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UPPER AND LOWER ALMOST QUASI (τ_1, τ_2)-CONTINUITY

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Abstract. This paper is concerned with the concepts of upper almost quasi (τ_1, τ_2)-continuous multifunctions and lower almost quasi (τ_1, τ_2)-continuous multifunctions. Furthermore, several characterizations of upper almost quasi (τ_1, τ_2)-continuous multifunctions and lower almost quasi (τ_1, τ_2)-continuous multifunctions are investigated.

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Weaker and stronger forms of open sets play an important role in topological spaces. By utilizing these sets several authors introduced and investigated various types of generalizations of continuity. In [16], the present authors studied some properties of (Λ, sp)-open sets. Viriyapong and Boonpok [60] investigated several characterizations of (Λ, sp)-continuous functions by using (Λ, sp)-open sets and (Λ, sp)-closed sets. Dungthaisong et al. [31] introduced and studied the concept of $g_{(m,n)}$-continuous functions. Duangphui et al. [30] introduced and investigated the notion of $(\mu, \mu')^{(m,n)}$-continuous functions. Furthermore, several characterizations of almost (Λ, p)-continuous functions, strongly $\theta(\Lambda, p)$-continuous functions, almost strongly $\theta(\Lambda, p)$-continuous functions, $\theta(\Lambda, p)$-continuous functions, weakly (Λ, b)-continuous functions, $\theta(\star)$-precontinuous functions, $\star$-continuous functions, θ-$\mathscr{I}$-continuous functions, almost (g, m)-continuous functions, pairwise M-continuous functions, (τ_1, τ_2)-continuous functions, almost (τ_1, τ_2)-continuous functions, weakly (τ_1, τ_2)-continuous functions, almost quasi (τ_1, τ_2)-continuous functions and weakly quasi (τ_1, τ_2)-continuous functions were

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presented in [53], [54], [3], [49], [15], [11], [12], [23], [27], [28], [4], [5], [6], [36] and [29], respectively. Marcus [40] introduced and investigated the notion of quasi continuous functions. Popa [47] introduced and studied the notion of almost quasi continuous functions. Neubrunnovaá [41] showed that quasi continuity is equivalent to semi-continuity due to Levine [38]. Popa and Stan [48] introduced and investigated the notion of weakly quasi continuous functions. Weak quasi continuity is implied by quasi continuity and weak continuity [39] which are independent of each other. It is shown in [44] that weak quasi continuity is equivalent to weak semi-continuity due to Arya and Bhamini [1] and Kar and Bhattacharyya [32]. Bânzara and Crivăţ [2] introduced and studied the concept of quasi continuous multifunctions.

In 1998, Popa and Noiri [45] introduced the concept of almost quasi continuous multifunctions and investigated some characterizations of such multifunctions. Noiri and Popa [43] introduced and studied the notion of weakly quasi continuous multifunctions. Moreover, several characterizations of weakly quasi continuous multifunctions have been obtained in [45]. Popa and Noiri [46] introduced and studied the concepts of upper and lower θ-quasi continuous multifunctions. Laprom et al. [37] introduced and investigated the concept of $\beta(\tau_1, \tau_2)$-continuous multifunctions. Viriyapong and Boonpok [61] introduced and studied the notion of $(\tau_1, \tau_2)\alpha$-continuous multifunctions. Furthermore, some characterizations of $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions, almost weakly (τ_1, τ_2)-continuous multifunctions, $\star$-continuous multifunctions, $\beta(\star)$-continuous multifunctions, weakly quasi (Λ, sp)-continuous multifunctions, α-$\star$-continuous multifunctions, almost α-$\star$-continuous multifunctions, almost quasi $\star$-continuous multifunctions, weakly α-$\star$-continuous multifunctions, $s\beta(\star)$-continuous multifunctions, weakly $s\beta(\star)$-continuous multifunctions, $\theta(\star)$-quasi continuous multifunctions, almost $\imath^\star$-continuous multifunctions, weakly (Λ, sp)-continuous multifunctions, $\alpha(\Lambda, sp)$-continuous multifunctions, almost $\alpha(\Lambda, sp)$-continuous multifunctions, almost $\beta(\Lambda, sp)$-continuous multifunctions, (τ_1, τ_2)-continuous multifunctions, almost (τ_1, τ_2)-continuous multifunctions, weakly (τ_1, τ_2)-continuous multifunctions, weakly quasi (τ_1, τ_2)-continuous multifunctions, s-$(\tau_1, \tau_2)p$-continuous multifunctions and c-(τ_1, τ_2)-continuous multifunctions were established in [22], [20], [25], [19], [59], [7], [10], [24], [13], [8], [9], [17], [21], [14], [34], [18], [56], [52], [35], [55], [50], [57] and [33], respectively. In particular, some characterizations of upper and lower θ-quasi continuous multifunctions were presented in [42]. Quite recently, Pue-on et al. [51] introduced and studied the concept of almost quasi (τ_1, τ_2)-continuous multifunctions. In this paper, we introduce the concepts of upper almost quasi (τ_1, τ_2)-continuous multifunctions and lower almost quasi (τ_1, τ_2)-continuous multifunctions. We also investigate several characterizations of upper almost quasi (τ_1, τ_2)-continuous multifunctions and lower almost quasi (τ_1, τ_2)-continuous multifunctions.

1. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [26] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [26] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [26] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [26] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) *$A \subseteq \tau_1\tau_2$-Cl(A) and $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Cl$(A)) = \tau_1\tau_2$-Cl(A).*
(2) *If $A \subseteq B$, then $\tau_1\tau_2$- Cl$(A) \subseteq \tau_1\tau_2$-Cl$(B)$.*
(3) *$\tau_1\tau_2$-Cl(A) is $\tau_1\tau_2$-closed.*
(4) *A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2$-Cl(A).*
(5) *$\tau_1\tau_2$-Cl$(X - A) = X - \tau_1\tau_2$-Int$(A)$.*

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$-*open* [61] (resp. $(\tau_1, \tau_2)p$-*open* [22], $(\tau_1, \tau_2)\beta$-*open* [22], $\alpha(\tau_1, \tau_2)$-*open* [58]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$, $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A)))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))))$. A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)s$-*open* [22] if $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$. The complement of a $(\tau_1, \tau_2)s$-open set is called $(\tau_1, \tau_2)s$-closed. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $(\tau_1, \tau_2)s$-closed sets of X containing A is called the $(\tau_1, \tau_2)s$-*closure* [22] of A and is denoted by (τ_1, τ_2)-sCl(A). The union of all $(\tau_1, \tau_2)s$-open sets of X contained in A is called the $(\tau_1, \tau_2)s$-*interior* [22] of A and is denoted by (τ_1, τ_2)-sInt(A).

Lemma 2. *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) *(τ_1, τ_2)-sCl$(A) = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A)) \cup A$ [20];*
(2) *(τ_1, τ_2)-sInt$(A) = \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A)) \cap A$ [51].*

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

2. Upper and Lower Almost Quasi (τ_1, τ_2)-Continuous Multifunctions

In this section, we introduce the notions of upper almost quasi (τ_1, τ_2)-continuous multifunctions and lower almost quasi (τ_1, τ_2)-continuous multifunctions. Furthermore, several characterizations and some properties concerning upper almost quasi (τ_1, τ_2)-continuous multifunctions and lower almost quasi (τ_1, τ_2)-continuous multifunctions are discussed.

Definition 1. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be upper almost quasi (τ_1, τ_2)-continuous at a point $x \in X$ if for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$ and each $\tau_1\tau_2$-open set U of X containing x, there exists a nonempty $\tau_1\tau_2$-open set G such that $G \subseteq U$ and $F(G) \subseteq (\sigma_1, \sigma_2)$-sCl$(V)$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be upper almost quasi (τ_1, τ_2)-continuous if F is upper almost quasi (τ_1, τ_2)-continuous at each point x of X.*

Theorem 1. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is upper almost quasi (τ_1, τ_2)-continuous at $x \in X$;*
(2) *for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$, there exists a $(\tau_1, \tau_2)s$-open set U of X containing x such that $F(U) \subseteq (\sigma_1, \sigma_2)$-sCl$(V)$;*
(3) *$x \in (\tau_1, \tau_2)$-sInt$(F^+((\sigma_1, \sigma_2)$-sCl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$;*
(4) *$x \in \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(F^+((\sigma_1, \sigma_2)$-sCl$(V))))$ for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$.*

Proof. (1) ⇒ (2): Let $\mathscr{U}(x)$ the family of all $\tau_1\tau_2$-open sets of X containing x. Let V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \subseteq V$. For each $H \in \mathscr{U}(x)$, there exists a nonempty $\tau_1\tau_2$-open set G_H such that $G_H \subseteq H$ and $F(G_H) \subseteq (\sigma_1, \sigma_2)$-sCl$(V)$. Let

$$W = \cup\{G_H \mid H \in \mathscr{U}(x)\}.$$

Then, W is $\tau_1\tau_2$-open in X, $x \in \tau_1\tau_2$-Cl(W) and $F(W) \subseteq (\sigma_1, \sigma_2)$-sCl$(V)$. Put $U = W \cup \{x\}$, then $W \subseteq U \subseteq \tau_1\tau_2$-Cl$(W)$. Thus, U is a $(\tau_1, \tau_2)s$-open set of X containing x such that $F(U) \subseteq (\sigma_1, \sigma_2)$-sCl$(V)$.

(2) ⇒ (3): Let V be any $\sigma_1\sigma_2$-open set of Y and $F(x) \subseteq V$. Then, there exists a $(\tau_1, \tau_2)s$-open set U of X containing x such that $F(U) \subseteq (\sigma_1, \sigma_2)$-sCl$(V)$. Thus, $x \in U \subseteq F^+((\sigma_1, \sigma_2)$-sCl$(V))$ and hence

$$x \in U \subseteq (\tau_1, \tau_2)\text{-sInt}(F^+((\sigma_1, \sigma_2)\text{-sCl}(V))).$$

(3) ⇒ (4): Let V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \subseteq V$.
By (3), $x \in (\tau_1, \tau_2)$-sInt$(F^+((\sigma_1, \sigma_2)$-sCl$(V)))$. Now, put $U = (\tau_1, \tau_2)$-sInt$(F^+((\sigma_1, \sigma_2)$-sCl$(V)))$. Then, we have U is $(\tau_1, \tau_2)s$-open in X and by Lemma 2,

$$x \in U \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(U)) \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^+((\sigma_1, \sigma_2)\text{-sCl}(V)))).$$

$(4) \Rightarrow (1)$: Let U be any $\tau_1\tau_2$-open set of X containing x and V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \subseteq V$. Then by (4),

$$x \in \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^+((\sigma_1, \sigma_2)\text{-sCl}(V))))$$

and hence $\tau_1\tau_2\text{-Int}(F^+((\sigma_1, \sigma_2)\text{-sCl}(V))) \cap U \neq \emptyset$. Put

$$W = \tau_1\tau_2\text{-Int}(F^+((\sigma_1, \sigma_2)\text{-sCl}(V))) \cap U.$$

Then, we have W is a nonempty $\tau_1\tau_2$-open set of X such that $W \subseteq U$ and $F(W) \subseteq (\sigma_1, \sigma_2)\text{-sCl}(V)$. This shows that F is upper almost quasi (τ_1, τ_2)-continuous at x. □

Definition 2. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be lower almost quasi (τ_1, τ_2)-continuous at a point $x \in X$ if for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$ and each $\tau_1\tau_2$-open set U of X containing x, there exists a nonempty $\tau_1\tau_2$-open set G such that $G \subseteq U$ and (σ_1, σ_2)-sCl$(V) \cap F(z) \neq \emptyset$ for each $z \in G$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be lower almost quasi (τ_1, τ_2)-continuous if F is lower almost quasi (τ_1, τ_2)-continuous at each point x of X.*

Theorem 2. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is lower almost quasi (τ_1, τ_2)-continuous at $x \in X$;*

(2) *for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a (τ_1, τ_2)s-open set U of X containing x such that*

$$(\sigma_1, \sigma_2)\text{-sCl}(V) \cap F(z) \neq \emptyset$$

for every $z \in U$;

(3) *$x \in (\tau_1, \tau_2)$-sInt$(F^-((\sigma_1, \sigma_2)\text{-sCl}(V)))$ for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$;*

(4) *$x \in \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(F^-((\sigma_1, \sigma_2)\text{-sCl}(V))))$ for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$.*

Proof. The proof is similar to that of Theorem 1. □

Theorem 3. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is upper almost quasi (τ_1, τ_2)-continuous;*

(2) *for each $x \in X$ and every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$, there exists a (τ_1, τ_2)s-open set U of X containing x such that $F(U) \subseteq (\sigma_1, \sigma_2)$-sCl$(V)$;*

(3) *$F^+(V)$ is (τ_1, τ_2)s-open in X for every (σ_1, σ_2)r-open set V of Y;*

(4) *$F^+(V) \subseteq (\tau_1, \tau_2)$-sInt$(F^+((\sigma_1, \sigma_2)\text{-sCl}(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;*

(5) *(τ_1, τ_2)-sCl$(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B))))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(B))$ for every subset B of Y;*

(6) *$F^+(V) \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(F^+((\sigma_1, \sigma_2)\text{-sCl}(V))))$ for every $\sigma_1\sigma_2$-open set V of Y.*

Proof. (1) ⇒ (2): It follows from Theorem 1.

(2) ⇒ (3): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y and $x \in F^+(V)$. Then, we have $F(x) \subseteq V$ and there exists a $(\tau_1, \tau_2)s$-open set U of X containing x such that $F(U) \subseteq V$. Thus, $x \in U \subseteq F^+(V)$ and hence $x \in (\tau_1, \tau_2)$-sInt$(F^+(V))$. Therefore, $F^+(V) \subseteq (\tau_1, \tau_2)$-sInt$(F^+(V))$. This shows that $F^+(V)$ is $(\tau_1, \tau_2)s$-open in X.

(3) ⇒ (4): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in F^+(V)$. Then, we have $F(x) \subseteq V \subseteq (\sigma_1, \sigma_2)$-sCl$(V)$. Thus, $x \in F^+((\sigma_1, \sigma_2)$-sCl$(V))$. By Lemma 2, we have (σ_1, σ_2)-sCl(V) is $(\sigma_1, \sigma_2)r$-open in Y and by (3), $F^+((\sigma_1, \sigma_2)$-sCl$(V))$ is $(\tau_1, \tau_2)s$-open in X and

$$x \in (\tau_1, \tau_2)\text{-sInt}(F^+((\sigma_1, \sigma_2)\text{-sCl}(V))).$$

Thus, $F^+(V) \subseteq (\tau_1, \tau_2)$-sInt$(F^+((\sigma_1, \sigma_2)$-sCl$(V)))$.

(4) ⇒ (5): Let B be any subset of Y. Then, we have $Y - \sigma_1\sigma_2$-Cl(B) is a $\sigma_1\sigma_2$-open set of Y. By (4) and Lemma 2, we have

$$\begin{aligned}
&X - F^-(\sigma_1\sigma_2\text{-Cl}(B)) \\
&= F^+(Y - \sigma_1\sigma_2\text{-Cl}(B)) \\
&\subseteq (\tau_1, \tau_2)\text{-sInt}(F^+((\sigma_1, \sigma_2)\text{-sCl}(Y - \sigma_1\sigma_2\text{-Cl}(B)))) \\
&= (\tau_1, \tau_2)\text{-sInt}(X - F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_1\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B))))) \\
&= X - (\tau_1, \tau_2)\text{-sCl}(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))))
\end{aligned}$$

and hence

$$(\tau_1, \tau_2)\text{-sCl}(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B))))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(B)).$$

(5) ⇒ (6): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, $Y - V$ is $\sigma_1\sigma_2$-closed in Y. By (5) and Lemma 2,

$$\begin{aligned}
\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(Y - V))))) &\subseteq F^-(Y - V) \\
&= X - F^+(V).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(Y - V))))) \\
&= \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^-(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))) \\
&= \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(X - F^+((\sigma_1, \sigma_2)\text{-sCl}(V)))) \\
&= X - \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^+((\sigma_1, \sigma_2)\text{-sCl}(V)))).
\end{aligned}$$

Thus, $F^+(V) \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(F^+((\sigma_1, \sigma_2)$-sCl$(V))))$.

(6) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \subseteq V$. By (6), we have

$$x \in F^+(V) \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^+((\sigma_1,\sigma_2)\text{-sCl}(V))))$$

and by Lemma 2, $x \in F^+(V) \subseteq (\tau_1,\tau_2)\text{-sInt}(F^+((\sigma_1,\sigma_2)\text{-sCl}(V)))$.
Put $U = (\tau_1,\tau_2)\text{-sInt}(F^+((\sigma_1,\sigma_2)\text{-sCl}(V)))$. Then, U is a $(\tau_1,\tau_2)s$-open set of X containing x such that $F(U) \subseteq (\sigma_1,\sigma_2)\text{-sCl}(V)$. This shows that F is upper almost quasi (τ_1,τ_2)-continuous. □

Theorem 4. *For a multifunction $F : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$, the following properties are equivalent:*

(1) *F is lower almost quasi (τ_1,τ_2)-continuous;*

(2) *for each $x \in X$ and every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $(\tau_1,\tau_2)s$-open set U of X containing x such that $(\sigma_1,\sigma_2)\text{-sCl}(V) \cap F(z) \neq \emptyset$ for every $z \in U$;*

(3) *$F^-(V)$ is $(\tau_1,\tau_2)s$-open in X for every $(\sigma_1,\sigma_2)r$-open set V of Y;*

(4) *$F^-(V) \subseteq (\tau_1,\tau_2)\text{-sInt}(F^-((\sigma_1,\sigma_2)\text{-sCl}(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;*

(5) *$(\tau_1,\tau_2)\text{-sCl}(F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B))))) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(B))$ for every subset B of Y;*

(6) *$F^-(V) \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^-((\sigma_1,\sigma_2)\text{-sCl}(V))))$ for every $\sigma_1\sigma_2$-open set V of Y.*

Proof. The proof is similar to that of Theorem 3. □

Theorem 5. *For a multifunction $F : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$, the following properties are equivalent:*

(1) *F is upper almost quasi (τ_1,τ_2)-continuous;*

(2) *$(\tau_1,\tau_2)\text{-sCl}(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$ for every $(\sigma_1,\sigma_2)\beta$-open set V of Y;*

(3) *$(\tau_1,\tau_2)\text{-sCl}(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$ for every $(\sigma_1,\sigma_2)s$-open set V of Y;*

(4) *$F^+(V) \subseteq (\tau_1,\tau_2)\text{-sInt}(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$ for every $(\sigma_1,\sigma_2)p$-open set V of Y.*

Proof. The proof is similar to that of Theorem 3. □

Theorem 6. *For a multifunction $F : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$, the following properties are equivalent:*

(1) *F is lower almost quasi (τ_1,τ_2)-continuous;*

(2) *$(\tau_1,\tau_2)\text{-sCl}(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$ for every $(\sigma_1,\sigma_2)\beta$-open set V of Y;*

(3) *$(\tau_1,\tau_2)\text{-sCl}(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$ for every $(\sigma_1,\sigma_2)s$-open set V of Y;*

(4) *$F^-(V) \subseteq (\tau_1,\tau_2)\text{-sInt}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$ for every $(\sigma_1,\sigma_2)p$-open set V of Y.*

Proof. The proof is similar to that of Theorem 5. □

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