A METHOD FOR DETECTING POLLUTION IN POPULATION DYNAMICS PROBLEM WITH INCOMPLETE DATA

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Abstract. Modeling population dynamics problems leads to mathematical systems with missing data. For instance, the problems of pollution in population dynamics have generally missing source terms as well as missing initial or boundary conditions. The paper is concerned with identifying unknown parameters arising in the state equation of some population dynamics system with incomplete initial condition.
To this aim the so-called sentinel method is used. We prove the existence of a sentinel by solving a new controllability result for a linear two stroke system for which Carleman inequalities are revisisited.
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1. Introduction

We consider a linear model describing the dynamics of population with age dependence, spatial structure and incomplete data. More precisely, let Ω be an open and bounded domain of $\mathbb{R}^N$, $N \in \{1, 2, 3\}$, with boundary Γ of C^∞. For the time $T > 0$ and the life expectancy of an individual $A > 0$, set $U = (0, T) \times (0, A)$, $Q = U \times \Omega$, $Q_A = (0, A) \times \Omega$, $\quad Q_T = (0, T) \times \Omega$, $\Sigma = U \times \Gamma$, $\Sigma_1 = U \times \Gamma_1$, where Γ_1 is a nonempty open subset of Γ. We denote by ν the outer normal on Γ. Then consider the following two stroke problem:

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$$\begin{cases} \frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} - \triangle y + \mu y &= \xi + \lambda \hat{\xi} & \text{in} & Q \\ y(0,a,x) &= y^0 + \tau \hat{y}^0 & \text{in} & Q_A \\ y(t,0,x) &= \int_0^A \beta(t,a,x) y(t,a,x) da & \text{in} & Q_T \\ y &= 0 & \text{on} & \Sigma_1 \\ \frac{\partial y}{\partial \nu} &= 0 & \text{on} & \Sigma \setminus \Sigma_1 \end{cases} \tag{1}$$

It is assumed that $\mu \geq 0$ and $\beta \geq 0$. The parameters of the problem have the following sense: the final time $T > 0$ is the horizon of the problem, the bound $A > 0$ is the expectation of life, the weight β is the natural fertility rate, the function $\mu = \mu(t,a,x)$ is the natural death rate of $a-$year old individuals at time $t > 0$ and in the position x, the functions ξ and y^0 are known with $\xi \in L^2(Q)$ and $y^0 \in L^2(\Omega)$. But, the terms: $\lambda \hat{\xi}$(so-called pollution term) and $\tau \hat{y}^0$ (so-called perturbation term) are unknown, $\hat{\xi}$ and $\hat{y}^0$ are renormalized and represent the size of pollution and perturbation

$$\|\hat{\xi}\|_{L^2(Q)} \leq 1, \ \|\hat{y}^0\|_{L^2(\Omega)} \leq 1, \text{ so that the reals } \lambda, \tau \text{ are small enough.}$$

In the model (1), we are interested in identifying the parameter λ in the state equation, independently from the variation $\tau \hat{y}^0$ around the initial data. To identify this parameter, we use The sentinel method of Lions [11]. In this paper we construct sentinels when the supports of the observation function and of the control function are included in two different open subsets of $\mathbb{R}^N$ (see Nakoulima [7]).

The sentinel concept relies on the following three objects: some state equation (for instance (1)), some observation function (2), and some control function w to be determined.

- **•: A state equation** represented here by (1) and we suppose that (1) has a unique solution denoted by $y = y(t,a,x,\lambda,\tau) = y(\lambda,\tau)$ depends on two parameters λ and τ in some relevant space. We assume the following:

$$(H1) \quad : \quad \begin{cases} \beta \in L^\infty(Q), \quad \beta(t,a,x) \geq 0 \text{ a.e. in } Q, \\ \sup_{(t,x)\in]0,T[\times\Omega} \int_{]0,A[} (\left|\beta^2(t,a,x)\right| + \left|\nabla \beta\right|^2 (t,a,x) da), \\ \exists \, a_0, a_1 \in (0, \ A) \text{ with } 0 < a_0 < a_1 \; s.t. \; \beta(.,a,.) = 0 \text{ for a} \in (0,a_0) \cup (a_1, \ A). \end{cases}$$

$$(H2) \quad : \quad \mu \in C([0, \ T] \times [0, \ A] \times \bar{\Omega}) \ \mu(t,a,x) \geq 0 \text{ a.e} \quad \text{in} \quad Q.$$

$$(H3) \quad : \quad \begin{cases} \forall t, \ 0 < t < A, \quad \forall x \in \Omega, \ \lim_{a \longrightarrow A} \int_0^A \mu(\iota, a - t + \iota, x) d\iota = +\infty \,, \\ \forall t, \ A < t < T, \quad \forall x \in \Omega, \ \lim_{a \longrightarrow A} \int_0^a \mu(t - a + \alpha, \alpha, x) d\alpha = +\infty, \\ \nabla \mu \in [L^\infty(Q)]^n. \end{cases}$$

- **An observation** Some non empty open subset $O \subset \Omega$, is called observatory set. The observation is y in O, for the time T. We denote by y_{obs} this observation

$$y_{obs} = m_0 \in L^2(U \times O) \tag{2}$$

- **A function** $S = S(\lambda, \tau)$ called "sentinel". Let $h_0 \in L^2(U \times O)$. Let on the other hand ω be some open and non empty subset of Ω such that $\omega \neq O$.
  For a control function $\mathrm{w} \in L^2(U \times \omega)$, we define the functional

$$S(\lambda, \tau) = \int_U \int_O h_0 y(\lambda, \tau) dt da dx + \int_U \int_\omega \mathrm{w} y(\lambda, \tau) dt da dx. \tag{3}$$

  We say that S defines a sentinel for the problem (1) if there exists w such that S is insensitive (at first order) with respect the to missing terms $\tau \hat{y}^0$, which means

$$\frac{\partial S}{\partial \tau}(0, 0) = 0 \qquad \forall\, \hat{y}^0, \tag{4}$$

  where here $(0; 0)$ corresponds to $\lambda = \tau = 0$ and w is of minimal norm in $L^2(U \times \omega)$. That is

$$\|\mathrm{w}\|_{L^2(U \times \omega)} = \min_{u \in L^2(U \times \omega)} \|u\|_{L^2(U \times \omega)}. \tag{5}$$

Several authors studied the sentinel problem. We refer to [3, 7, 10, 11] and the reference therein. In [9], B. Ainseba and al. used the method of sentinels to identify parameters of pollution in a rever. O Bodart and al. applied it in [8] to identify an unknown boundary. In [3], G. M. Mophou and O. Nakoulima studied the problem of sentinels with given sensitivity. Recently, the author S. Sawadogo introduced in [11] the distributed sentinels into the equation of the dynamics of populations to study a population subject to a migratory phenomenon. In this paper, we apply sentinel method to identify parameter in population dynamics with age dependence, spatiale structure and incomplete data. The problem is as follows: Given $h_0 \in L^2(U \times O)$, find a control w in $L^2(U \times \omega)$ such that if $y = y(\lambda, \tau)$ is solution of (1) and S is defined by (3), then (4), and (5) hold.

The remainder of this paper is as follows: In section 2, we establish the equivalence between sentinel problem and null controllability problem. Section 3 and section 4 are devoted respectively to preliminary results and proof of main result. In section 5 we formulate the information given by sentinel.

2. Null controllability problem

We show in this section that the existence of the sentinel comes to null controllability property. We begin by transforming the insensibility condition (4).

Set

$$y_\tau = \frac{d}{d\tau} y(\lambda, \tau)|_{\lambda = \tau = 0}.$$

Then the function y_τ is solution of

$$\begin{cases} \dfrac{\partial y_\tau}{\partial t}+\dfrac{\partial y_\tau}{\partial a}-\triangle y_\tau+\mu y_\tau & = & 0 & \text{in} & Q, \\ y_\tau(0,a,x) & = & \hat{y}^0 & \text{in} & Q_A, \\ y_\tau(t,0,x) & = & \displaystyle\int_0^A \beta(t,a,x)y_\tau(t,a,x)da & \text{in} & Q_T, \\ y_\tau & = & 0 & \text{on} & \Sigma_1, \\ \dfrac{\partial y_\tau}{\partial \nu} & = & 0 & \text{on} & \Sigma\setminus\Sigma_1. \end{cases} \tag{6}$$

Problem (6) is linear and has a unique solution y_τ. The insensibility condition (4) holds if and only if

$$\int_Q (h_0\chi_O+\mathrm{w}\chi_\omega)y_\tau dtdadx=0 \tag{7}$$

We can transform (7) by introducing the classical adjoint state. More precisely, we define the function $q=q(t;a;x)$ as the solution of the backward problem:

$$\begin{cases} -\dfrac{\partial q}{\partial t}-\dfrac{\partial q}{\partial a}-\triangle q+\mu q & = & \beta(t,a,x)q(t,0,x) & & \\ & + & h_0\chi_O+\mathrm{w}\chi_\omega & \text{in} & Q, \\ q & = & 0 & \text{on} & \Sigma_1, \\ \dfrac{\partial q}{\partial \nu} & = & 0 & \text{on} & \Sigma\setminus\Sigma_1 \\ q(T,a,x) & = & 0 & \text{in} & Q_A, \\ q(t,A,x) & = & 0 & \text{in} & Q_T. \end{cases} \tag{8}$$

As for the problem (6), the problem (8) has a unique solution q . The function q depends on the control w that we shall determine:

Indeed, if we multiply the first equation in (8) by y_τ , and we integrate by parts over Q , we obtain

$$\int_Q (h_0\chi_O+\mathrm{w}\chi_\omega)y_\tau dtdadx=\int_0^A\int_\omega q(0,a,x)\hat{y}^0dadx\ \forall\ \hat{y}^0\in L^2(Q_A)$$

So, the condition (4) (or (7)) is equivalent to

$$q(0,a,x;v)=0 \qquad \text{in } Q_A \tag{9}$$

Thus, the sentinel problem (3), (4), (5) is equivalent to the following null controllability problem:: Given $h_0\in L^2(U\times O)$, find a control w in $L^2(U\times\omega)$ such that if q is the solution of (8), then (5) and (9) hold.

In the following we set:

$$\begin{cases} L &= \dfrac{\partial}{\partial t}+\dfrac{\partial}{\partial a}-\Delta+\mu I, \\ L^* &= -\dfrac{\partial}{\partial t}-\dfrac{\partial}{\partial a}-\Delta+\mu I, \\ \mathcal{V} &= \left\{\rho\in C^\infty\left(\overline{Q}\right), \rho_{|_{\Sigma_1}}=\left(\frac{\partial\rho}{\partial a}+\frac{\partial\rho}{\partial t}\right)_{|_{\Sigma_1}}=0,\ \frac{\partial\rho}{\partial\nu}_{|_{\Sigma\setminus\Sigma_1}}=0.\right\} \end{cases} \tag{10}$$

3. Existence of a sentinel

We begin with some observability inequality, which will be proved in detail in the last section. Then we have:

Theoreme 3.1. *Let be $\rho\in\mathcal{V}$, then there exists a positive constant $C=C(\Omega,\omega,O,A,T)$ such that*

$$\int_Q\frac{1}{\theta^2}|\rho|^2dtdadx\le C\left[\int_Q|L\rho|^2dtdadx+\int_0^T\int_0^A\int_\omega|\rho|^2dtdadx,\right] \tag{11}$$

where $\theta\in C^2(Q)$ positive with $\frac{1}{\theta}$ bounded.

According to the RHS of (11), we consider the space $\mathcal{V}$ endowed with the bilinear form $a(.;.)$ defined by:

$$a(u;v)=\int_Q LuLvdtdadx+\int_0^T\int_0^A\int_\omega uvdtdadx. \tag{12}$$

According to Theorem 3.1 , this symetric bilinear form is a scalar product on $\mathcal{V}$.
Let V be the completion of $\mathcal{V}$ with respect to the norm

$$v\mapsto\|v\|_V=\sqrt{a(u,v)}, \tag{13}$$

then, V is a Hilbert space for the scalar product $a(v;\hat{v})$ and the associated norm.

Remark 3.1. *We can precise the structure of the elements of V . Indeed, let $H_\theta(Q)$ be the weigthed Hilbert space defined by*

$$H_\theta(Q)=\left\{\rho\in L^2(Q)\ \text{ such that}: \int_Q\frac{1}{\theta^2}|\rho|^2dtdadx<\infty,\right\}$$

endowed with the natural norm $\|\rho\|_\theta=\left(\int_Q\frac{1}{\theta^2}|\rho|^2dtdadx\right)^{\frac12}$. This shows that V is imbedded continuously in $H_\theta(Q)$: $\|\rho\|_\theta\le C\|\rho\|_V$.

Now if $h_0\in L^2(Q)$ and $\theta h_0\in L^2(Q)$ $(i.e.:h_0\in L^2_\theta(Q)$, then from (12) and the Cauchy-Schwartz inequality, we deduce that the linear form defined on V by

$$\rho\mapsto\int_Q h_0\chi_O\rho dtdadx,$$

is continuous. Therefore, from the Lax-Milgram theorem there exits a unique $u\in V$ solution of the variational equation:

$$a(u;v)=\int_Q h_0\chi_O vdtdadx \qquad \forall v\in V. \tag{14}$$

Theoreme 3.2. *Assume that* $h_0 \in L^2_\theta(Q)$*, and let* u *be the unique solution of* (14). *We set*

$$\mathrm{w} = -u\chi_\omega \tag{15}$$

and

$$q = Lu. \tag{16}$$

Then, the pair $(\mathrm{w}; q)$ *is such that* (8) − (9) *hold* (*i.e there is some insensitive sentinel defined by* (3) − (4)).

4. Preliminary results: Observability inequality

The proof for the observability inequality in theorem 3.1 will hold from Carleman estimates that we carefully show in the following results. Let us consider an auxillary function $\psi \in C^2(\overline{\Omega})$ which satisfies the following conditions:

$$\forall x \in \Omega, \psi(x) > 0 \ \ and \ \ \forall x \in \Gamma, \ \ \psi(x) = 0 \ \ \forall x \in \overline{\Omega \setminus \omega_0} \ \ |\nabla \psi| \neq 0. \tag{17}$$

where ω_0 designates any open set such that $\omega_0 \Subset \omega$. Such a function ψ exists according to A.Fursikov and O.Yu.Imanuvilov [7]. For any positive parameter value λ we define the following weight functions:

$$\varphi(t,a,x) = \frac{e^{\lambda\psi(x)}}{at(T-t)(A-a)}, \tag{18}$$

$$\tilde{\varphi}(t,a,x) = \frac{e^{-\lambda\psi(x)}}{at(T-t)(A-a)}, \tag{19}$$

$$\eta(t,a,x) = \frac{e^{2\lambda|\psi|_\infty} - e^{\lambda\psi(x)}}{at(T-t)(A-a)}, \tag{20}$$

$$\tilde{\eta}(t,a,x) = \frac{e^{2\lambda|\psi|_\infty} - e^{-\lambda\psi(x)}}{at(T-t)(A-a)}. \tag{21}$$

Then

$$\nabla\tilde{\varphi} = -\lambda\tilde{\varphi}\nabla\psi, \quad \nabla\tilde{\eta} = \lambda\tilde{\varphi}\nabla\psi \tag{22}$$

and

$$\nabla\varphi = \lambda\varphi\nabla\psi, \quad \nabla\eta = -\lambda\varphi\nabla\psi \tag{23}$$

We also notice the following properties:

$$\begin{gathered} |\varphi_t| \leq C\varphi^2, \quad |\varphi_a| \leq C\varphi^2, \quad |\eta_t| \leq C\varphi^2, \quad |\eta_{tt}| \leq C\varphi^3, \\ |\eta_a| \leq C\varphi^2, \quad |\eta_{at}| \leq C\varphi^3, \quad |\eta_{aa}| \leq C\varphi^3. \end{gathered} \tag{24}$$

The following theorem states the Carleman inequalities concerning (11):

Proposition 4.1. *There exist constants $s_0 > 0$, $\lambda_0 > 0$ and $C > 0$ depending on Ω, ω, ψ, and A, T, such that for all $s > s_0$, $\lambda > \lambda_0$, and for any function $\rho \in \mathcal{V}$ given by (10) , we have*

$$\begin{aligned}
&2s^3\lambda^4\int_Q \varphi^3 e^{-2s\eta}|\rho|^2 dtdadx + 4s^2\lambda\int_{\Sigma\setminus\Sigma_1}\varphi\left(\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial a}\right)\frac{\partial\psi}{\partial\nu}e^{-2s\eta}|\rho|^2 dtdad\Gamma\\
&-4s^3\lambda^3\int_{\Sigma_1}\varphi^3|\nabla\psi|^2\frac{\partial\psi}{\partial\nu}e^{-2s\eta}|\rho|^2 dtdad\Gamma - 4s^2\lambda^3\int_{\Sigma\setminus\Sigma_1}\varphi^2|\nabla\psi|^2\frac{\partial\psi}{\partial\nu}e^{-2s\eta}|\rho|^2 dtdad\Gamma\\
&-2s\lambda\int_{\Sigma\setminus\Sigma_1}\varphi\rho\left(\frac{\partial\rho}{\partial t}+\frac{\partial\rho}{\partial a}\right)\frac{\partial\psi}{\partial\nu}e^{-2s\eta}dtdad\Gamma - 4s\lambda\int_{\Sigma_1}\varphi\nabla\psi e^{-2s\eta}\nabla\rho\frac{\partial\rho}{\partial\nu}dtdad\Gamma \qquad (25)\\
&+2s\lambda\int_{\Sigma}\varphi\left(\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial a}\right)\frac{\partial\psi}{\partial\nu}e^{-2s\eta}|\nabla\rho|^2 dtdad\Gamma\\
&\leq C\left(\int_Q e^{-2s\eta}|\frac{\partial\rho}{\partial t}+\frac{\partial\rho}{\partial a}-\Delta\rho+\mu\rho|^2 dtdadx + s^3\lambda^4\int_0^T\int_0^A\int_\omega \varphi^3 e^{-2s\eta}|\rho|^2 dtdadx\right).
\end{aligned}$$

Proof. Here we will use a classic method for the proof of this proposition. For simplicity, we will give the proof in three steps.

Step 1. Change of variables and plan of what follows. In this step, we set the differential equations satisfied by a new function w, which will be ρ up to a weight function. Let us set $w = e^{-s\eta}\rho, \forall\rho \in \nu$, $f = L\rho$ and $g = e^{-s\eta}f$. Using the definition of η it then follows that

$$w(0, a, x) = w(T, a, x) = 0,$$

$$w(t, 0, x) = w(t, A, x) = 0,$$

and

$$w = 0 \text{ on } \Sigma_1.$$

After some calcutions we get

$$P_1w + P_2w = g_{s,\lambda}\,, \qquad (26)$$

where

$$P_1w = w_t + w_a + 2s\lambda\varphi\nabla\psi\nabla w + 2s\lambda^2|\nabla\psi|^2\varphi w \qquad (27)$$

$$P_2w = -\Delta w - s^2\lambda^2\varphi^2|\nabla\psi|^2 w + s\eta_t w + s\eta_a w \qquad (28)$$

$$g_{s,\lambda} = g - s\lambda\Delta\psi\varphi w + s\lambda^2|\nabla|^2\varphi w - \mu w. \qquad (29)$$

We have from (24)

$$||P_1w||^2_{L^2(Q)} + ||P_2w||^2_{L^2(Q)} + 2\iiint_Q P_1wP_2w dtdadx = ||g_{s,\lambda}||^2_{L^2(Q)} \qquad (30)$$

In the following steps, with definition of $\mathcal{V}$ we will see that $2\iiint_Q P_1wP_2wdtdadx$ is positive up to several terms that can be controlled whenever we make an appropriate choice of the parameters s and

λ. More precisely, in the second step we will calculate $2\iiint_Q P_1wP_2wdtdadx$. This will give inequality with global terms of $\|w\|$ and $|\nabla w|^2$ on the left-hand side, while two local terms of $|\frac{\partial w}{\partial\nu}|^2$ and $|\nabla w|^2$ will appear on the right-hand side . In the third step we will add three terms (involving w_t,w_a, and Δw) to the left of (26). This will help us to eliminate the local term containing ∇w that appears on the right-hand side and will provide a Carleman inequality for the function w . Finally, we will turn back to the original function ρ and deduce the inequality (25).

Step 2. In this step, we will develop the terms $I_{k,l}$ appearing in $\iiint_Q P_1wP_2wdtdadx$. For this, we will integrate by parts several times with respect to the space and time variables, so derivatives of the weight functions will be involved. We will use the definitions (18) $-$ (23) and estimates (24) $-$ (25) . We then have the following results:

$$\begin{aligned} I_{1,1} = & -\iiint_Q w_t\Delta wdtdadx \\ = & -\iiint_\Sigma w_t\frac{\partial w}{\partial\nu}dtdad\Gamma + 0, \end{aligned} \tag{31}$$

$$\begin{aligned} I_{2,1} = & -\iiint_Q w_a\Delta wdtdadx \\ = & -\iiint_\Sigma w_a\frac{\partial w}{\partial\nu}dtdad\Gamma + 0, \end{aligned} \tag{32}$$

$$I_{1,2} = -s^2\lambda^2\iiint_Q|\nabla\psi|^2\varphi^2w_twdtdadx = B \tag{33}$$

$$I_{2,2} = -s^2\lambda^2\iiint_Q|\nabla\psi|^2\varphi^2w_awdtdadx = B \tag{34}$$

$$\begin{aligned} I_{1,3} = & \; s\iiint_Q\eta_tw_twdtdadx \\ = & -\frac{1}{2}s\iiint_Q\eta_{tt}w^2dtdadx = B \end{aligned} \tag{35}$$

$$\begin{aligned} I_{1,4} = & \; s\iiint_Q\eta_aw_twdtdadx \\ = & -\frac{1}{2}s\iiint_Q\eta_{at}w^2dtdadx = B \end{aligned} \tag{36}$$

$$\begin{aligned} I_{2,3} = & \; s\iiint_Q\eta_tw_awdtdadx \\ = & -\frac{1}{2}s\iiint_Q\eta_{ta}w^2dtdadx = B \end{aligned} \tag{37}$$

$$\begin{aligned} I_{2,4} = & \; s\iiint_Q\eta_aw_awdtdadx \\ = & -\frac{1}{2}s\iiint_Q\eta_{aa}w^2dtdadx = B \end{aligned} \tag{38}$$

And

$$\begin{aligned} I_{3,1} =& -2s\lambda \iiint_Q \varphi \nabla\psi \nabla w \Delta w dtdadx \\ =& -2s\lambda \iiint_\Sigma \varphi \nabla\psi.\nabla w \frac{\partial w}{\partial \nu} dtdad\Gamma + 2s\lambda^2 \iiint_Q \varphi \left|\nabla\psi.\nabla w\right|^2 dtdadx \\ &+ s\lambda \iiint_\Sigma \varphi \frac{\partial \psi}{\partial \nu} \left|\nabla w\right|^2 dtdad\Gamma - s\lambda^2 \iiint_Q \varphi \left|\nabla\psi\right|^2 \left|\nabla w\right|^2 dtdadx + A \end{aligned} \tag{39}$$

$$\begin{aligned} I_{3,2} =& -2s^3\lambda^3 \iiint_Q |\nabla\psi|^2 \varphi^3 \nabla\psi \nabla w.w dtdadx \\ =& -s^3\lambda^3 \iiint_Q \varphi^3 |\nabla\psi|^2 \nabla\psi \nabla |w|^2 dtdadx \\ =& -s^3\lambda^3 \iiint_\Sigma \varphi^3 |\nabla\psi|^2 \frac{\partial \psi}{\partial \nu} |w|^2 dtdad\Gamma + 3s^3\lambda^4 \iiint_Q \varphi^3 |\nabla\psi|^4 |w|^2 dtdadx + B \end{aligned} \tag{40}$$

$$\begin{aligned} I_{3,3} =& 2s^2\lambda \iiint_Q \varphi \nabla\psi \nabla w \eta_t w dtdadx \\ =& s^2\lambda \iiint_\Sigma \varphi \eta_t \frac{\partial \psi}{\partial \nu} |w|^2 dtdad\Gamma + B \end{aligned} \tag{41}$$

$$\begin{aligned} I_{3,4} =& 2s^2\lambda \iiint_Q \varphi \nabla\psi \nabla w \eta_a w dtdadx \\ =& s^2\lambda \iiint_\Sigma \varphi \eta_a \frac{\partial \psi}{\partial \nu} |w|^2 dtdad\Gamma + B \end{aligned} \tag{42}$$

$$\begin{aligned} I_{4,1} =& -2s\lambda^2 \iiint_Q \varphi |\nabla\psi|^2 \Delta w.w dtdadx \\ =& -2s\lambda^2 \iiint_\Sigma \varphi |\nabla\psi|^2 \frac{\partial w}{\partial \nu}.w dtdad\Gamma + 2s\lambda^2 \iiint_Q \varphi |\nabla\psi|^2 |\nabla w|^2 dtdadx + A + B \end{aligned} \tag{43}$$

$$I_{4,2} = -2s^3\lambda^4 \iiint_Q \varphi^3 |\nabla\psi|^4 |w|^2 dtdadx \tag{44}$$

$$I_{4,3} = 2s^2\lambda^2 \iiint_Q \varphi \eta_t |\nabla\psi|^2 |w|^2 dtdadx = B \tag{45}$$

$$I_{4,4} = 2s^2\lambda^2 \iiint_Q \varphi \eta_a |\nabla\psi|^2 |w|^2 dtdadx = B \tag{46}$$

Summing all the terms, it follows:

$$\begin{aligned} 2\iiint_Q P_1 w P_2 w dtdadx =& A + B + 2s\lambda^2 \iiint_Q \varphi |\nabla\psi|^2 |\nabla w|^2 dtdadx \\ &+ 2s^3\lambda^4 \iiint_Q \varphi^3 |\nabla\psi|^4 |w|^2 dtdadx + 4s\lambda^2 \iiint_Q \varphi |\nabla\psi.\nabla w|^2 dtdadx \\ &-2\iiint_\Sigma \left(\frac{\partial w}{\partial t} + \frac{\partial w}{\partial a}\right) \frac{\partial w}{\partial \nu} dtdad\Gamma - 4s\lambda \iiint_\Sigma \varphi \nabla\psi.\nabla w \frac{\partial w}{\partial \nu} dtdad\Gamma \\ &+2s\lambda \iiint_\Sigma \varphi \frac{\partial \psi}{\partial \nu} \left|\nabla w\right|^2 dtdad\Gamma - 2s^3\lambda^3 \iiint_\Sigma \varphi^3 |\nabla\psi|^2 \frac{\partial \psi}{\partial \nu} |w|^2 dtdad\Gamma \\ &+2s^2\lambda \iiint_\Sigma \left(\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial a}\right) \frac{\partial \psi}{\partial \nu} |w|^2 dtdad\Gamma - 4s\lambda^2 \iiint_\Sigma \varphi |\nabla\psi|^2 \frac{\partial w}{\partial \nu}.w dtdad\Gamma. \end{aligned} \tag{47}$$

But $|\nabla\psi| \neq 0$ on $\overline{\Omega - \omega_0}$, hence there is $\delta > 0$ such that

$$|\nabla\psi| \geq \delta \text{ on } \overline{\Omega - \omega_0}. \tag{48}$$

On the other hand

$$\iiint_Q |g_{s,\lambda}|^2 dtdadx \leq \iiint_Q e^{-2s\eta}|g|^2 dtdadx + B$$

so that

$$||P_1 w||^2_{L^2(Q)} + ||P_2 w||^2_{L^2(Q)} + 2\iiint_Q P_1 w P_2 w dtdadx \leq \iiint_Q e^{-2s\eta}|g|^2 dtdadx + B.$$

Consequently:

$$\begin{aligned}
&||P_1 w||^2_{L^2(Q)} + ||P_2 w||^2_{L^2(Q)} + 2s\lambda^2\delta^2 \iiint_Q \varphi|\nabla w|^2 dtdadx \\
&+2s^3\lambda^4\delta^4 \iiint_Q \varphi|w|^2 dtdadx - 2\iiint_\Sigma \left(\frac{\partial w}{\partial t} + \frac{\partial w}{\partial a}\right)\frac{\partial w}{\partial \nu} dtdad\Gamma \\
&-4s\lambda \iiint_\Sigma \varphi\nabla\psi.\nabla w \frac{\partial w}{\partial \nu} dtdad\Gamma + 2s\lambda \iiint_\Sigma \varphi\frac{\partial \psi}{\partial \nu}|\nabla w|^2 dtdad\Gamma \\
&-2s^3\lambda^3 \iiint_\Sigma \varphi^3|\nabla\psi|^2\frac{\partial \psi}{\partial \nu}|w|^2 dtdad\Gamma + 2s^2\lambda \iiint_\Sigma \left(\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial a}\right)\frac{\partial \psi}{\partial \nu}|w|^2 dtdad\Gamma \\
&-4s\lambda^2 \iiint_\Sigma \varphi|\nabla\psi|^2\frac{\partial w}{\partial \nu}.w dtdad\Gamma + A + B \\
&\leq \iiint_Q e^{-2s\eta}|g|^2 dtdadx + B + 2s\lambda^2\delta^2 \int_0^T\int_0^A\int_{\omega_0} \varphi|\nabla w|^2 dtdadx \\
&+2s^3\lambda^4\delta^4 \int_0^T\int_0^A\int_{\omega_0} \varphi|w|^2 dtdadx.
\end{aligned}$$

We can eliminate A and B by choosing s and λ large enough. And we observe that: by multiplying (28) by $\theta^2\varphi w$ and by integrating by part over Q, after some calculations we obtains:

$$\int_0^T\int_0^A\int_\omega \varphi\theta^2|\nabla w|^2 dtdadx$$

$$\leq C\left(\iiint_Q P_2 w\theta^2 w dtdadx + \int_0^T\int_0^A\int_\omega \varphi^{\frac{1}{2}}\theta^2|\nabla w|^2\varphi^{\frac{1}{2}} w dtdadx + s^2\lambda^2 \int_0^T\int_0^A\int_\omega \varphi|w|^2 dtdadx.\right)$$

For $\theta \in \mathcal{D}(\omega)$ such that $0 \leq \theta \leq 1$ and $\theta(x) = 1$ on ω_0,using Cauchy-Schwarz and Young inequalities the inequality under above gives

$$2s\lambda^2\delta^2 \int_0^T\int_0^A\int_{\omega_0} \varphi|\nabla w|^2 dtdadx \leq \frac{1}{2}\int_0^T\int_0^A\int_\Omega |P_2 w|^2 dtdadx + Cs^3\lambda^4\delta^4 \int_0^T\int_0^A\int_\omega \varphi|w|^2 dtdadx.$$

Step 3. Now, we should write the inequality below in terms of the solution ρ, since

$$|w|^2 = e^{-2s\eta}|u|^2.$$

So

$$\frac{1}{2}||P_1w||^2_{L^2(Q)}+\frac{1}{2}||P_2w||^2_{L^2(Q)}+2s\lambda^2\iiint_Q\varphi|\nabla w|^2dtdadx$$
$$+2s^3\lambda^4\iiint_Q\varphi e^{-2s\eta}|\rho|^2dtdadx-2\iiint_\Sigma\left(\frac{\partial w}{\partial t}+\frac{\partial w}{\partial a}\right)\frac{\partial w}{\partial \nu}dtdad\Gamma$$
$$-4s\lambda\iiint_\Sigma\varphi\nabla\psi.\nabla w\frac{\partial w}{\partial\nu}dtdad\Gamma+2s\lambda\iiint_\Sigma\varphi\frac{\partial\psi}{\partial\nu}\left|\nabla w\right|^2dtdad\Gamma$$
$$-2s^3\lambda^3\iiint_\Sigma\varphi^3|\nabla\psi|^2\frac{\partial\psi}{\partial\nu}e^{-2s\eta}|\rho|^2dtdad\Gamma+2s^2\lambda\iiint_\Sigma\left(\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial a}\right)\frac{\partial\psi}{\partial\nu}e^{-2s\eta}|\rho|^2dtdad\Gamma$$
$$-4s\lambda^2\iiint_\Sigma\varphi|\nabla\psi|^2\frac{\partial w}{\partial\nu}.wdtdad\Gamma$$
$$\leq C\left(\iiint_Q e^{-2s\eta}|g|^2dtdadx+s^3\lambda^4\int_0^T\int_0^A\int_\omega\varphi^3e^{-2s\eta}|\rho|^2dtdadx.\right)$$

Now from

$$\nabla\rho=e^{s\eta}\left(\nabla w-s\lambda\varphi\nabla\psi w\right),$$

we deduce

$$\int_Q\varphi e^{-2s\eta}|\nabla\rho|^2dtdadx\leq C\left(\int_Q\varphi|\nabla w|^2dtdadx+s^2\lambda^2\int_Q\varphi|w|^2dtdadx.\right)$$

We then use the explicit form of P_1w and P_2w, and get

$$\frac{1}{s}\int_Q\frac{1}{\varphi}\left|\frac{\partial w}{\partial t}+\frac{\partial w}{\partial a}\right|^2dtdadx\leq C\left(\iiint_Q e^{-2s\eta}|g|^2dtdadx+s^3\lambda^4\int_0^T\int_0^A\int_\omega\varphi^3|w|^2dtdadx.\right)$$

and

$$\frac{1}{s}\int_Q\frac{1}{\varphi}\left|\Delta w\right|^2dtdadx\leq C\left(\iiint_Q e^{-2s\eta}|g|^2dtdadx+s^3\lambda^4\int_0^T\int_0^A\int_\omega\varphi^3|w|^2dtdadx.\right)$$

We sum up to finally have

$$+2s^3\lambda^4\iiint_Q\varphi e^{-2s\eta}|\rho|^2dtdadx+4s^2\lambda\iiint_\Sigma\left(\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial a}\right)\frac{\partial\psi}{\partial\nu}e^{-2s\eta}|\rho|^2dtdad\Gamma$$
$$-4s^3\lambda^3\iiint_\Sigma\varphi^3|\nabla\psi|^2\frac{\partial\psi}{\partial\nu}e^{-2s\eta}|\rho|^2dtdad\Gamma-4s^2\lambda^3\iiint_\Sigma\varphi^2|\nabla\psi|^2\frac{\partial\psi}{\partial\nu}e^{-2s\eta}|\rho|^2dtdad\Gamma$$
$$+\iiint_\Sigma\left(2s\frac{\partial\eta}{\partial t}+2s\frac{\partial\eta}{\partial a}-4s\lambda^2\varphi|\nabla\psi|^2-4s^2\lambda^2\varphi^2|\nabla\psi|^2\right)e^{-2s\eta}\frac{\partial\rho}{\partial\nu}\rho dtdad\Gamma$$
$$-2s\lambda\iiint_\Sigma\varphi\frac{\partial\psi}{\partial\nu}e^{-2s\eta}\left(\frac{\partial\rho}{\partial t}+\frac{\partial\rho}{\partial a}\right)\rho dtdad\Gamma-4s\lambda\iiint_\Sigma\varphi\nabla\psi e^{-2s\eta}\nabla\rho\frac{\partial\rho}{\partial\nu}dtdad\Gamma$$
$$+2s\lambda\iiint_\Sigma\varphi\frac{\partial\psi}{\partial\nu}e^{-2s\eta}|\nabla\rho|^2dtdad\Gamma-2\iiint_\Sigma e^{-2s\eta}\left(\frac{\partial\rho}{\partial t}+\frac{\partial\rho}{\partial a}\right)\frac{\partial\rho}{\partial\nu}dtdad\Gamma$$
$$\leq C\left(\iiint_Q e^{-2s\eta}|g|^2dtdadx+s^3\lambda^4\int_0^T\int_0^A\int_\omega\varphi^3|\rho|^2dtdadx.\right)$$

Using the fact that $\rho_{|_{\Sigma_1}}=\left(\frac{\partial\rho}{\partial a}+\frac{\partial\rho}{\partial t}\right)|_{\Sigma_1}=0$ and $\frac{\partial\rho}{\partial\nu}|_{\Sigma\setminus\Sigma_1}=0$ we obtain (25) □

Now we proceed as the following. For $s \geq s_0$, $\lambda \geq \lambda_0$ and using the notation given by (17) and (19), we define

$$\tilde{w}(t,a,x) = e^{-s\tilde{\eta}(t,a,x)}\rho(t,a,x). \tag{49}$$

We easily notice that

$$\tilde{w}(0,a,x) = \tilde{w}(T,a,x) = 0, \tag{50}$$

$$\tilde{w}(t,0,x) = \tilde{w}(t,A,x) = 0. \tag{51}$$

Calculating $\tilde{P}\tilde{w} = e^{-s\tilde{\eta}}g = e^{-s\tilde{\eta}}\left[(\partial_t + \partial_a - \Delta + \mu I)(e^{s\tilde{\eta}}\tilde{w})\right]$, using notation (47), we set

$$\tilde{P}_1\tilde{w} + \tilde{P}_2\tilde{w} = g_{s,\lambda}\,, \tag{52}$$

where

$$\tilde{P}_1\tilde{w} = \tilde{w}_t + \tilde{w}_a - 2s\lambda\tilde{\varphi}\nabla\psi\nabla\tilde{w} + 2s\lambda^2|\nabla\psi|^2\tilde{\varphi}\tilde{w} \tag{53}$$

$$\tilde{P}_2\tilde{w} = -\Delta\tilde{w} - s^2\lambda^2\tilde{\varphi}^2|\nabla\psi|^2\tilde{w} + s\tilde{\eta}_t\tilde{w} + s\tilde{\eta}_a\tilde{w} \tag{54}$$

$$g_{s,\lambda} = e^{-s\tilde{\eta}}g + s\lambda\Delta\psi\tilde{\varphi}\tilde{w} + s\lambda^2|\nabla\psi|^2\tilde{\varphi}\tilde{w} - \mu\tilde{w} \tag{55}$$

Proposition 4.2. *There exist constants $s_0 > 0$, $\lambda_0 > 0$ and $C > 0$ depending on Ω, ω, ψ, and A, T, such that for all $s > s_0$, $\lambda > \lambda_0$, and for any function $\rho \in \mathcal{V}$ given by (10) , we have*

$$\begin{aligned}
&2s^3\lambda^4\int_Q \tilde{\varphi}^3 e^{-2s\tilde{\eta}}|\rho|^2 dtdadx - 4s^2\lambda\int_{\Sigma\setminus\Sigma_1}\tilde{\varphi}\left(\tfrac{\partial\tilde{\eta}}{\partial t} + \tfrac{\partial\tilde{\eta}}{\partial a}\right)\tfrac{\partial\psi}{\partial\nu}e^{-2s\tilde{\eta}}|\rho|^2 dtdad\Gamma \\
&+4s^3\lambda^3\int_{\Sigma_1}\tilde{\varphi}^3|\nabla\psi|^2\tfrac{\partial\psi}{\partial\nu}e^{-2s\tilde{\eta}}|\rho|^2 dtdad\Gamma + 4s^2\lambda^3\int_{\Sigma\setminus\Sigma_1}\tilde{\varphi}^2|\nabla\psi|^2\tfrac{\partial\psi}{\partial\nu}e^{-2s\tilde{\eta}}|\rho|^2 dtdad\Gamma \\
&+2s\lambda\int_{\Sigma\setminus\Sigma_1}\tilde{\varphi}\rho\left(\tfrac{\partial\rho}{\partial t} + \tfrac{\partial\rho}{\partial a}\right)\tfrac{\partial\psi}{\partial\nu}e^{-2s\tilde{\eta}}dtdad\Gamma + 4s\lambda\int_{\Sigma_1}\varphi\nabla\psi e^{-2s\tilde{\eta}}\nabla\rho\tfrac{\partial\rho}{\partial\nu}dtdad\Gamma \\
&\qquad -2s\lambda\int_{\Sigma}\tilde{\varphi}\left(\tfrac{\partial\tilde{\eta}}{\partial t} + \tfrac{\partial\tilde{\eta}}{\partial a}\right)\tfrac{\partial\psi}{\partial\nu}e^{-2s\tilde{\eta}}|\nabla\rho|^2 dtdad\Gamma \\
&\leq C\left(\int_Q e^{-2s\tilde{\eta}}|\tfrac{\partial\rho}{\partial t} + \tfrac{\partial\rho}{\partial a} - \Delta\rho + \mu\rho|^2 dtdadx + s^3\lambda^4\int_0^T\int_0^A\int_\omega \tilde{\varphi}^3 e^{-2s\tilde{\eta}}|\rho|^2 dtdadx\right).
\end{aligned} \tag{56}$$

Proof. The proof is similar to the one of Proposition 4.1, so we let it to the reader. □

Finally, we give below the conclusion to theorem 3.1. We obtain from Proposition 4.1 and Proposition 4.2 the following observability inequality:

Corollaire 4.1. *There is a positive constant $C = C(\Omega, \omega, \psi, T, A)$ such that we have*

$$\int_Q \frac{1}{\theta^2}|\rho|^2 dtdadx \leq C\left[\int_Q \left|\frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial a} - \Delta\rho\right|^2 dtdadx + \int_0^T\int_0^A\int_\omega |\rho|^2 dtdadx,\right], \tag{57}$$

where $\frac{1}{\theta^2} = \varphi^3 e^{-2s\eta} + \tilde{\varphi}^3 e^{-2s\tilde{\eta}}$ is a bounded weight function.

Proof. Summing the terms in (25) and (56) we get the following:

$$\begin{aligned}
&2s^3\lambda^4\int_Q \left(\varphi^3 e^{-2s\eta}+\tilde{\varphi}^3 e^{-2s\tilde{\eta}}\right)|\rho|^2 dtdadx\\
&-4s^2\lambda\int_{\Sigma\setminus\Sigma_1}\left[\varphi\left(\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial a}\right)e^{-2s\eta}-\tilde{\varphi}\left(\frac{\partial\tilde{\eta}}{\partial t}+\frac{\partial\tilde{\eta}}{\partial a}\right)e^{-2s\tilde{\eta}}\right]\frac{\partial\psi}{\partial\nu}|\rho|^2 dtdad\Gamma\\
&-4s^3\lambda^3\int_{\Sigma_1}\left(\varphi^3 e^{-2s\eta}-\tilde{\varphi}^3 e^{-2s\tilde{\eta}}\right)|\nabla\psi|^2\frac{\partial\psi}{\partial\nu}|\rho|^2 dtdad\Gamma\\
&-4s^2\lambda^3\int_{\Sigma\setminus\Sigma_1}\left(\varphi^2 e^{-2s\eta}-\tilde{\varphi}^2 e^{-2s\tilde{\eta}}\right)|\nabla\psi|^2\frac{\partial\psi}{\partial\nu}|\rho|^2 dtdad\Gamma\\
&+2s\lambda\int_{\Sigma\setminus\Sigma_1}\left(\varphi e^{-2s\eta}-\tilde{\varphi} e^{-2s\tilde{\eta}}\right)\left(\frac{\partial\rho}{\partial t}+\frac{\partial\rho}{\partial a}\right)\frac{\partial\psi}{\partial\nu}\rho\, dtdad\Gamma\\
&-4s\lambda\int_{\Sigma_1}\left(\varphi e^{-2s\eta}-\tilde{\varphi} e^{-2s\tilde{\eta}}\right)\nabla\psi\nabla\rho\frac{\partial\rho}{\partial\nu} dtdad\Gamma\\
&+2s\lambda\int_{\Sigma}\left[\varphi\left(\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial a}\right)e^{-2s\eta}-\tilde{\varphi}\left(\frac{\partial\tilde{\eta}}{\partial t}+\frac{\partial\tilde{\eta}}{\partial a}\right)e^{-2s\tilde{\eta}}\right]\frac{\partial\psi}{\partial\nu}|\nabla\rho|^2 dtdad\Gamma\\
&\leq C\left[\int_Q\left(e^{-2s\eta}+e^{-2s\tilde{\eta}}\right)\left|\frac{\partial\rho}{\partial t}+\frac{\partial\rho}{\partial a}-\Delta\rho\right|^2 dtdadx\right.\\
&\left.+s^3\lambda^4\int_0^T\int_0^A\int_\omega\left(\varphi^3 e^{-2s\eta}+\tilde{\varphi}^3 e^{-2s\tilde{\eta}}\right)|\rho|^2 dtdadx\right].
\end{aligned} \tag{58}$$

Now, it suffices to notice that $\varphi=\tilde{\varphi}$ and $\eta=\tilde{\eta}$ on Σ □

5. Informations given by the sentinel

Because of (4) we can write

$$S(\lambda,\tau)-S(0,0)\approx\lambda\frac{\partial S}{\partial\lambda}(0,\,0),\text{ for }\quad\lambda,\tau\quad\text{small.}$$

In (3), $S(\lambda;\tau)$ is observed and using (2)

$$S(\lambda;\tau)=\int_Q(h_0\chi_O+\mathrm{w}\chi_\omega)m_0 dtdadx.$$

So that (4) becomes

$$\lambda\frac{\partial S}{\partial\lambda}(0,\,0)\approx\int_Q(h_0\chi_O+\mathrm{w}\chi_\omega)(m_0-y_0)dtdadx, \tag{59}$$

where $y_0=y(\lambda=0,\tau=0)$.

From (3) we have

$$\frac{\partial S}{\partial\lambda}(0,\,0)=\int_Q(h_0\chi_O+\mathrm{w}\chi_\omega)y_\lambda dtdadx, \tag{60}$$

where here χ_O and χ_ω denote the characteristic functions of O and ω respectively.

The derivative $y_\lambda=(\frac{\partial y}{\partial\lambda}(0,\,0))$ only depends on $\hat{\xi}$ and other known data. Consequently, the estimates (59) contains the informations on $\lambda\hat{\xi}$ (see for details remark 5.1 below).

Remark 5.1. *The knowledge of the optimal control* w *provides informations about the pollution term* $\lambda\hat{\xi}$.

and let $y_\lambda = \frac{\partial y}{\partial \lambda}(0,0)$ be the solution of

$$\begin{cases} Ly_\lambda = \hat{\xi} & \text{in} \quad Q, \\ y_\lambda(0,a,x) = 0 & \text{in} \quad Q_A, \\ y_\lambda(t,0,x) = \int_0^A \beta(t,a,x) y_\lambda(t,a,x)da & \text{in} \quad Q_T, \\ y_\lambda = 0 & \text{on} \quad \Sigma_1, \\ \frac{\partial y_\lambda}{\partial \nu} = 0 & \text{on} \quad \Sigma \setminus \Sigma_1. \end{cases} \tag{61}$$

Multiplying (8) by y_λ and integrating by parts over Q, we get

$$\int_Q q\hat{\xi} dtdadx = \int_Q (h_0\chi_O + \mathrm{w}\chi_\omega) y_\lambda dtdadx. \tag{62}$$

So that from (57), (58) and (60) we deduce

$$\int_Q \lambda\hat{\xi} dtdadx = \int_Q (h_0\chi_O + w\chi_\omega)(m_0 - y_0) dtdadx.$$

Authors' Contributions

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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