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JACOBSON RADICAL IN UNITAL KRASNER TERNARY HYPERRINGS

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Abstract. Let R be Krasner ternary hyperring with unital element and I be a hyper-ideal of R. The intersection of all maximal hyper-ideals of R containing I is called the Jacobson hyper radical of I . Define the property $\mathscr{J}$ on the class of hyper-ideals of R by, I possesses property $\mathscr{J}$ if and only if I coincides with its Jacobson hyper radical. In this short paper, it is shown that the property $\mathscr{J}$ exhibits a radical behavior on the class of hyper-ideals of R.

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1. Introduction

Hyper structure theory which is also known as multi valued algebra in other literature was introduced in 1934 by the French Mathematician F. Marty [9] at the 8th Congress of Scandinavian Mathematicians where he defined the concept of hyper-operation on groups. In the usual algebraic structure, the binary operation of two elements of a set is again an element of the same set, while in an algebraic hyper structure, the hyper-operation of two elements, is a subset of the same set. If this hyper-operation sends two elements to a singleton, then the hyper-operation coincides with the classical binary operation. In literature, a number of different hyper structure theories are widely studied since these represent a suitable and natural generalization of classical algebraic structures such as groups, rings and modules and for their applications to many areas of pure and applied mathematics and computer science. Some of these applications can be found in [6] and [7]. On the other hand, in classical algebraic structures, for instance in rings, a classic way of obtaining reasonable structural results is to determine all rings up to isomorphism and to single out some "bad" properties called radical properties of these rings and study only those rings that do not have these properties. Several radical properties in rings are Jacobson radical, regularity radical, nilradical, and Wedderburn radical. In 2015, Castillo and Vilela [2] defined the concept of regularity in the classes of Krasner Ternary hyperrings and proved that this property

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is radical. We follow this procedure and defined Jacobson property and prove that this property is radical on the classes of hyper-ideals in Krasner ternary hyperrings.

2. Preliminaries

This section presents the preliminaries in this paper.

Definition 2.1. [4] Let H be a nonempty set and let $\mathcal{P}^*(H)$ be the set of all nonempty subsets of H. A map $* : H \times H \to \mathcal{P}^*(H)$ is called a *hyper-operation* on the set H. The couple $(H, *)$ is called a *hyper groupoid* . If A and B are nonempty subsets of H, we define

i. $A * B = \bigcup_{a\in A, b\in B} (a * b)$;

ii. $x * A = \bigcup_{a\in A} (a * x)$;

iii. $B * x = \bigcup_{b\in B} (b * x)$.

Definition 2.2. [4] A nonempty set H with a hyper-operation " $+$ " is said to be a *canonical hyper group* if the following are satisfied:

i. For every $x, y \in H, x + y = y + x$;

ii. for every $x, y, z \in H, x + (y + z) = (x + y) + z$;

iii. $\exists 0_H \in H$ (called the neutral element of H) such that $x + 0_H = \{x\}$ and $0_H + x = \{x\}$ for all $x \in H$;

iv. for each $x \in H$ there exists a unique element denoted by $-x \in H$ such that $0_H \in [x + (-x)] \cap [(-x) + x]$;

v. for every $x, y, z \in H$, $z \in x + y$ implies $y \in -x + z$ and $x \in -y + z$. The pair $(H, +)$ is called a *canonical hyper group*

Definition 2.3. [1] Let H be a nonempty set. A *ternary multiplication* on the set H is a function $* : H \times H \times H \to H$.

Example 2.4. Consider the set $\mathbb{N}$. Define a ternary multiplication " $*$ " on $\mathbb{N}$ by $a * b * c = \max\{a, b, c\}$ for all $a, b, c \in \mathbb{N}$. Then " $*$ " is a ternary multiplication on $\mathbb{N}$.

Definition 2.5. [1] A *Krasner ternary hyperring* is an algebraic structure $(R, +, \cdot)$ consisting of a nonempty set R, a hyper-operation " $+$ " and ternary multiplication " $\cdot$ " satisfying the following:

i. $(R, +)$ is a canonical hyper group;

ii. $(a \cdot b \cdot c) \cdot d \cdot e = a \cdot (b \cdot c \cdot d) \cdot e = a \cdot b \cdot (c \cdot d \cdot e)$;

iii. $(a + b) \cdot c \cdot d = a \cdot c \cdot d + b \cdot c \cdot d$;

iv. $a \cdot (b + c) \cdot d = a \cdot b \cdot d + a \cdot c \cdot d$;

v. $a \cdot b \cdot (c + d) = a \cdot b \cdot c + a \cdot b \cdot d$;

vi. $a \cdot 0_R \cdot 0_R = 0_R \cdot a \cdot 0_R = 0_R \cdot 0_R \cdot a = 0_R$, where 0_R is the neutral element.

for every $a, b, c, d, e \in R$, if without ambiguity, $a \cdot b \cdot c$ can be written as juxtaposition abc. unless otherwise specified.

Definition 2.6. [3] If R is a Krasner ternary hyperring and if there exists an element $1_R \in R$ such that $1_R \neq 0_R$ and $1_R 1_R x = 1_R x 1_R = x 1_R 1_R = x$ for all $x \in R$, then 1_R is called a *unital element* of R.

Definition 2.7. [1] Let $(R, +, \cdot)$ be a Krasner ternary hyperring. A subset non-empty S of R is called a *Krasner subternary hyperring* if $(S, +, \cdot)$ is also a Krasner ternary hyperring.

Definition 2.8. [1] A Krasner subternary hypering I of a Krasner ternary hyperring R is called

i. *right hyper-ideal* of R if $abi \in I$ for all $a, b \in R$ and $i \in I$;
ii. *left hyper-ideal* of R if $iab \in I$ for all $a, b \in R$ and $i \in I$;
iii. *lateral hyper-ideal* of R if $aib \in I$ for all $a, b \in R$ and $i \in I$;
iv. *hyper-ideal* of R if it is a right, left, and lateral hyper-ideal.

Definition 2.9. [3] A hyper-ideal M of a Krasner ternary hyperring R is said to be *maximal* if the following are satisfied:

i. M is properly contained in R;
ii. if I is a hyper-ideal of R such that $M \subset I$, then $I = R$.

All throughout this paper, the class of all maximal hyper-ideals of R will be denoted by $spec(R)$ i.e.,

$$\text{spec}\,(R) = \{M_k \mid M_k \text{ is a maximal hyper-ideal of } R\}$$

and if I is a hyper-ideal of R, the class of maximal hyper-ideals containing I will be denoted by $spec(I)$ i.e., ,

$$spec\,(I) = \big\{M_k \mid M_k \text{ is a maximal hyper-ideal of } R \text{ and } I \subseteq M_k\big\}.$$

Examples of the above structures can be found in [1], [2], and [3].

Definition 2.10. [2] Let $(R_1, +, \cdot)$ and $(R_2, \oplus, *)$ be two Krasner ternary hyperrings with neutral elements 0_{R_1} and 0_{R_2} respectively. A mapping $\phi : R_1 \to R_2$ is called a *homomorphism* if the following are satisfied:

i. $\phi(a + b) = \phi(a) \oplus \phi(b)$ for all $a, b \in R_1$;
ii. $\phi(a \cdot b \cdot c) = \phi(a) * \phi(b) * \phi(c)$ for all $a, b, c \in R_1$;
iii. $\phi(0_{R_1}) = 0_{R_2}$.

Definition 2.11. [2] Let R be a Krasner ternary hyperring . A property $\mathscr{P}$ is called a *radical property* of R if it satisfies the following conditions:

i. The image of a $\mathscr{P}$-hyper-ideal in the homomorphic image of R is a $\mathscr{P}$-hyper-ideal.

ii. Every Krasner ternary hyperring R contains a $\mathscr{P}$-hyper radical.

iii. Every $\mathscr{P}$-hyper-ideal of the quotient Krasner ternary hyperring $R/\mathscr{P}(R)$ is zero.

iv. The $\mathscr{P}$-hyper radical of $\mathscr{P}(R)$ is $\mathscr{P}(R)$, i.e.

$$\mathscr{P}(\mathscr{P}(R)) = \mathscr{P}(R).$$

The next results are needed in this study.

Theorem 2.12. [1] *Let R_1 and R_2 be Krasner ternary hyperrings, $f : R_1 \to R_2$ a homomorphism;*

i. If f is surjective and I is a hyper-ideal of R_1, then $f(I)$ is a hyper-ideal of R_2.

ii. If B is a hyper-ideal of R_2, then $f^{-1}(B)$ is a hyper-ideal of R_1.

Theorem 2.13. [1] *Let I and J be hyper-ideals of a Krasner ternary hyperring R. Then*

i. $I + J$ is a hyper-ideal of R and I and J are hyper-ideals of $I + J$;

ii. $I \cap J$ is a hyper-ideal of both I and J .

Theorem 2.14. [1](**Correspondence Theorem**) *If I and J are hyper-ideals of a Krasner ternary hyperring R, then there is a one-to-one correspondence between the set of hyper-ideals of R which contains I and the set of all hyper ideals of R/I, given by $J \to J/I$. Hence, every hyper-ideal in R/I is of the form J/I, where J is a hyper-ideal of R containing I.*

Theorem 2.15. [2] *Let R be a Krasner ternary hyperring and $\mathcal{I} = \left\{I_\omega\right\}_{\omega\in\Omega}$ be the family of left [resp. right and lateral] hyper-ideals of R. Then*

$$M = \bigcup \left\{ \sum_{t=1}^{n} I_t \mid a_t \in I_t, n \in \mathbb{N} \right\}$$

is a hyper-ideal of R.

Theorem 2.16. [3]. *Let R be a Krasner ternary hyperring with unital element 1_R and $x \in R$. Then $\langle x\rangle_l = \bigcup \left\{ \sum_{i=1}^{n} \mid r_i s_i x, n \in \mathbb{N} \right\}$ is a hyper-ideal of R called the left principal hyper-ideal generated by x.*

Theorem 2.17. [3]. *Let R be a non trivial Krasner ternary hyperring with unital element 1_R. Then a maximal right [resp. left, lateral] hyper-ideal always exists and every hyper-ideal in R is contained in a maximal right hyper-ideal [resp. left, lateral].*

3. Main Results

This section presents the main results generated in this study.

If R is a Krasner ternary hyperring, then by Theorem 2.19 (ii), the intersection of hyper-ideals of R is a hyper-ideal R. Thus we have the following remark.

Remark 3.1. Let R be a Krasner ternary hyperring and I be a hyper-ideal of R. Then

$$\bigcap_{M_k \in spec(I)} M_k$$

is a hyper-ideal of R.

Definition 3.2. Let R be a Krasner ternary hyperring with unital element 1_R and I be a hyper-ideal of R. The intersection of all maximal ideals of R that contains I is called the *Jacobson hyper radical of* I denoted by $\mathscr{J}(I)$ and the intersection of all maximal ideals of R is called the *Jacobson Hyperradical of* R *and will be denoted by* $\mathscr{J}(R)$.

Define the property $\mathscr{J}$ on the class of hyper-ideals of R by I possesses property $\mathscr{J}$ if and only if $\mathscr{J}(I) = I$, in this case we call I the *Jacobson hyper-ideal of* R.

The next result will characterize maximal hyper-ideals in terms of their Jacobson hyper radicals

Theorem 3.3. *Let* R *be Krasner ternary hyperring and* M *a hyper-ideal of* R*. Then,* $\mathscr{J}(M) = M$ *if and only if* M *is maximal.*

Proof. Suppose M is is maximal, then $M \in spec(M)$. Let $x \in M$, since $M \subseteq M_k$ for all $M_k \in spec(M)$, it follows that

$$x \in \bigcap_{M_k \in spec(M)} M_k = \mathscr{J}(M).$$

Consequently, $M \subseteq \mathscr{J}(M)$. On the other hand, if $x \in \mathscr{J}(M)$, then x belongs to every maximal hyper-ideal containing M. Since M is maximal and $M \subseteq M$ implies $x \in M$, consequently $\mathscr{J}(M) \subseteq M$. Therefore, $\mathscr{J}(M) = M$.

Conversely, suppose M is not maximal. By Theorem 2.17, there exist a maximal ideal M_k such that $M \subset M_k$. Since $M = \mathscr{J}(M)$, $\mathscr{J}(M) \subset M_k$ hence, there exist $x \in M_k$ such that $x \notin \mathscr{J}(M)$, it follows that there exist a maximal ideal M_i containing M such that $x \notin M_i$, consequently $M_i \subset x + M_i$. Now, consider the hyper-ideal $\langle x \rangle_l$ in Theorem 2.16. Then $M_i \subset x + M_i \subset \langle x \rangle_l + M_i$. By maximality of M_l, $\langle x \rangle_l + M_i = R$. Since $1_R \notin M_i$, implies that $1_R \in \langle x \rangle_l$ thus, $\langle x \rangle_l = R$. Now, $x \in M_k$ implies that $\langle x \rangle_l \subseteq M_k$, thus $M_k = R$. A contradiction since M_k is maximal therefore, M must be maximal. ∎

Theorem 3.4. *Let* R *be a Krasner ternary hyperring. Then the image of a Jacobson hyper hyper-ideal of* R *is a Jacobson hyper-ideal in the homomorphic image of* R*.*

Proof. Let R and S be Krasner ternary hyperrings, I a Jacobson hyper-ideal of R, and $f : R \to S$ be a homomorphism then, $f : R \to f(R)$ is a surjective homomorphism. By Theorem 2.12 (i), $f(I)$ is a hyper-ideal of $f(R)$. Let M be a hyper-ideal of $f(R)$ such that $f(I) \subset M$. Then $f^{-1}(f(I)) \subseteq f^{-1}(M)$. Since in general, $I \subseteq f^{-1}(f(I))$, it follows that $I \subseteq f^{-1}(M)$. By Theorem 2.12 (ii), $f^{-1}(M)$ is a hyper-ideal of R_1. Since I is maximal, $f^{-1}(M) = R$. Hence $f(f^{-1}(M)) = f(R)$, but $f(f^{-1}(M)) \subseteq M$, it follows that $f(R) \subseteq M$. Since $M \subseteq f(R)$ consequently, $M = f(R)$. Thus, $f(I)$ is maximal in $f(R)$. Therefore, by Theorem 3.3, $f(I)$ is a Jacobson hyper-ideal of $f(R)$. ■

Theorem 3.5. *Let R be a Krasner ternary hyperring and I and J are Jacobson hyper-ideals of R, then $I + J$ is Jacobson hyper-ideal of R.*

Proof. Let N be a hyper-ideal of R such that $I + J \subset N$. Then $I \subseteq N$ and $J \subseteq N$. Since both I and J are maximal implies $N = R$, hence $I + J$
is maximal. Therefore, by Theorem 3.3, $I + J$ is a Jacobson hyper-ideal of R. ■

By induction on n, we have the following remark.

Remark 3.6. Let R be a Krasner ternary hyperring and $\mathcal{I} = \left\{I_\omega\right\}_{\omega\in\Omega}$ be the family of Jacobson hyper-ideals of R. Then $\sum_{k=1}^{n} I_k$ for each $I_k \in \mathcal{I}$ is a Jacobson hyper-ideal of R.

Theorem 3.7. *Any Krasner ternary hyperring R with unital element has a Jacobson hyper radical $\mathscr{J}(R)$.*

Proof. Let R be a Krasner ternary hyperring and $\mathcal{I} = \left\{I_\omega\right\}_{\omega\in\Omega}$ where Ω is an indexing set be the family of all Jacobson hyper-ideals of R. By Theorem 2.15, $\bigcup\left\{\sum_{k=1}^{n} I_k \mid I_k \in \mathcal{I}, n \in \mathbb{N}\right\}$ is a hyper-ideal of R. Now let I be a Jacobson hyper-ideal of R. Set $I_1 = I$. Then by Remark 3.6, $\sum_{k=1}^{m} I_k$ is a Jacobson hyper-ideal of R and

$$I \subseteq \sum_{k=1}^{m} I_k \subseteq \bigcup\left\{\sum_{k=1}^{n} I_k \mid I_k \in \mathcal{I}, n \in \mathbb{N}\right\}.$$

Consequently, since I is arbitrary,

$$\bigcup\left\{\sum_{k=1}^{n} I_k \mid I_k \in \mathcal{I}, n \in \mathbb{N}\right\}$$

contains all Jacobson hyper-ideals of R. Therefore,

$$\mathscr{J}(R) = \bigcup\left\{\sum_{k=1}^{n} I_k \mid I_k \in \mathcal{I}, n \in \mathbb{N}\right\}.$$

■

Theorem 3.8. *Let R be a Krasner ternary hyperring with unital element. Then, every hyper-ideal of the quotient $R/\mathscr{J}(R)$ is zero.*

Proof. Let K be a Jacobson hyper-ideal of $R/\mathscr{J}(R)$. By Theorem 2.14, $K = J/\mathscr{J}(R)$ for some hyper-ideal J with $\mathscr{J}(R) \subseteq J$. Now, by Theorem 2.17, there always exist a maximal hyper-ideal M that contains J, hence $J \subseteq M \subseteq \mathscr{J}(R)$. Consequently, $J = \mathscr{J}(R)$. Therefore,

$$K = J/\mathscr{J}(R) = \mathscr{J}(R) = 0_{R/\mathscr{J}(R)}.$$

■

Corollary 3.9. *Let R be a Krasner ternary hyperring with unital element 1_R. Then,*

$$\mathscr{J}(\mathscr{J}(R)) = \mathscr{J}(R)$$

.

Proof. By Theorem 3.7, $\mathscr{J}(R)$ is a hyper-ideal of R which is maximal since it contains all maximal hyper-ideals of R. Therefore, by Theorem 3.3, $\mathscr{J}(\mathscr{J}(R)) = \mathscr{J}(R)$. ■

Corollary 3.10. *$\mathscr{J}$ is a radical property on the classes of hyper-ideals in unital Krasner ternary hyperrings. .*

Proof. This is a direct consequence of Theorem 3.4, Theorem 3.7, Theorem 3.8, Corollary 3.10, and Definition 2.11. ■

4. Conclusion

In this study, we have investigated the Jacobson radical in the context of Krasner ternary hyperrings. In particular, it is demonstrated that the the property of being a maximal hyper-ideal is a radical property in the class of Krasner ternary hyperrings. This result provides a deeper understanding of the structure of Jacobson radicals can be extended in such non-classical algebraic structure.

Conflicts of Interest. The author declares that there are no conflicts of interest regarding the publication of this paper.

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