega),\mu_{\xi_{(T,I)}}(\gamma*\omega)\},\min\{\mu_{\zeta_{(T,I)}}(\omega),\mu_{\xi_{(T,I)}}(\omega)\}\}\\
&=\min\{\mu_{\zeta_{(T,I)}\cap\xi_{(T,I)}}(\gamma*\omega),\mu_{\zeta_{(T,I)}\cap\xi_{(T,I)}}(\omega)\},\\
&\vartheta_{\zeta_{(T,I)}\cap\xi_{(T,I)}}(\gamma)\\
&=\max\{\vartheta_{\zeta_{(T,I)}}(\gamma),\vartheta_{\xi_{(T,I)}}(\gamma)\}\\
&\leq\max\{\max\{\vartheta_{\zeta_{(T,I)}}(\gamma*\omega),\vartheta_{\zeta_{(T,I)}}(\omega)\},\max\{\vartheta_{\xi_{(T,I)}}(\gamma*\omega),\vartheta_{\xi_{(T,I)}}(\omega)\}\}\\
&=\max\{\max\{\vartheta_{\zeta_{(T,I)}}(\gamma*\omega),\vartheta_{\xi_{(T,I)}}(\gamma*\omega)\},\max\{\vartheta_{\zeta_{(T,I)}}(\omega),\vartheta_{\xi_{(T,I)}}(\omega)\}\}\\
&=\max\{\vartheta_{\zeta_{(T,I)}\cap\xi_{(T,I)}}(\gamma*\omega),\vartheta_{\zeta_{(T,I)}\cap\xi_{(T,I)}}(\omega)\},
\end{aligned}$$

$$\begin{aligned}
&\mu_{\zeta_F\cap\xi_F}(\gamma)\\
&=\max\{\mu_{\zeta_F}(\gamma),\mu_{\xi_F}(\gamma)\}\\
&\leq\max\{\max\{\mu_{\zeta_F}(\gamma*\omega),\mu_{\zeta_F}(\omega)\},\max\{\mu_{\xi_F}(\gamma*\omega),\mu_{\xi_F}(\omega)\}\}\\
&=\max\{\max\{\mu_{\zeta_F}(\gamma*\omega),\mu_{\xi_F}(\gamma*\omega)\},\max\{\mu_{\zeta_F}(\omega),\mu_{\xi_F}(\omega)\}\}\\
&=\max\{\mu_{\zeta_F\cap\xi_F}(\gamma*\omega),\mu_{\zeta_F\cap\xi_F}(\omega)\},\\
&\vartheta_{\zeta_F\cap\xi_F}(\gamma)\\
&=\min\{\vartheta_{\zeta_F}(\gamma),\vartheta_{\xi_F}(\gamma)\}\\
&\geq\min\{\min\{\vartheta_{\zeta_F}(\gamma*\omega),\vartheta_{\zeta_F}(\omega)\},\min\{\vartheta_{\xi_F}(\gamma*\omega),\vartheta_{\xi_F}(\omega)\}\}\\
&=\min\{\min\{\vartheta_{\zeta_F}(\gamma*\omega),\vartheta_{\xi_F}(\gamma*\omega)\},\min\{\vartheta_{\zeta_F}(\omega),\vartheta_{\xi_F}(\omega)\}\}\\
&=\min\{\vartheta_{\zeta_F\cap\xi_F}(\gamma*\omega),\vartheta_{\zeta_F\cap\xi_F}(\omega)\}.
\end{aligned}$$

Therefore, $\mu_{\zeta_{T,I,F}}\cap\xi_{T,I,F}$ is an intuitionistic neutrosophic $\hat{Z}$-ideal of $\mathfrak{M}$. □

Also, generalizing the previous theorem, we have:

Theorem 3.2. *Let $\{\varsigma_{(T_i,I_i,F_i)}\mid i\in\Omega\}$ be a family of intuitionistic neutrosophic $\hat{Z}$-ideals of $\mathfrak{M}$. Then $\bigcap_{i\in\Omega}\varsigma_{(T_i,I_i,F_i)}$ is an intuitionistic neutrosophic $\hat{Z}$-ideal of $\mathfrak{M}$.*

Theorem 3.3. *An intuitionistic neutrosophic set $\varsigma=(\mu_{\varsigma_{(T,I,F)}},\vartheta_{\varsigma_{(T,I,F)}})$ is an intuitionistic neutrosophic $\hat{Z}$-ideal of $\mathfrak{M}$ if and only if $\mu_{\varsigma_{(T,I,F)}}$ and $(\vartheta_{\varsigma_{(T,I,F)}})^c$ are neutrosophic $\hat{Z}$-ideals of $\mathfrak{M}$.*

Proof. Let $\varsigma = (\mu_{\varsigma_{(T,I,F)}}, \vartheta_{\varsigma_{(T,I,F)}})$ be an intuitionistic neutrosophic $\hat{Z}$-ideal of $\mathfrak{M}$. We immediately get that $\mu_{\varsigma_{(T,I,F)}}$ is a neutrosophic $\hat{Z}$-ideal of $\mathfrak{M}$. Now, for any $\gamma, \omega \in \mathfrak{M}$, we have

$$\begin{aligned}
(\vartheta_{\varsigma_{(T,I)}})^c(0) &= 1 - \vartheta_{\varsigma_{(T,I)}}(0) \\
&\geq 1 - \max\{\vartheta_{\varsigma_{(T,I)}}(\gamma * \omega), \vartheta_{\varsigma_{(T,I)}}(\omega)\} \\
&= \min\{1 - \vartheta_{\varsigma_{(T,I)}}(\gamma * \omega), 1 - \vartheta_{\varsigma_{(T,I)}}(\omega)\} \\
&= \min\{(\vartheta_{\varsigma_{(T,I)}})^c(\gamma * \omega), (\vartheta_{\varsigma_{(T,I)}})^c(\omega)\},
\end{aligned}$$

$\Longrightarrow (\vartheta_{\varsigma_{(T,I)}})^c(\gamma) \geq \min\{(\vartheta_{\varsigma_{(T,I)}})^c(\gamma * \omega), (\vartheta_{\varsigma_{(T,I)}})^c(\omega)\}$, and

$$\begin{aligned}
(\vartheta_{\varsigma_F})^c(0) &= 1 - \vartheta_{\varsigma_F}(0) \\
&\leq 1 - \min\{\vartheta_{\varsigma_F}(\gamma * \omega), \vartheta_{\varsigma_F}(\omega)\} \\
&= \max\{1 - \vartheta_{\varsigma_F}(\gamma * \omega), 1 - \vartheta_{\varsigma_F}(\omega)\} \\
&= \max\{(\vartheta_{\varsigma_F})^c(\gamma * \omega), (\vartheta_{\varsigma_F})^c(\omega)\},
\end{aligned}$$

$\Longrightarrow (\vartheta_{\varsigma_F})^c(\gamma) \leq \max\{(\vartheta_{\varsigma_F})^c(\gamma * \omega), (\vartheta_{\varsigma_F})^c(\omega)\}$. This implies that $(\vartheta_{\varsigma_{(T,I,F)}})^c$ is a neutrosophic $\hat{Z}$-ideal of $\mathfrak{M}$.

Conversely, assume that $\mu_{\varsigma_{(T,I,F)}}$ and $(\vartheta_{\varsigma_{(T,I,F)}})^c$ are neutrosophic $\hat{Z}$-ideals of $\mathfrak{M}$. Let $\varsigma = (\mu_{\varsigma_{(T,I,F)}}, \vartheta_{\varsigma_{(T,I,F)}})$ be an intuitionistic neutrosophic $\hat{Z}$-ideal of $\mathfrak{M}$. For every $\gamma, \omega \in \mathfrak{M}$, we have

$$\begin{aligned}
\mu_{\varsigma_{(T,I)}}(0) &\geq \mu_{\varsigma_{(T,I)}}(\gamma), \\
\mu_{\varsigma_F}(0) &\leq \mu_{\varsigma_F}(\gamma), \\
1 - \vartheta_{\varsigma_{(T,I)}}(0) = (\vartheta_{\varsigma_{(T,I)}})^c(0) &\geq (\vartheta_{\varsigma_{(T,I)}})^c(\gamma) = 1 - \vartheta_{\varsigma_{(T,I)}}(\gamma) \\
\Rightarrow \vartheta_{\varsigma_{(T,I)}}(0) &\leq \vartheta_{\varsigma_{(T,I)}}(\gamma), \\
1 - \vartheta_{\varsigma_F}(0) = (\vartheta_{\varsigma_F})^c(0) &\leq (\vartheta_{\varsigma_F})^c(\gamma) = 1 - \vartheta_{\varsigma_F}(\gamma) \\
\Rightarrow \vartheta_{\varsigma_F}(0) &\geq \vartheta_{\varsigma_F}(\gamma),
\end{aligned}$$

$$\begin{aligned}
\mu_{\varsigma_{(T,I)}}(\gamma) &\geq \min\{\mu_{\varsigma_{(T,I)}}(\gamma * \omega), \mu_{\varsigma_{(T,I)}}(\omega)\}, \\
\mu_{\varsigma_F}(\gamma) &\leq \max\{\mu_{\varsigma_F}(\gamma * \omega), \mu_{\varsigma_F}(\omega)\}, \\
1 - \vartheta_{\varsigma_{(T,I)}}(\gamma) &= (\vartheta_{\varsigma_{(T,I)}})^c(\gamma) \\
&\geq \min\{(\vartheta_{\varsigma_{(T,I)}})^c(\gamma * \omega), (\vartheta_{\varsigma_{(T,I)}})^c(\omega)\} \\
&= \min\{1 - \vartheta_{\varsigma_{(T,I)}}(\gamma * \omega), 1 - \vartheta_{\varsigma_{(T,I)}}(\omega)\} \\
&= 1 - \max\{\vartheta_{\varsigma_{(T,I)}}(\gamma * \omega), \vartheta_{\varsigma_{(T,I)}}(\omega)\} \\
\Rightarrow \vartheta_{\varsigma_{(T,I)}}(\gamma) &\leq \max\{\vartheta_{\varsigma_{(T,I)}}(\gamma * \omega), \vartheta_{\varsigma_{(T,I)}}(\omega)\},
\end{aligned}$$

$$
\begin{aligned}
1-\vartheta_{\varsigma_F}(\gamma) &= (\vartheta_{\varsigma_F})^c(\gamma)\\
&\le \max\{(\vartheta_{\varsigma_F})^c(\gamma*\omega),(\vartheta_{\varsigma_F})^c(\omega)\}\\
&= \max\{1-\vartheta_{\varsigma_F}(\gamma*\omega),1-\vartheta_{\varsigma_F}(\omega)\}\\
&= 1-\min\{\vartheta_{\varsigma_F}(\gamma*\omega),\vartheta_{\varsigma_F}(\omega)\}\\
\Rightarrow \vartheta_{\varsigma_F}(\gamma) &\ge \min\{\vartheta_{\varsigma_F}(\gamma*\omega),\vartheta_{\varsigma_F}(\omega)\}.
\end{aligned}
$$

Therefore, $\varsigma=(\mu_{\varsigma_{(T,I,F)}},\vartheta_{\varsigma_{(T,I,F)}})$ is an intuitionistic neutrosophic $\hat{Z}$-ideal of $\mathfrak{M}$. □

Theorem 3.4. *Let $\varsigma=(\mu_{\varsigma_{(T,I,F)}},\vartheta_{\varsigma_{(T,I,F)}})$ be an intuitionistic neutrosophic set in $\mathfrak{M}$. Then $\varsigma=(\mu_{\varsigma_{(T,I,F)}},\vartheta_{\varsigma_{(T,I,F)}})$ is an intuitionistic neutrosophic $\hat{Z}$-ideal of $\mathfrak{M}$ if and only if*

$$\boxtimes\varsigma=(\mu_{\varsigma_{(T,I,F)}},(\mu_{\varsigma_{(T,I,F)}})^c)\ \ \textit{and}\ \ \otimes\varsigma=((\vartheta_{\varsigma_{(T,I,F)}})^c,\vartheta_{\varsigma_{(T,I,F)}})$$

are intuitionistic neutrosophic $\hat{Z}$-ideals of $\mathfrak{M}$.

Proof. The proof is similar to Theorem 3.3, so we omit it. □

4. Homomorphisms of intuitionistic neutrosophic $\hat{Z}$-ideals

Homomorphisms play a central role in algebra by preserving structure across algebraic systems. In this section, we explore how intuitionistic neutrosophic $\hat{Z}$-ideals behave under homomorphisms. Specifically, we examine the conditions under which the image and inverse image of such ideals remain within the same class, ensuring their structural consistency across mappings. Preserving ideal properties under homomorphic transformations provides a robust foundation for extending the theory to more general algebraic frameworks.

Definition 4.1. *Let*

$$\varsigma=\{\langle\gamma,\mu_{\varsigma_T}(\gamma),\mu_{\varsigma_I}(\gamma),\mu_{\varsigma_F}(\gamma),\vartheta_{\varsigma_T}(\gamma),\vartheta_{\varsigma_I}(\gamma),\vartheta_{\varsigma_F}(\gamma)\rangle\mid\gamma\in\mathfrak{M}\}$$

be an intuitionistic neutrosophic set in $\mathfrak{M}$, and let $\mathfrak{h}$ be a mapping from a set $\mathfrak{M}$ into a set $\mathfrak{Y}$. Then its image of ς under $\mathfrak{h}$, denoted by $\mathfrak{h}(\varsigma)$, is defined as

$$\mathfrak{h}(\varsigma)=\{\langle\gamma,\mathfrak{h}_{\sup}\mu_{\varsigma_T}(\gamma),\mathfrak{h}_{\sup}\mu_{\varsigma_I}(\gamma),\mathfrak{h}_{\inf}\mu_{\varsigma_F}(\gamma),\mathfrak{h}_{\inf}\vartheta_{\varsigma_T}(\gamma),\mathfrak{h}_{\inf}\vartheta_{\varsigma_I}(\gamma),\mathfrak{h}_{\sup}\vartheta_{\varsigma_F}(\gamma)\rangle\mid\gamma\in\mathfrak{Y}\}$$

where

$$\mathfrak{h}_{\sup}(\mu_{\varsigma_T})(\omega)=\begin{cases}\sup\limits_{\gamma\in\mathfrak{h}^{-1}(\omega)}\mu_{\varsigma_T}(\gamma), & \textit{if } \mathfrak{h}^{-1}(\omega)\neq\emptyset,\\ 1, & \textit{otherwise,}\end{cases}$$

$$\mathfrak{h}_{\sup}(\mu_{\varsigma_I})(\omega)=\begin{cases}\sup\limits_{\gamma\in\mathfrak{h}^{-1}(\omega)}\mu_{\varsigma_I}(\gamma), & \textit{if } \mathfrak{h}^{-1}(\omega)\neq\emptyset\\ 1, & \textit{otherwise,}\end{cases}$$

$$\mathfrak{h}_{\inf}(\mu_{\varsigma_F})(\omega) = \begin{cases} \inf\limits_{\gamma \in \mathfrak{h}^{-1}(\omega)} \mu_{\varsigma_F}(\gamma), & \text{if } \mathfrak{h}^{-1}(\omega) \neq \emptyset \\ 0, & \text{otherwise}, \end{cases}$$

$$\mathfrak{h}_{\inf}(\vartheta_{\varsigma_T})(\omega) = \begin{cases} \inf\limits_{\gamma \in \mathfrak{h}^{-1}(\omega)} \vartheta_{\varsigma_T}(\gamma), & \text{if } \mathfrak{h}^{-1}(\omega) \neq \emptyset \\ 1, & \text{otherwise}, \end{cases}$$

$$\mathfrak{h}_{\inf}(\vartheta_{\varsigma_I})(\omega) = \begin{cases} \inf\limits_{\gamma \in \mathfrak{h}^{-1}(\omega)} \vartheta_{\varsigma_I}(\gamma), & \text{if } \mathfrak{h}^{-1}(\omega) \neq \emptyset \\ 1, & \text{otherwise}, \end{cases}$$

$$\mathfrak{h}_{\sup}(\vartheta_{\varsigma_F})(\omega) = \begin{cases} \sup\limits_{\gamma \in \mathfrak{h}^{-1}(\omega)} \vartheta_{\varsigma_F}(\gamma), & \text{if } \mathfrak{h}^{-1}(\omega) \neq \emptyset \\ 0, & \text{otherwise}. \end{cases}$$

Theorem 4.1. *Let $\mathfrak{h}$ be a homomorphism from the $\hat{Z}$-algebras $(\mathfrak{M}, *, 0)$ into $(\mathfrak{Y}, *, 0)$ and $\varsigma_{(T,I,F)} = \{\langle \gamma, \mu_{\varsigma_{(T,I,F)}}(\gamma), \vartheta_{\varsigma_{(T,I,F)}}(\gamma)\rangle \mid \gamma \in \mathfrak{M}\}$ be an intuitionistic neutrosophic $\hat{Z}$-ideal of $\mathfrak{M}$ with the sup-sup-inf property. Then the image of ς under $\mathfrak{h}$,*

$$\mathfrak{h}(\varsigma) = \{\langle \omega, \mu^{\mathfrak{h}}_{\varsigma_{(T,I,F)}}(\omega), \vartheta^{\mathfrak{h}}_{\varsigma_{(T,I,F)}}(\omega)\rangle \mid \omega \in \mathfrak{Y}\},$$

is an intuitionistic neutrosophic $\hat{Z}$-ideal of $\mathfrak{Y}$.

Proof. Let $a, b \in \mathfrak{Y}$ with $\gamma_0 \in \mathfrak{h}^{-1}(a)$ and $\omega_0 \in \mathfrak{h}^{-1}(b)$. Then

$$\mu_{\varsigma_{(T,I)}}(\gamma_0) = \sup_{\mathfrak{t} \in \mathfrak{h}^{-1}(a)} \mu_{\varsigma_{(T,I)}}(\mathfrak{t}), \mu_{\varsigma_{(T,I)}}(\omega_0) = \sup_{\mathfrak{t} \in \mathfrak{h}^{-1}(b)} \mu_{\varsigma_{(T,I)}}(\mathfrak{t}),$$

$$\mu_{\varsigma_F}(\gamma_0) = \inf_{\mathfrak{t} \in \mathfrak{h}^{-1}(a)} \mu_{\varsigma_F}(\mathfrak{t}), \mu_{\varsigma_F}(\omega_0) = \inf_{\mathfrak{t} \in \mathfrak{h}^{-1}(b)} \mu_{\varsigma_F}(\mathfrak{t}),$$

$$\vartheta_{\varsigma_T}(\gamma_0) = \inf_{\mathfrak{t} \in \mathfrak{h}^{-1}(a)} \vartheta_{\varsigma_T}(\mathfrak{t}), \vartheta_{\varsigma_T}(\omega_0) = \inf_{\mathfrak{t} \in \mathfrak{h}^{-1}(b)} \vartheta_{\varsigma_T}(\mathfrak{t}),$$

$$\vartheta_{\varsigma_F}(\gamma_0) = \sup_{\mathfrak{t} \in \mathfrak{h}^{-1}(a)} \vartheta_{\varsigma_F}(\mathfrak{t}), \vartheta_{\varsigma_F}(\omega_0) = \sup_{\mathfrak{t} \in \mathfrak{h}^{-1}(b)} \vartheta_{\varsigma_F}(\mathfrak{t}).$$

Now,

$$\begin{aligned} \mu^{\mathfrak{h}}_{\varsigma_{(T,I)}}(0') &= \sup_{\mathfrak{t} \in \mathfrak{h}^{-1}(0')} \mu_{\varsigma_{(T,I)}}(\mathfrak{t}) \geq \mu_{\varsigma_{(T,I)}}(0) \geq \mu_{\varsigma_{(T,I)}}(\gamma_0) \\ &= \sup_{\mathfrak{t} \in \mathfrak{h}^{-1}(a)} \mu_{\varsigma_{(T,I)}}(\mathfrak{t}) = \mu^{\mathfrak{h}}_{\varsigma_T}(a), \\ \mu^{\mathfrak{h}}_{\varsigma_F}(0') &= \inf_{\mathfrak{t} \in \mathfrak{h}^{-1}(0')} \mu_{\varsigma_F}(\mathfrak{t}) \leq \mu_{\varsigma_F}(0) \leq \mu_{\varsigma_F}(\gamma_0) \\ &= \inf_{\mathfrak{t} \in \mathfrak{h}^{-1}(a)} \mu_{\varsigma_F}(\mathfrak{t}) = \mu^{\mathfrak{h}}_{\varsigma_F}(a), \\ \vartheta^{\mathfrak{h}}_{\varsigma_{(T,I)}}(0') &= \inf_{\mathfrak{t} \in \mathfrak{h}^{-1}(0')} \vartheta_{\varsigma_{(T,I)}}(\mathfrak{t}) \leq \vartheta_{\varsigma_{(T,I)}}(0) \leq \vartheta_{\varsigma_{(T,I)}}(\gamma_0) \end{aligned}$$

$$= \inf_{\mathfrak{t}\in\mathfrak{h}^{-1}(a)} \vartheta_{\varsigma_{(T,I)}}(\mathfrak{t}) = \vartheta^{\mathfrak{h}}_{\varsigma_T}(a),$$

$$\vartheta^{\mathfrak{h}}_{\varsigma_F}(0') = \sup_{\mathfrak{t}\in\mathfrak{h}^{-1}(0')} \vartheta_{\varsigma_F}(\mathfrak{t}) \geq \vartheta_{\varsigma_F}(0) \geq \vartheta_{\varsigma_F}(\gamma_0)$$

$$= \sup_{\mathfrak{t}\in\mathfrak{h}^{-1}(a)} \vartheta_{\varsigma_F}(\mathfrak{t}) = \vartheta^{\mathfrak{h}}_{\varsigma_F}(a),$$

$$\begin{aligned}
\min\{\mu^{\mathfrak{h}}_{\varsigma_{(T,I)}}(a*b), \mu^{\mathfrak{h}}_{\varsigma_{(T,I)}}(b)\} &= \min\{\sup_{\mathfrak{t}\in\mathfrak{h}^{-1}(a*b)} \mu_{\varsigma_{(T,I)}}(\mathfrak{t}), \sup_{\mathfrak{t}\in\mathfrak{h}^{-1}(b)} \mu_{\varsigma_{(T,I)}}(\mathfrak{t})\} \\
&\leq \min\{\mu_{\varsigma_{(T,I)}}(\gamma_0 * \omega_0), \mu_{\varsigma_{(T,I)}}(\omega_0)\} \\
&\leq \mu_{\varsigma_{(T,I)}}(\gamma_0) \\
&= \sup_{\mathfrak{t}\in\mathfrak{h}^{-1}(a)} \mu_{\varsigma_{(T,I)}}(\mathfrak{t}) \\
&= \mu^{\mathfrak{h}}_{\varsigma_{(T,I)}}(a) \\
\Rightarrow \mu^{\mathfrak{h}}_{\varsigma_{(T,I)}}(a) &\geq \min\{\mu^{\mathfrak{h}}_{\varsigma_{(T,I)}}(a*b), \mu^{\mathfrak{h}}_{\varsigma_T}(b)\},
\end{aligned}$$

$$\begin{aligned}
\max\{\mu^{\mathfrak{h}}_{\varsigma_F}(a*b), \mu^{\mathfrak{h}}_{\varsigma_F}(b)\} &= \max\{\inf_{\mathfrak{t}\in\mathfrak{h}^{-1}(a*b)} \mu_{\varsigma_F}(\mathfrak{t}), \inf_{\mathfrak{t}\in\mathfrak{h}^{-1}(b)} \mu_{\varsigma_F}(\mathfrak{t})\} \\
&\geq \max\{\mu_{\varsigma_F}(\gamma_0 * \omega_0), \mu_{\varsigma_F}(\omega_0)\} \\
&\geq \mu_{\varsigma_F}(\gamma_0) \\
&= \inf_{\mathfrak{t}\in\mathfrak{h}^{-1}(a)} \mu_{\varsigma_F}(\mathfrak{t}) \\
&= \mu^{\mathfrak{h}}_{\varsigma_F}(a) \\
\Rightarrow \mu^{\mathfrak{h}}_{\varsigma_F}(a) &\leq \max\{\mu^{\mathfrak{h}}_{\varsigma_F}(a*b), \mu^{\mathfrak{h}}_{\varsigma_F}(b)\},
\end{aligned}$$

$$\begin{aligned}
\max\{\vartheta^{\mathfrak{h}}_{\varsigma_{(T,I)}}(a*b), \vartheta^{\mathfrak{h}}_{\varsigma_{(T,I)}}(b)\} &= \max\{\inf_{\mathfrak{t}\in\mathfrak{h}^{-1}(a*b)} \vartheta_{\varsigma_{(T,I)}}(\mathfrak{t}), \inf_{\mathfrak{t}\in\mathfrak{h}^{-1}(b)} \vartheta_{\varsigma_{(T,I)}}(\mathfrak{t})\} \\
&\geq \max\{\vartheta_{\varsigma_{(T,I)}}(\gamma_0 * \omega_0), \vartheta_{\varsigma_{(T,I)}}(\omega_0)\} \\
&\geq \vartheta_{\varsigma_{(T,I)}}(\gamma_0) \\
&= \inf_{\mathfrak{t}\in\mathfrak{h}^{-1}(a)} \vartheta_{\varsigma_{(T,I)}}(\mathfrak{t}) \\
&= \vartheta^{\mathfrak{h}}_{\varsigma_{(T,I)}}(a) \\
\Rightarrow \vartheta^{\mathfrak{h}}_{\varsigma_{(T,I)}}(a) &\leq \max\{\vartheta^{\mathfrak{h}}_{\varsigma_{(T,I)}}(a*b), \vartheta^{\mathfrak{h}}_{\varsigma_T}(b)\},
\end{aligned}$$

$$\begin{aligned}
\max\{\vartheta^{\mathfrak{h}}_{\varsigma_F}(a*b), \vartheta^{\mathfrak{h}}_{\varsigma_F}(b)\} &= \min\{\sup_{\mathfrak{t}\in\mathfrak{h}^{-1}(a*b)} \vartheta_{\varsigma_F}(\mathfrak{t}), \sup_{\mathfrak{t}\in\mathfrak{h}^{-1}(b)} \vartheta_{\varsigma_F}(\mathfrak{t})\} \\
&\leq \min\{\vartheta_{\varsigma_F}(\gamma_0 * \omega_0), \vartheta_{\varsigma_F}(\omega_0)\} \\
&\leq \vartheta_{\varsigma_F}(\gamma_0)
\end{aligned}$$

$$= \sup_{\mathfrak{t} \in \mathfrak{h}^{-1}(a)} \vartheta_{\varsigma_F}(\mathfrak{t})$$
$$= \vartheta^{\mathfrak{h}}_{\varsigma_F}(a)$$
$$\Rightarrow \vartheta^{\mathfrak{h}}_{\varsigma_F}(a) \geq \min\{\vartheta^{\mathfrak{h}}_{\varsigma_F}(a * b), \vartheta^{\mathfrak{h}}_{\varsigma_F}(b)\}.$$

Hence,

$$\mathfrak{h}(\varsigma) = \{\langle \omega, \mu^{\mathfrak{h}}_{\varsigma_{(T,I,F)}}(\omega), \vartheta^{\mathfrak{h}}_{\varsigma_{(T,I,F)}}(\omega)\rangle \mid \omega \in \mathfrak{Y}\},$$

is an intuitionistic neutrosophic $\hat{Z}$-ideal of $\mathfrak{Y}$. □

Theorem 4.2. *Let $\mathfrak{h}$ be a homomorphism from the $\hat{Z}$-algebras $(\mathfrak{M}, *, 0)$ into $(\mathfrak{Y}, *, 0)$ and $\varsigma_{(T,I,F)} = \{\langle \omega, \mu_{\varsigma_{(T,I,F)}}(\omega), \vartheta_{\varsigma_{(T,I,F)}}(\omega)\rangle \mid \omega \in \mathfrak{Y}\}$ be an intuitionistic neutrosophic $\hat{Z}$-ideal of $\mathfrak{Y}$. Then, the inverse image of ς,*

$$\mathfrak{h}^{-1}(\varsigma) = \{\langle \gamma, \mu^{\mathfrak{h}^{-1}}_{\varsigma_{(T,I,F)}}(\gamma), \vartheta^{\mathfrak{h}^{-1}}_{\varsigma_{(T,I,F)}}(\gamma)\rangle \mid \gamma \in \mathfrak{M}\}$$

is an intuitionistic neutrosophic $\hat{Z}$-ideal of $\mathfrak{M}$.

Proof. Let $\gamma, \omega \in \mathfrak{M}$. Then

$$\mu^{\mathfrak{h}^{-1}}_{\varsigma_{(T,I)}}(0) = \mu_{\varsigma_{(T,I)}}(\mathfrak{h}(0)) \geq \mu_{\varsigma_{(T,I)}}(\mathfrak{h}(\gamma)) = \mu^{\mathfrak{h}^{-1}}_{\varsigma_T}(\gamma),$$
$$\mu^{\mathfrak{h}^{-1}}_{\varsigma_F}(0) = \mu_{\varsigma_F}(\mathfrak{h}(0)) \leq \mu_{\varsigma_F}(\mathfrak{h}(\gamma)) = \mu^{\mathfrak{h}^{-1}}_{\varsigma_F}(\gamma),$$
$$\vartheta^{\mathfrak{h}^{-1}}_{\varsigma_{(T,I)}}(0) = \vartheta_{\varsigma_T}(\mathfrak{h}(0)) \leq \vartheta_{\varsigma_T}(\mathfrak{h}(\gamma)) = \vartheta^{\mathfrak{h}^{-1}}_{\varsigma_T}(\gamma),$$
$$\vartheta^{\mathfrak{h}^{-1}}_{\varsigma_F}(0) = \vartheta_{\varsigma_F}(\mathfrak{h}(0)) \geq \vartheta_{\varsigma_F}(\mathfrak{h}(\gamma)) = \vartheta^{\mathfrak{h}^{-1}}_{\varsigma_F}(\gamma),$$

$$\mu^{\mathfrak{h}^{-1}}_{\varsigma_{(T,I)}}(\gamma) = \mu_{\varsigma_{(T,I)}}(\mathfrak{h}(\gamma))$$
$$\geq \min\{\mu_{\varsigma_{(T,I)}}(\mathfrak{h}(\gamma) * \mathfrak{h}(\omega)), \mu_{\varsigma_{(T,I)}}(\mathfrak{h}(\omega))\}$$
$$= \min\{\mu_{\varsigma_{(T,I)}}(\mathfrak{h}(\gamma * \omega)), \mu_{\varsigma_{(T,I)}}(\mathfrak{h}(\omega))\}$$
$$= \min\{\mu^{\mathfrak{h}^{-1}}_{\varsigma_{(T,I)}}(\gamma * \omega), \mu^{\mathfrak{h}^{-1}}_{\varsigma_{(T,I)}}(\omega)\},$$

$$\mu^{\mathfrak{h}^{-1}}_{\varsigma_F}(\gamma) = \mu_{\varsigma_F}(\mathfrak{h}(\gamma))$$
$$\leq \max\{\mu_{\varsigma_F}(\mathfrak{h}(\gamma) * \mathfrak{h}(\omega)), \mu_{\varsigma_F}(\mathfrak{h}(\omega))\}$$
$$= \max\{\mu_{\varsigma_F}(\mathfrak{h}(\gamma * \omega)), \mu_{\varsigma_F}(\mathfrak{h}(\omega))\}$$
$$= \max\{\mu^{\mathfrak{h}^{-1}}_{\varsigma_F}(\gamma * \omega), \mu^{\mathfrak{h}^{-1}}_{\varsigma_F}(\omega)\},$$

$$\begin{aligned}\vartheta^{\mathfrak{h}^{-1}}_{\varsigma_{(T,I)}}(\gamma) &= \vartheta_{\varsigma_{(T,I)}}(\mathfrak{h}(\gamma))\\ &\leq \max\{\vartheta_{\varsigma_{(T,I)}}(\mathfrak{h}(\gamma) * \mathfrak{h}(\omega)), \vartheta_{\varsigma_{(T,I)}}(\mathfrak{h}(\omega))\}\\ &= \max\{\vartheta_{\varsigma_{(T,I)}}(\mathfrak{h}(\gamma * \omega)), \vartheta_{\varsigma_{(T,I)}}(\mathfrak{h}(\omega))\}\\ &= \max\{\vartheta^{\mathfrak{h}^{-1}}_{\varsigma_{(T,I)}}(\gamma * \omega), \vartheta^{\mathfrak{h}^{-1}}_{\varsigma_{(T,I)}}(\omega)\},\end{aligned}$$

$$\begin{aligned}\vartheta^{\mathfrak{h}^{-1}}_{\varsigma_F}(\gamma) &= \vartheta_{\varsigma_F}(\mathfrak{h}(\gamma))\\ &\geq \min\{\vartheta_{\varsigma_F}(\mathfrak{h}(\gamma) * \mathfrak{h}(\omega)), \vartheta_{\varsigma_F}(\mathfrak{h}(\omega))\}\\ &= \min\{\vartheta_{\varsigma_F}(\mathfrak{h}(\gamma * \omega)), \vartheta_{\varsigma_F}(\mathfrak{h}(\omega))\}\\ &= \min\{\vartheta^{\mathfrak{h}^{-1}}_{\varsigma_F}(\gamma * \omega), \vartheta^{\mathfrak{h}^{-1}}_{\varsigma_F}(\omega)\}.\end{aligned}$$

Hence,

$$\mathfrak{h}^{-1}(\varsigma) = \{\langle \gamma, \mu^{\mathfrak{h}^{-1}}_{\varsigma_{(T,I,F)}}(\gamma), \vartheta^{\mathfrak{h}^{-1}}_{\varsigma_{(T,I,F)}}(\gamma)\rangle \mid \gamma \in \mathfrak{M}\}$$

is an intuitionistic neutrosophic $\hat{Z}$-ideal of $\mathfrak{M}$. □

Theorem 4.3. *Let $\mathfrak{h} : (\mathfrak{M}, *, 0) \to (\mathfrak{Y}, *', 0')$ be a $\hat{Z}$-epimorphism of $\hat{Z}$-algebras. Let $\varsigma_{(T,I,F)}$ be an intuitionistic neutrosophic set of $\mathfrak{Y}$. If $\mathfrak{h}^{-1}(\varsigma_{(T,I,F)})$ is an intuitionistic neutrosophic $\hat{Z}$-ideal of $\mathfrak{M}$, then $\varsigma_{(T,I,F)}$ is an intuitionistic neutrosophic $\hat{Z}$-ideal of $\mathfrak{Y}$.*

Proof. Let $\omega \in \mathfrak{Y}$. Then there exists $\gamma \in \mathfrak{M}$ with $\mathfrak{h}(\gamma) = \omega$. Thus,

$$\begin{aligned}&\mu_{\varsigma_{(T,I)}}(\omega) = \mu_{\varsigma_{(T,I)}}(\mathfrak{h}(\gamma)) = \mu^{\mathfrak{h}^{-1}}_{\varsigma_{(T,I)}}(\gamma) \leq \mu^{\mathfrak{h}^{-1}}_{\varsigma_{(T,I)}}(0) = \mu_{\varsigma_{(T,I)}}(\mathfrak{h}(0)) = \mu_{\varsigma_{(T,I)}}(0')\\ &\Rightarrow \mu_{\varsigma_{(T,I)}}(0') \geq \mu_{\varsigma_{(T,I)}}(\omega),\\ &\mu_{\varsigma_F}(\omega) = \mu_{\varsigma_F}(\mathfrak{h}(\gamma)) = \mu^{\mathfrak{h}^{-1}}_{\varsigma_F}(\gamma) \geq \mu^{\mathfrak{h}^{-1}}_{\varsigma_F}(0) = \mu_{\varsigma_F}(\mathfrak{h}(0)) = \mu_{\varsigma_F}(0')\\ &\Rightarrow \mu_{\varsigma_F}(0') \leq \mu_{\varsigma_F}(\omega),\end{aligned}$$

$$\begin{aligned}&\vartheta_{\varsigma_{(T,I)}}(\omega) = \vartheta_{\varsigma_{(T,I)}}(\mathfrak{h}(\gamma)) = \vartheta^{\mathfrak{h}^{-1}}_{\varsigma_{(T,I)}}(\gamma) \leq \vartheta^{\mathfrak{h}^{-1}}_{\varsigma_T}(0) = \vartheta_{\varsigma_T}(\mathfrak{h}(0)) = \vartheta_{\varsigma_T}(0')\\ &\Rightarrow \vartheta_{\varsigma_T}(0') \geq \vartheta_{\varsigma_T}(\omega),\\ &\vartheta_{\varsigma_F}(\omega) = \vartheta_{\varsigma_F}(\mathfrak{h}(\gamma)) = \vartheta^{\mathfrak{h}^{-1}}_{\varsigma_F}(\gamma) \geq \vartheta^{\mathfrak{h}^{-1}}_{\varsigma_F}(0) = \vartheta_{\varsigma_F}(\mathfrak{h}(0)) = \vartheta_{\varsigma_F}(0')\\ &\Rightarrow \vartheta_{\varsigma_F}(0') \leq \vartheta_{\varsigma_F}(\omega),\end{aligned}$$

$$\begin{aligned}\mu_{\varsigma_{(T,I)}}(\mathfrak{h}(a)) &= \mu^{\mathfrak{h}^{-1}}_{\varsigma_{(T,I)}}(a)\\ &\geq \min\{\mu^{\mathfrak{h}^{-1}}_{\varsigma_{(T,I)}}(a * b), \mu^{\mathfrak{h}^{-1}}_{\varsigma_{(T,I)}}(b)\}\\ &= \min\{\mu_{\varsigma_{(T,I)}}(\mathfrak{h}(a * b)), \mu_{\varsigma_{(T,I)}}(\mathfrak{h}(b))\}\\ &= \min\{\mu_{\varsigma_{(T,I)}}(\mathfrak{h}(a) *' \mathfrak{h}(b)), \mu_{\varsigma_{(T,I)}}(\mathfrak{h}(b))\}\end{aligned}$$

$$= \min\{\mu_{\varsigma_{(T,I)}}(a *' b), \mu_{\varsigma_T}(\mathfrak{h}(b))\},$$

$$\begin{aligned}
\mu_{\varsigma_F}(\mathfrak{h}(a)) &= \mu_{\varsigma_F}^{\mathfrak{h}^{-1}}(a) \\
&\leq \max\{\mu_{\varsigma_F}^{\mathfrak{h}^{-1}}(a * b), \mu_{\varsigma_F}^{\mathfrak{h}^{-1}}(b)\} \\
&= \max\{\mu_{\varsigma_F}(\mathfrak{h}(a * b)), \mu_{\varsigma_F}(\mathfrak{h}(b))\} \\
&= \max\{\mu_{\varsigma_F}(\mathfrak{h}(a) *' \mathfrak{h}(b)), \mu_{\varsigma_F}(\mathfrak{h}(b))\} \\
&= \max\{\mu_{\varsigma_F}(a *' b), \mu_{\varsigma_F}(\mathfrak{h}(b))\},
\end{aligned}$$

$$\begin{aligned}
\vartheta_{\varsigma_{(T,I)}}(\mathfrak{h}(a)) &= \vartheta_{\varsigma_{(T,I)}}^{\mathfrak{h}^{-1}}(a) \\
&\leq \max\{\vartheta_{\varsigma_{(T,I)}}^{\mathfrak{h}^{-1}}(a * b), \vartheta_{\varsigma_{(T,I)}}^{\mathfrak{h}^{-1}}(b)\} \\
&= \max\{\vartheta_{\varsigma_{(T,I)}}(\mathfrak{h}(a * b)), \vartheta_{\varsigma_{(T,I)}}(\mathfrak{h}(b))\} \\
&= \max\{\vartheta_{\varsigma_{(T,I)}}(\mathfrak{h}(a) *' \mathfrak{h}(b)), \vartheta_{\varsigma_{(T,I)}}(\mathfrak{h}(b))\} \\
&= \max\{\vartheta_{\varsigma_{(T,I)}}(a *' b), \vartheta_{\varsigma_T}(\mathfrak{h}(b))\},
\end{aligned}$$

$$\begin{aligned}
\vartheta_{\varsigma_F}(\mathfrak{h}(a)) &= \vartheta_{\varsigma_F}^{\mathfrak{h}^{-1}}(a) \\
&\geq \min\{\vartheta_{\varsigma_F}^{\mathfrak{h}^{-1}}(a * b), \vartheta_{\varsigma_F}^{\mathfrak{h}^{-1}}(b)\} \\
&= \min\{\vartheta_{\varsigma_F}(\mathfrak{h}(a * b)), \vartheta_{\varsigma_F}(\mathfrak{h}(b))\} \\
&= \min\{\vartheta_{\varsigma_F}(\mathfrak{h}(a) *' \mathfrak{h}(b)), \vartheta_{\varsigma_F}(\mathfrak{h}(b))\} \\
&= \min\{\vartheta_{\varsigma_F}(a *' b), \vartheta_{\varsigma_F}(\mathfrak{h}(b))\}.
\end{aligned}$$

Hence, $\varsigma_{(T,I,F)}$ is an intuitionistic neutrosophic $\hat{Z}$-ideal of $\mathfrak{Y}$. □

Theorem 4.4. *Let $\mathfrak{h}$ be a $\hat{Z}$-endomorphism of a $\hat{Z}$-algebra $(\mathfrak{M}, *, 0)$. If $\varsigma_{(T,I,F)}$ is an intuitionistic neutrosophic $\hat{Z}$-ideal of $\mathfrak{M}$, then $\varsigma_{(T,I,F)}^{\mathfrak{h}}$ is also an intuitionistic neutrosophic $\hat{Z}$-ideal of $\mathfrak{M}$.*

Proof. Let $\gamma, \omega \in \mathfrak{M}$. Then

$$\begin{aligned}
\mu_{\varsigma_{(T,I)}}^{\mathfrak{h}}(0) &= \mu_{\varsigma_{(T,I)}}(\mathfrak{h}(0)) \geq \mu_{\varsigma_{(T,I)}}(\mathfrak{h}(\gamma)) = \mu_{\varsigma_{(T,I)}}^{\mathfrak{h}}(\gamma), \\
\mu_{\varsigma_F}^{\mathfrak{h}}(0) &= \mu_{\varsigma_F}(\mathfrak{h}(0)) \leq \mu_{\varsigma_F}(\mathfrak{h}(\gamma)) = \mu_{\varsigma_F}^{\mathfrak{h}}(\gamma), \\
\vartheta_{\varsigma_T}^{\mathfrak{h}}(0) &= \vartheta_{\varsigma_T}(\mathfrak{h}(0)) \leq \vartheta_{\varsigma_T}(\mathfrak{h}(\gamma)) = \vartheta_{\varsigma_T}^{\mathfrak{h}}(\gamma), \\
\vartheta_{\varsigma_F}^{\mathfrak{h}}(0) &= \vartheta_{\varsigma_F}(\mathfrak{h}(0)) \geq \vartheta_{\varsigma_F}(\mathfrak{h}(\gamma)) = \vartheta_{\varsigma_F}^{\mathfrak{h}}(\gamma),
\end{aligned}$$

$$\begin{aligned}
\mu_{\varsigma_{(T,I)}}^{\mathfrak{h}}(\gamma) &= \mu_{\varsigma_{(T,I)}}(\mathfrak{h}(\gamma)) \\
&\geq \min\{\mu_{\varsigma_{(T,I)}}(\mathfrak{h}(\gamma) * \mathfrak{h}(\omega)), \mu_{\varsigma_{(T,I)}}(\mathfrak{h}(\omega))\}
\end{aligned}$$

$$= \min\{\mu_{\varsigma_{(T,I)}}(\mathfrak{h}(\gamma * \omega)), \mu_{\varsigma_{(T,I)}}(\mathfrak{h}(\omega))\}$$
$$= \min\{\mu^{\mathfrak{h}}_{\varsigma_{(T,I)}}(\gamma * \omega), \mu^{\mathfrak{h}}_{\varsigma_T}(\omega)\},$$

$$\begin{aligned}\mu^{\mathfrak{h}}_{\varsigma_F}(\gamma) &= \mu_{\varsigma_F}(\mathfrak{h}(\gamma))\\ &\leq \max\{\mu_{\varsigma_F}(\mathfrak{h}(\gamma) * \mathfrak{h}(\omega)), \mu_{\varsigma_F}(\mathfrak{h}(\omega))\}\\ &= \max\{\mu_{\varsigma_F}(\mathfrak{h}(\gamma * \omega)), \mu_{\varsigma_F}(\mathfrak{h}(\omega))\}\\ &= \max\{\mu^{\mathfrak{h}}_{\varsigma_F}(\gamma * \omega), \mu^{\mathfrak{h}}_{\varsigma_F}(\omega)\},\end{aligned}$$

$$\begin{aligned}\vartheta^{\mathfrak{h}}_{\varsigma_{(T,I)}}(\gamma) &= \vartheta_{\varsigma_{(T,I)}}(\mathfrak{h}(\gamma))\\ &\leq \max\{\vartheta_{\varsigma_{(T,I)}}(\mathfrak{h}(\gamma) * \mathfrak{h}(\omega)), \vartheta_{\varsigma_{(T,I)}}(\mathfrak{h}(\omega))\}\\ &= \max\{\vartheta_{\varsigma_{(T,I)}}(\mathfrak{h}(\gamma * \omega)), \vartheta_{\varsigma_{(T,I)}}(\mathfrak{h}(\omega))\}\\ &= \max\{\vartheta^{\mathfrak{h}}_{\varsigma_{(T,I)}}(\gamma * \omega), \vartheta^{\mathfrak{h}}_{\varsigma_{(T,I)}}(\omega)\},\end{aligned}$$

$$\begin{aligned}\vartheta^{\mathfrak{h}}_{\varsigma_F}(\gamma) &= \vartheta_{\varsigma_F}(\mathfrak{h}(\gamma))\\ &\geq \min\{\vartheta_{\varsigma_F}(\mathfrak{h}(\gamma) * \mathfrak{h}(\omega)), \vartheta_{\varsigma_F}(\mathfrak{h}(\omega))\}\\ &= \min\{\vartheta_{\varsigma_F}(\mathfrak{h}(\gamma * \omega)), \vartheta_{\varsigma_F}(\mathfrak{h}(\omega))\}\\ &= \min\{\vartheta^{\mathfrak{h}}_{\varsigma_F}(\gamma * \omega), \vartheta^{\mathfrak{h}}_{\varsigma_F}(\omega)\}.\end{aligned}$$

Hence, $\varsigma^{\mathfrak{h}}_{(T,I,F)}$ is an intuitionistic neutrosophic $\hat{Z}$-ideal of $\mathfrak{M}$. □

5. Preservation of intuitionistic neutrosophic $\hat{Z}$-ideals under Cartesian products

Cartesian products provide a mechanism to construct new algebraic structures from existing ones, enabling a broader analysis of ideal behavior across combined systems. In this section, we examine the preservation of intuitionistic neutrosophic $\hat{Z}$-ideals under Cartesian products of $\hat{Z}$-algebras. We establish conditions under which the component-wise definition of these ideals ensures that their Cartesian product also forms an intuitionistic neutrosophic $\hat{Z}$-ideal. This investigation demonstrates the stability and scalability of the proposed ideal framework within product algebraic systems.

Definition 5.1. *Let*

$$\varsigma = \{\langle \gamma, \mu_{\varsigma_T}(\gamma), \mu_{\varsigma_I}(\gamma), \mu_{\varsigma_F}(\gamma), \vartheta_{\varsigma_T}(\gamma), \vartheta_{\varsigma_I}(\gamma), \vartheta_{\varsigma_F}(\gamma)\rangle \mid \gamma \in \mathfrak{M}\}$$

and

$$\xi = \{\langle \omega, \mu_{\xi_{(T,I)}}(\omega), \mu_{\xi_I}(\omega), \mu_{\xi_F}(\omega), \vartheta_{\xi_{(T,I)}}(\omega), \vartheta_{\xi_I}(\omega), \vartheta_{\xi_F}(\omega)\rangle \mid \omega \in \mathfrak{Y}\}$$

be intuitionistic neutrosophic sets on $\mathfrak{M}$ and $\mathfrak{Y}$, respectively. The Cartesian product of ς and ξ is defined as

$$\varsigma \times \xi = \{\langle(\gamma, \omega), \mu_{(\varsigma\times\xi)_{(T,I,F)}}(\gamma, \omega), \vartheta_{(\varsigma\times\xi)_{(T,I,F)}}(\gamma, \omega)\rangle \mid \gamma \in \mathfrak{M}, \omega \in \mathfrak{Y}\},$$

where $\mu_{(\varsigma\times\xi)_{(T,I,F)}} : \mathfrak{M} \times \mathfrak{Y} \to [0, 1]$ and $\vartheta_{(\varsigma\times\xi)_{(T,I,F)}} : \mathfrak{M} \times \mathfrak{Y} \to [0, 1]$ such that

$$\begin{aligned}
\mu_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(0, 0) &= \mu_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(\gamma_1, \gamma_2),\\
\vartheta_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(0, 0) &= \vartheta_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(\gamma_1, \gamma_2),\\
\mu_{(\varsigma_F\times\xi_F)}(0, 0) &= \mu_{(\varsigma_F\times\xi_F)}(\gamma_1, \gamma_2),\\
\vartheta_{(\varsigma_F\times\xi_F)}(0, 0) &= \vartheta_{(\varsigma_F\times\xi_F)}(\gamma_1, \gamma_2),\\
\mu_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(\gamma, \omega) &= \min\{\mu_{\varsigma_{(T,I)}}(\gamma * \omega), \mu_{\xi_{(T,I)}}(\omega)\},\\
\vartheta_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(\gamma, \omega) &= \max\{\vartheta_{\varsigma_{(T,I)}}(\gamma * \omega), \vartheta_{\xi_T}(\omega)\},\\
\mu_{(\varsigma_F\times\xi_F)}(\gamma, \omega) &= \max\{\mu_{\varsigma_F}(\gamma * \omega), \mu_{\xi_F}(\omega)\},\\
\vartheta_{(\varsigma_F\times\xi_F)}(\gamma, \omega) &= \min\{\vartheta_{\varsigma_F}(\gamma * \omega), \vartheta_{\xi_F}(\omega)\}.
\end{aligned}$$

Theorem 5.1. *Let $\varsigma_{(T,I,F)}$ and $\xi_{(T,I,F)}$ be two intuitionistic neutrosophic $\hat{Z}$-ideals in a $\hat{Z}$-algebra $\mathfrak{M}$. Then $\varsigma_{(T,I,F)} \times \xi_{(T,I,F)}$ is an intuitionistic neutrosophic $\hat{Z}$-ideal of $\mathfrak{M} \times \mathfrak{M}$.*

Proof. Take $(\gamma_1, \gamma_2) \in \mathfrak{M} \times \mathfrak{M}$. Then

$$\begin{aligned}
\mu_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(0, 0) &= \min\{\mu_{\varsigma_{(T,I)}}(0), \mu_{\xi_{(T,I)}}(0)\}\\
&\geq \min\{\mu_{\varsigma_{(T,I)}}(\gamma_1), \mu_{\xi_{(T,I)}}(\gamma_2)\}\\
&= \mu_{(\varsigma_{(T,I)}\times\xi_T)}(\gamma_1, \gamma_2),\\
\mu_{(\varsigma_F\times\xi_F)}(0, 0) &= \max\{\mu_{\varsigma_F}(0), \mu_{\xi_F}(0)\}\\
&\leq \max\{\mu_{\varsigma_F}(\gamma_1), \mu_{\xi_F}(\gamma_2)\}\\
&= \mu_{(\varsigma_F\times\xi_F)}(\gamma_1, \gamma_2),\\
\vartheta_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(0, 0) &= \max\{\vartheta_{\varsigma_{(T,I)}}(0), \vartheta_{\xi_{(T,I)}}(0)\}\\
&\leq \max\{\vartheta_{\varsigma_{(T,I)}}(\gamma_1), \vartheta_{\xi_{(T,I)}}(\gamma_2)\}\\
&= \vartheta_{(\varsigma_{(T,I)}\times\xi_T)}(\gamma_1, \gamma_2),\\
\vartheta_{(\varsigma_F\times\xi_F)}(0, 0) &= \min\{\vartheta_{\varsigma_F}(0), \vartheta_{\xi_F}(0)\}\\
&\geq \min\{\vartheta_{\varsigma_F}(\gamma_1), \vartheta_{\xi_F}(\gamma_2)\}\\
&= \vartheta_{(\varsigma_F\times\xi_F)}(\gamma_1, \gamma_2).
\end{aligned}$$

Now, take $(\gamma_1, \gamma_2), (\omega_1, \omega_2) \in \mathfrak{M} \times \mathfrak{M}$. Then

$$\mu_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(\gamma_1, \gamma_2)$$

$$= \min\{\mu_{\varsigma_{(T,I)}}(\gamma_1), \mu_{\xi_{(T,I)}}(\gamma_2)\}$$

$$\geq \min\{\min\{\mu_{\varsigma_{(T,I)}}(\gamma_1 * \omega_1), \mu_{\varsigma_{(T,I)}}(\omega_1)\}, \min\{\mu_{\xi_{(T,I)}}(\gamma_2 * \omega_2), \mu_{\xi_{(T,I)}}(\omega_2)\}\}$$

$$= \min\{\min\{\mu_{\varsigma_{(T,I)}}(\gamma_1 * \omega_1), \mu_{\xi_{(T,I)}}(\gamma_2 * \omega_2)\}, \min\{\mu_{\varsigma_{(T,I)}}(\omega_1), \mu_{\xi_{(T,I)}}(\omega_2)\}\}$$

$$= \min\{\mu_{(\varsigma_{(T,I)} \times \xi_{(T,I)})}(\gamma_1 * \omega_1, \gamma_2 * \omega_2), \mu_{(\varsigma_{(T,I)} \times \xi_{(T,I)})}(\omega_1, \omega_2)\}$$

$$= \min\{\mu_{(\varsigma_{(T,I)} \times \xi_{(T,I)})}((\gamma_1, \gamma_2) * (\omega_1, \omega_2)), \mu_{(\varsigma_{(T,I)} \times \xi_T)}(\omega_1, \omega_2)\},$$

$$\mu_{(\varsigma_F \times \xi_F)}(\gamma_1, \gamma_2)$$

$$= \max\{\mu_{\varsigma_F}(\gamma_1), \mu_{\xi_F}(\gamma_2)\}$$

$$\leq \max\{\max\{\mu_{\varsigma_F}(\gamma_1 * \omega_1), \mu_{\varsigma_F}(\omega_1)\}, \max\{\mu_{\xi_F}(\gamma_2 * \omega_2), \mu_{\xi_F}(\omega_2)\}\}$$

$$= \max\{\max\{\mu_{\varsigma_F}(\gamma_1 * \omega_1), \mu_{\xi_F}(\gamma_2 * \omega_2)\}, \max\{\mu_{\varsigma_F}(\omega_1), \mu_{\xi_F}(\omega_2)\}\}$$

$$= \max\{\mu_{(\varsigma_F \times \xi_F)}(\gamma_1 * \omega_1, \gamma_2 * \omega_2), \mu_{(\varsigma_F \times \xi_F)}(\omega_1, \omega_2)\}$$

$$= \max\{\mu_{(\varsigma_F \times \xi_F)}((\gamma_1, \gamma_2) * (\omega_1, \omega_2)), \mu_{(\varsigma_F \times \xi_F)}(\omega_1, \omega_2)\},$$

$$\vartheta_{(\varsigma_{(T,I)} \times \xi_{(T,I)})}(\gamma_1, \gamma_2)$$

$$= \max\{\vartheta_{\varsigma_{(T,I)}}(\gamma_1), \vartheta_{\xi_{(T,I)}}(\gamma_2)\}$$

$$\leq \max\{\max\{\vartheta_{\varsigma_{(T,I)}}(\gamma_1 * \omega_1), \vartheta_{\varsigma_{(T,I)}}(\omega_1)\}, \max\{\vartheta_{\xi_{(T,I)}}(\gamma_2 * \omega_2), \vartheta_{\xi_{(T,I)}}(\omega_2)\}\}$$

$$= \max\{\max\{\vartheta_{\varsigma_{(T,I)}}(\gamma_1 * \omega_1), \vartheta_{\xi_{(T,I)}}(\gamma_2 * \omega_2)\}, \max\{\vartheta_{\varsigma_{(T,I)}}(\omega_1), \vartheta_{\xi_{(T,I)}}(\omega_2)\}\}$$

$$= \max\{\vartheta_{(\varsigma_{(T,I)} \times \xi_{(T,I)})}(\gamma_1 * \omega_1, \gamma_2 * \omega_2), \vartheta_{(\varsigma_{(T,I)} \times \xi_{(T,I)})}(\omega_1, \omega_2)\}$$

$$= \max\{\vartheta_{(\varsigma_{(T,I)} \times \xi_{(T,I)})}((\gamma_1, \gamma_2) * (\omega_1, \omega_2)), \vartheta_{(\varsigma_{(T,I)} \times \xi_{(T,I)})}(\omega_1, \omega_2)\},$$

$$\vartheta_{(\varsigma_F \times \xi_F)}(\gamma_1, \gamma_2)$$

$$= \min\{\vartheta_{\varsigma_F}(\gamma_1), \vartheta_{\xi_F}(\gamma_2)\}$$

$$\geq \min\{\min\{\vartheta_{\varsigma_F}(\gamma_1 * \omega_1), \vartheta_{\varsigma_F}(\omega_1)\}, \min\{\vartheta_{\xi_F}(\gamma_2 * \omega_2), \vartheta_{\xi_F}(\omega_2)\}\}$$

$$= \min\{\min\{\vartheta_{\varsigma_F}(\gamma_1 * \omega_1), \vartheta_{\xi_F}(\gamma_2 * \omega_2)\}, \min\{\vartheta_{\varsigma_F}(\omega_1), \vartheta_{\xi_F}(\omega_2)\}\}$$

$$= \min\{\vartheta_{(\varsigma_F \times \xi_F)}(\gamma_1 * \omega_1, \gamma_2 * \omega_2), \vartheta_{(\varsigma_F \times \xi_F)}(\omega_1, \omega_2)\}$$

$$= \min\{\vartheta_{(\varsigma_F \times \xi_F)}((\gamma_1, \gamma_2) * (\omega_1, \omega_2)), \vartheta_{(\varsigma_F \times \xi_F)}(\omega_1, \omega_2)\}.$$

Hence, $\varsigma_{(T,I,F)} \times \xi_{(T,I,F)}$ is an intuitionistic neutrosophic $\hat{Z}$-ideal of $\mathfrak{M} \times \mathfrak{M}$. □

Theorem 5.2. *Let $\varsigma_{(T,I,F)}$ and $\xi_{(T,I,F)}$ be two intuitionistic neutrosophic sets in a $\hat{Z}$-algebra $\mathfrak{M}$. If $\varsigma_{(T,I,F)} \times \xi_{(T,I,F)}$ is an intuitionistic neutrosophic $\hat{Z}$-ideal of $\mathfrak{M} \times \mathfrak{M}$, then the following are true: for all $\omega, \gamma \in \mathfrak{M}$,*

$$\mu_{\varsigma_{(T,I)}}(0) \geq \mu_{\xi_{(T,I)}}(\omega), \quad \mu_{\xi_{(T,I)}}(0) \geq \mu_{\varsigma_T}(\gamma),$$
$$\mu_{\varsigma_F}(0) \leq \mu_{\xi_F}(\omega), \quad \mu_{\xi_F}(0) \leq \mu_{\varsigma_F}(\gamma),$$
$$\vartheta_{\varsigma_{(T,I)}}(0) \leq \vartheta_{\xi_{(T,I)}}(\omega), \quad \vartheta_{\xi_{(T,I)}}(0) \leq \vartheta_{\varsigma_{(T,I)}}(\gamma),$$
$$\vartheta_{\varsigma_F}(0) \geq \vartheta_{\xi_F}(\omega), \quad \vartheta_{\xi_F}(0) \geq \vartheta_{\varsigma_F}(\gamma).$$

Proof. Assume that $\mu_{\varsigma_{(T,I)}}(\gamma) > \mu_{\xi_{(T,I)}}(0)$ for some $\gamma \in \mathfrak{M}$. Then

$$\begin{aligned}\mu_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(\gamma,0) &= \min\{\mu_{\varsigma_{(T,I)}}(\gamma), \mu_{\xi_{(T,I)}}(0)\}\\ &> \min\{\mu_{\varsigma_{(T,I)}}(0), \mu_{\xi_{(T,I)}}(0)\}\\ &= \mu_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(0,0),\end{aligned}$$

which is a contradiction.

We can prove the remaining cases similarly to the above. □

Theorem 5.3. *Let $\varsigma_{(T,I,F)}$ and $\xi_{(T,I,F)}$ be two intuitionistic neutrosophic sets in a $\hat{Z}$-algebra $\mathfrak{M}$. If $\varsigma_{(T,I,F)} \times \xi_{(T,I,F)}$ is an intuitionistic neutrosophic $\hat{Z}$-ideal of $\mathfrak{M} \times \mathfrak{M}$, then either $\varsigma_{(T,I,F)}$ or $\xi_{(T,I,F)}$ is an intuitionistic neutrosophic $\hat{Z}$-ideal of $\mathfrak{M}$.*

Proof. By Theorem 5.2, we have

(1) $\mu_{\varsigma_{(T,I)}}(0) \geq \mu_{\xi_{(T,I)}}(\omega)$ and $\vartheta_{\varsigma_{(T,I)}}(0) \leq \vartheta_{\xi_T}(\omega)$
(2) $\mu_{\varsigma_F}(0) \leq \mu_{\xi_F}(\omega)$ and $\vartheta_{\varsigma_F}(0) \geq \vartheta_{\xi_F}(\omega)$.

Now, consider the Cartesian product:

$$\mu_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(0,\omega) = \min\{\mu_{\varsigma_{(T,I)}}(0), \mu_{\xi_{(T,I)}}(\omega)\} = \mu_{\xi_{(T,I)}}(\omega),$$
$$\mu_{(\varsigma_F\times\xi_F)}(0,\omega) = \max\{\mu_{\varsigma_F}(0), \mu_{\xi_F}(\omega)\} = \mu_{\xi_F}(\omega),$$
$$\vartheta_{(\varsigma_T\times\xi_T)}(0,\omega) = \max\{\vartheta_{\varsigma_T}(0), \vartheta_{\xi_T}(\omega)\} = \vartheta_{\xi_T}(\omega),$$
$$\vartheta_{(\varsigma_F\times\xi_F)}(0,\omega) = \min\{\vartheta_{\varsigma_F}(0), \vartheta_{\xi_F}(\omega)\} = \vartheta_{\xi_F}(\omega).$$

Since $\varsigma_{(T,I,F)} \times \xi_{(T,I,F)}$ is an intuitionistic neutrosophic $\hat{Z}$-ideal of $\mathfrak{M} \times \mathfrak{M}$, the claim follows:
(i) $\mu_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(\gamma_1,\omega_1) \geq \min\{\mu_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}((\gamma_1,\omega_1)*(\gamma_2,\omega_2)), \mu_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(\gamma_2,\omega_2)\}$.
Now,

$$\mu_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}((\gamma_1,\omega_1)*(\gamma_2,\omega_2)) \geq \min\{\mu_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(\gamma_1,\omega_1), \mu_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(\gamma_2,\omega_2)\},$$
$$\mu_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(\gamma_1*\omega_1, \gamma_2*\omega_2) \geq \min\{\mu_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(\gamma_1,\omega_1), \mu_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(\gamma_2,\omega_2)\}.$$

Putting $\gamma_1 = \gamma_2 = 0$ in the above equations, we obtain

$$\mu_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(0,\omega_1) \geq \min\{\mu_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(0,\omega_1 * \omega_2), \mu_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(0,\omega_2)\},$$
$$\mu_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(0,\omega_1 * \omega_2) \geq \min\{\mu_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(0,\omega_1), \mu_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(0,\omega_2)\},$$
$$\mu_{\xi_{(T,I)}}(\omega_1) \geq \min\{\mu_{\xi_{(T,I)}}(\omega_1 * \omega_2), \mu_{\xi_{(T,I)}}(\omega_2)\},$$
$$\mu_{\xi_{(T,I)}}(\omega_1 * \omega_2) \geq \min\{\mu_{\xi_{(T,I)}}(\omega_1), \mu_{\xi_{(T,I)}}(\omega_2)\}.$$

In other cases,

$$\mu_{(\varsigma_F\times\xi_F)}(\gamma_1,\omega_1) \leq \max\{\mu_{(\varsigma_F\times\xi_F)}((\gamma_1,\omega_1) * (\gamma_2,\omega_2)), \mu_{(\varsigma_F\times\xi_F)}(\gamma_2,\omega_2)\}.$$

Now,

$$\mu_{(\varsigma_F\times\xi_F)}((\gamma_1,\omega_1) * (\gamma_2,\omega_2)) \leq \max\{\mu_{(\varsigma_F\times\xi_F)}(\gamma_1,\omega_1), \mu_{(\varsigma_F\times\xi_F)}(\gamma_2,\omega_2)\},$$
$$\mu_{(\varsigma_F\times\xi_F)}(\gamma_1 * \omega_1, \gamma_2 * \omega_2) \leq \max\{\mu_{(\varsigma_F\times\xi_F)}(\gamma_1,\omega_1), \mu_{(\varsigma_F\times\xi_F)}(\gamma_2,\omega_2)\}.$$

Putting $\gamma_1 = \gamma_2 = 0$ in the above relations, we get

$$\mu_{(\varsigma_F\times\xi_F)}(0,\omega_1) \leq \max\{\mu_{(\varsigma_F\times\xi_F)}(0,\omega_1 * \omega_2), \mu_{(\varsigma_F\times\xi_F)}(0,\omega_2)\},$$
$$\mu_{(\varsigma_F\times\xi_F)}(0,\omega_1 * \omega_2) \leq \max\{\mu_{(\varsigma_F\times\xi_F)}(0,\omega_1), \mu_{(\varsigma_F\times\xi_F)}(0,\omega_2)\},$$
$$\mu_{\xi_F}(\omega_1) \leq \max\{\mu_{\xi_F}(\omega_1 * \omega_2), \mu_{\xi_F}(\omega_2)\},$$
$$\mu_{\xi_F}(\omega_1 * \omega_2) \leq \max\{\mu_{\xi_F}(\omega_1), \mu_{\xi_F}(\omega_2)\}.$$

(ii) For $\vartheta_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}$, we have

$$\vartheta_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(\gamma_1,\omega_1) \leq \max\{\vartheta_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}((\gamma_1,\omega_1) * (\gamma_2,\omega_2)), \vartheta_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(\gamma_2,\omega_2)\},$$
$$\vartheta_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}((\gamma_1,\omega_1) * (\gamma_2,\omega_2)) \leq \max\{\vartheta_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(\gamma_1,\omega_1), \vartheta_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(\gamma_2,\omega_2)\},$$
$$\vartheta_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(\gamma_1 * \omega_1, \gamma_2 * \omega_2) \leq \max\{\vartheta_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(\gamma_1,\omega_1), \vartheta_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(\gamma_2,\omega_2)\}.$$

Putting $\gamma_1 = \gamma_2 = 0$, we obtain

$$\vartheta_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(0,\omega_1) \leq \max\{\vartheta_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(0,\omega_1 * \omega_2), \vartheta_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(0,\omega_2)\},$$
$$\vartheta_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(0,\omega_1 * \omega_2) \leq \max\{\vartheta_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(0,\omega_1), \vartheta_{(\varsigma_{(T,I)}\times\xi_{(T,I)})}(0,\omega_2)\},$$
$$\vartheta_{\xi_{(T,I)}}(\omega_1) \leq \max\{\vartheta_{\xi_{(T,I)}}(\omega_1 * \omega_2), \vartheta_{\xi_{(T,I)}}(\omega_2)\},$$
$$\vartheta_{\xi_{(T,I)}}(\omega_1 * \omega_2) \leq \max\{\vartheta_{\xi_{(T,I)}}(\omega_1), \vartheta_{\xi_{(T,I)}}(\omega_2)\}.$$

Also,

$$\vartheta_{(\varsigma_F\times\xi_F)}(\gamma_1,\omega_1) \geq \min\{\vartheta_{(\varsigma_F\times\xi_F)}((\gamma_1,\omega_1) * (\gamma_2,\omega_2)), \vartheta_{(\varsigma_F\times\xi_F)}(\gamma_2,\omega_2)\},$$
$$\vartheta_{(\varsigma_F\times\xi_F)}((\gamma_1,\omega_1) * (\gamma_2,\omega_2)) \geq \min\{\vartheta_{(\varsigma_F\times\xi_F)}(\gamma_1,\omega_1), \vartheta_{(\varsigma_F\times\xi_F)}(\gamma_2,\omega_2)\},$$

$$\vartheta_{(\varsigma_F \times \xi_F)}(\gamma_1 * \omega_1, \gamma_2 * \omega_2) \geq \min\{\vartheta_{(\varsigma_F \times \xi_F)}(\gamma_1, \omega_1), \vartheta_{(\varsigma_F \times \xi_F)}(\gamma_2, \omega_2)\}.$$

Putting $\gamma_1 = \gamma_2 = 0$ in the above relation, we obtain

$$\vartheta_{(\varsigma_F \times \xi_F)}(0, \omega_1) \geq \min\{\vartheta_{(\varsigma_F \times \xi_F)}(0, \omega_1 * \omega_2), \vartheta_{(\varsigma_F \times \xi_F)}(0, \omega_2)\}$$

$$\vartheta_{(\varsigma_F \times \xi_F)}(0, \omega_1 * \omega_2) \geq \min\{\vartheta_{(\varsigma_F \times \xi_F)}(0, \omega_1), \vartheta_{(\varsigma_F \times \xi_F)}(0, \omega_2)\}$$

$$\vartheta_{\xi_F}(\omega_1) \geq \min\{\vartheta_{\xi_F}(\omega_1 * \omega_2), \vartheta_{\xi_F}(\omega_2)\},$$

$$\vartheta_{\xi_F}(\omega_1 * \omega_2) \geq \min\{\vartheta_{\xi_F}(\omega_1), \vartheta_{\xi_F}(\omega_2)\}.$$

Therefore, either $\varsigma_{(T,I,F)}$ or $\xi_{(T,I,F)}$ is an intuitionistic neutrosophic $\hat{Z}$-ideal of $\mathfrak{M}$. □

6. Conclusion

This study has introduced and formalized the notion of intuitionistic neutrosophic $\hat{Z}$-ideals within the framework of $\hat{Z}$-algebras, providing an expressive algebraic model that simultaneously handles truth-membership, indeterminacy-membership, and falsity-membership. The proposed structure generalizes existing ideal concepts—including fuzzy, intuitionistic fuzzy, and neutrosophic $\hat{Z}$-ideals—while preserving fundamental algebraic properties under homomorphisms and Cartesian products.

The developed framework demonstrates both mathematical rigor and structural flexibility, positioning intuitionistic neutrosophic $\hat{Z}$-ideals as a unifying model that can subsume or approximate other generalized ideals under specific conditions.

Future research may explore connections with more advanced structures such as cubic fuzzy, interval-valued neutrosophic, and soft $\hat{Z}$-ideals. Extending the theory to topological $\hat{Z}$-algebras could further reveal continuity and closure properties under uncertainty. Additionally, potential applications in decision-making, optimization, and artificial intelligence highlight the practical relevance of this model. Algorithmic approaches for the computation and classification of these ideals, as well as their integration into graph theory, rings, lattices, and modules, represent promising directions for continued study.

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