GENERALIZED (ϕ, ψ)-DERIVATIONS ON σ-IDEAL OF A σ-PRIME Γ-RING ACTING AS HOMOMORPHISM AND ANTI-HOMOMORPHISM

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Abstract. Throughout the paper, M consider be a 2-torsion free σ-prime Γ-ring and U be a non-zero σ-ideal of M. We define $G : M \to M$ as a generalized derivation associated with $(\phi,\, \psi)$- derivation g if $G(x\,\alpha\,y) = G(x)\,\alpha\,\phi(y) + \psi(x)\,\alpha\,g(y)$, for all $x, y \in M$, and $\alpha \in \Gamma$, where ϕ and ψ are two endomorphisms of a σ-prime Γ-ring M, and g is defined as $g(x\,\alpha\,y) = g(x)\,\alpha\,\phi(y) + \psi(x)\,\alpha\,g(y)$, for all $x, y \in M$, and $\alpha \in \Gamma$. Then we prove that if G acts as a homomorphism or as an anti-homomorphism on U and if $(G,\, g)$ is a generalized $(\phi,\, \psi)$- derivation with the assumption that $\sigma g = g\sigma$, where ψ-is an automorphism of M such that $\sigma\psi = \psi\sigma$, then $g = 0$ or $G = \psi$.

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1. Introduction

As an immense generalization of the theory of rings, the concept of gamma ring was introduced. From its first appearance, the extensions and generalizations of various important results in the theory of classical rings to the theory of Γ-rings have attracted wider attention as an emerging area of research for modern algebraists to enrich the world of algebra. Nowadays, all over the world, a good number of prominent mathematicians have worked on this interesting area of research to develop many basic characterizations of gamma rings and have extended numerous significant results in this context in the last few decades.

As a generalization of ring theories, N. Nobusawa [14] first introduced the concept of Γ-rings. After W. E. Barnes [16] generalized the notion of Nobusawa's Γ-rings and gave a new definition of Γ-ring. The notion of generalized derivation was obtained by B. Hvala [4].

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L. Oukhite, S. Salhi and L. Taoufiq [8] worked on Jordan generalized derivations of σ-prime rings and proved that every Jordan generalized derivation on U of R is a generalized derivation on U of R, where U is a σ-square closed Lie ideal of a 2-torsion free σ-prime ring R. Some significant results were developed on Lie ideals and generalized derivations in σ-prime rings by M. S. Khan and M. A. Khan [13]. Some characterizations of Lie ideals of σ-prime Γ-rings have been developed by M. M. Rahman and A. C. Paul [10]. M. M. Rahman and A. C. Paul [11] also studied Jordan derivations on Lie ideals of Prime Γ-Rings.

Recently, in classical ring theory, H. E. Bell and L. C. Kappe [6] initiated the study of mapping, which acts as a homomorphism or as an anti-homomorphism on a prime ring, and proved that if d is a derivation of prime ring R that acts as a homomorphism and an anti-homomorphism on a nonzero ideal I of R, then $d = 0$ on R. Thereafter, E. Albas and N. Argac [5] extended this result to generalized derivations of the prime ring. L. Oukhtite and S. Salhi [9] also worked on σ-prime rings with a special kind of automorphism.

A. Asma, N. Rehman, and A. Shakir [2] extended the result of Bell and Kappe [6] on square closed Lie ideals. They proved that if d is a derivation of a 2-torsion-free prime ring R that acts as a homomorphism or as an anti-homomorphism on a nonzero square closed Lie ideal U of R, then either $d = 0$ or $U \subseteq Z(R)$. A. C. Paul and S. Chakraborty [1] studied σ-prime Γ-rings and proved that if a derivation d acts as a homomorphism and an anti-homomorphism in a σ-ideal U of a σ-prime Γ-ring M, then $d = 0$ or $U \subseteq Z(M)$.

A. Ali and K. Kumar [3] established the above-mentioned result for generalized (θ, ϕ)-derivations as a homomorphism and an anti-homomorphism on σ-prime rings. In Γ-ring, K. K. Dey and A. C. Paul [7] proved that if D is a generalized derivation of a prime Γ-ring M with an associated derivation d of M which acts as a homomorphism and an anti-homomorphism on non-zero ideal I of M, then $d = 0$ or M is commutative. M. R. Khan and M. M. Hasnain [12] worked on some theorems for sigma prime rings with differential identities on sigma ideals.

In this paper, the above-mentioned results following [9, 12, 15] in classical rings are extended to those in gamma rings with generalized (ϕ, ψ)-derivations on σ-ideal of a σ-prime Γ-ring acting as a homomorphism and an anti-homomorphism. Our objective is to prove that for a nonzero σ-ideal U of a 2-torsion free σ-prime Γ-ring M and (G, g) is a generalized (ϕ, ψ)-derivation with the assumption that $\sigma g = g\sigma$, where ψ is an automorphism of M such that $\sigma\psi = \psi\sigma$ and if G acts as a homomorphism or as an anti-homomorphism on U, then $g = 0$ or $G = \psi$.

Throughout this paper, M will consider an associative σ-prime Γ-ring with center $Z(M)$ and $[x, y]_\alpha = x\,\alpha\,y - y\,\alpha\,x$, which is known as the commutator of x and y with respect to α. Then the basic commutator identities become $[x\,\beta\,y, xz]_\gamma = [x, z]_\gamma \beta\, y + x\,\beta[y, z]_\gamma$ and $[x, y\,\beta\,z]_\gamma = [x, y]_\gamma \beta\, z + y\,\beta[x, z]_\gamma$, $\forall\, x, y, z \in M$ and $\beta, \gamma \in \Gamma$; where $x\,\alpha\,y\,\beta\,z = x\,\beta\,y\,\alpha\,z$, $\forall\, x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Also, we suppose that for a

Lie ideal U of a 2-torsion-free σ-prime Γ-ring M satisfying the condition $a\,\alpha\,b\,\beta\,c = a\,\beta\,b\,\alpha\,c$, for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$.

2. Preliminaries

Now we recall some important definitions that are useful for us within the scope of this study, as follows:

Definition 1. **Γ-Ring:** Let M and Γ be additive abelian groups. If there is a mapping $M \times \Gamma \times M \to M$ such that the conditions:

- $(x + y)\,\alpha\,z = x\,\alpha\,z + y\,\alpha\,z,\ x(\alpha + \beta)y = x\,\alpha\,y + x\,\beta\,y,\ x\,\alpha\,(y + z) = x\,\alpha\,y + x\,\alpha\,z$
- $(x\,\alpha\,y)\beta\,z = x\,\alpha(y\,\beta\,z)$

are satisfied for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, then M is called a Γ-ring.

Example 1. Let R be a ring having unity element 1 and $M = M_{3,2}(R)$ be the set of all 3×2 matrices over R. If we take $\Gamma = M_{2,3}(R)$, then M is a Γ-ring under the operations of addition and multiplication of matrices.

Example 2. If R is a ring of characteristic 2 having a unity element 1. Let $M = M_{1,2}(R)$ and $\Gamma = \{(\binom{n.1}{0}) : n \in Z\}$, then M is a Γ-ring. If we assume $N = \{(x,\, x) : x \in R\} \subseteq M$, then N is also a Γ-ring of M.

Definition 2. **Prime Γ-Ring:** A Γ-ring M is said to be a prime Γ-ring if $x\,\Gamma\,M\,\Gamma\,y = 0$ (with $x, y \in M$) implies $x = 0$ or $y = 0$. Similarly, M is said to be prime if the zero ideal is prime.

Definition 3. **Completely Prime Γ-Ring:** A Γ-ring M is said to be a completely prime Γ-ring if $x\,\Gamma\,y = 0$ (with $x, y \in M$) implies $x = 0$ or $y = 0$.

Definition 4. **Semiprime Γ-Ring:** A Γ-ring M is said to be a semiprime Γ-ring if $x\,\Gamma\,M\,\Gamma\,x = 0$ (with $x \in M$) implies $x = 0$.

Definition 5. **Completely Semiprime Γ-Ring:** A Γ-ring M is said to be a completely semiprime Γ-ring if $x\,\Gamma\,x = 0$ (with $x \in M$) implies $x = 0$.

Definition 6. **σ-Prime Γ-Ring:** A Γ-ring M equipped with an involution said to be a σ-prime Γ-ring if for all $x, y \in M$, $x\,\Gamma\,M\,\Gamma\,y = 0 = x\,\Gamma\,M\,\Gamma\sigma(y)$ implies $x = 0$ or $y = 0$.

Definition 7. **Ideal of Γ-Ring:** A subset S of the Γ-ring M is a right ideal (or left ideal) of M if S is an additive subgroup of M and $S\,\Gamma\,M = \{s\,\alpha\,m : s \in S,\ \alpha \in \Gamma,\ m \in M\}$ (or $M\,\Gamma\,S = \{m\,\alpha\,s : s \in S,\ \alpha \in \Gamma,\ m \in M\}$) is contained in S. If S is both a left and a right ideal of M, then S is a two-sided ideal of M or simply an ideal of a Γ-ring M.

Example 3. Let M be a Γ-ring and A is a additive subgroup of M. Then $A = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in M \right\}$ is a right ideal of M.

Definition 8. **σ-Ideal:** An ideal U is a σ-ideal if U is invariant under σ; that is, $\sigma(U) = U$.

Definition 9. **Prime Ideal:** An ideal P of a Γ-ring M is said to be prime ideal if for any ideals A and B of M, $A\,\Gamma\,B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

Definition 10. **Semiprime Ideal:** An ideal P of a Γ-ring M is said to be semiprime ideal if for any ideals A of M, $A\,\Gamma\,A \subseteq P$ implies $A \subseteq P$.

Definition 11. **Square Closed Lie Ideal and Admissible Lie Ideal:** An additive subgroup $U \subset M$ is said to be a Lie ideal of M if whenever $u \in U,\ m \in M$ and $\alpha \in \Gamma$, then $[u,\, m]_\alpha \in U$. A Lie ideal is called a square closed Lie ideal if $u\,\alpha\,u \in U$, for all $u \in U$; $\alpha \in \Gamma$. Furthermore, if the Lie ideal U is square closed and $U \nsubseteq Z(M)$ where $Z(M)$ denotes the center of M, then U is called an admissible Lie ideal of M.

Example 4. ([11], Example 2): Let R be a ring of characteristic 2 having a unity element of 1. Let $M = M_{1,2}(R)$ and $\Gamma = \left\{ \binom{n.1}{n.1} : n \in Z, 2 \nmid n \right\}$, then M is a Γ-ring. We have

$$\begin{aligned}(x,\, x)\begin{pmatrix} n \\ n \end{pmatrix}(a,\, b) - (a,\, b)\begin{pmatrix} n \\ n \end{pmatrix}(x,\, x) =& (xna - bnx,\, xnb - anx) \\ =& (xna - 2bnx + bnx,\, bnx - 2anx + xna) \\ =& (xna + bnx,\, bnx + xna) \in N\end{aligned}$$

Therefore, N is a Lie ideal of M.

Definition 12. **Jordan Product:** Let M be a Γ-ring and $x, y \in M,\ \alpha \in \Gamma$, then $(x \circ y) = x\,\alpha\,y + y\,\alpha\,x$ is called Jordan product of x and y with respect to α.

Definition 13. **Commutator:** Let M be a Γ-ring for $x, y \in M, \alpha \in \Gamma$ a new product, known as Lie product defined by $[x,\, y]_\alpha = x\,\alpha\,y - y\,\alpha\,x$ and it is called the commutator of x and y with respect to α.

Definition 14. **Commutative Γ-Ring:** A Γ-ring M is said to be commutative Γ-ring if $x\,\alpha\,y = y\,\alpha\,x$ for all $x, y \in M$ and $\alpha \in \Gamma$.

Example 5. Let $M = M_{1,2}(R)$ and $\Gamma = \{\binom{n.1}{0} : n \in Z\}$. Then M is a commutative Γ-ring under the operations of matrix addition and matrix multiplication with characteristic $M = 2$.

Definition 15. **Center of a Γ-Ring:** Let M be a Γ-ring, then the set $Z_\Gamma = \{z \in M : z\,\alpha\,m = m\,\alpha\,z,\ \forall m \in M,$ and $\alpha \in \Gamma\}$ is called the center of a Γ-ring M and it is denoted by $Z(M)$. It is a clear consequence that if $Z(M) = M$, then M is commutative.

Definition 16. **2-torsion Free:** A Γ-ring M is said to be 2-torsion free or of characteristic not equal to 2, denoted as $char.M \neq 2$, if $2x = 0$ implies $x = 0$, for all $x \in M$.

Definition 17. **Derivation:** Let M be a Γ-ring and $d : M \to M$ an additive map. Then d is called a derivation if $d(x\,\alpha\,y) = d(x)\,\alpha\,y + x\,\alpha\,d(y)$, where $x, y \in M$ and $\alpha \in \Gamma$.

Definition 18. **Generalized Derivation:** An additive mapping $D : M \to M$ is said to be a generalized derivation if there exists a derivation $d : M \to M$ such that $D(x\,\alpha\,y) = D(x)\,\alpha\,y + x\,\alpha\,d(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$.

Example 6. Let M be a Γ-ring and $F : M \to M$ an additive map defined by $F((x,\, y)) = (f(x),\, f(y))$. Then F is a generalized derivation on M.

Definition 19. **Jordan Derivation:** Let M be a Γ-ring and $d : M \to M$ an additive map. Then d is called a Jordan derivation if $d(x\,\alpha\,x) = d(x)\,\alpha\,x + x\,\alpha\,d(x)$, where $x \in M$ and $\alpha \in \Gamma$.

Definition 20. **Generalized Jordan Derivation:** An additive mapping $D : M \to M$ is said to be a generalized Jordan derivation if there exists a derivation $d : M \to M$ then $D(x\,\alpha\,x) = D(x)\,\alpha\,x + x\,\alpha\,d(x)$, for all $x \in M$ and $\alpha \in \Gamma$.

Example 7. Let M be a Γ-ring and $N = \{(x,\, x) : x \in R\}$ be the subset of M. The map $F : N \to N$ defined by $F((x,\, x)) = (f(x),\, f(x))$ is a generalized Jordan derivation on N.

Definition 21. **Homomorphism:** Let M be a Γ-ring. An additive mapping $\phi : M \to M$ is said to be a homomorphism if $\phi(x\,\alpha\,y) = \phi(x)\,\alpha\,\phi(y)$, for all $x, y \in M$ and $\alpha \in \Gamma$.

Example 8. Let M be a Γ-ring and define $\phi : M \to M'$ by $\phi(x) = x,\ \forall\, x \in M$.

Example 9. Let M be a Γ-ring and $M' = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in M \right\}$. Then define $\phi : M \to M'$ by $\phi(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in M', \forall a \in M$ also $\alpha = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} \in M', \forall \alpha \in \Gamma$. Then $\phi(a\,\alpha\,b) = \begin{pmatrix} acb & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} = \phi(a)\,\alpha\,\phi(b)$, here $a, b, c \in M$ and $\alpha \in \Gamma$.

Definition 22. **Isomorphism:** If ϕ is one-one and onto then an additive mapping $\phi : M \to M$ is called an isomorphism of M into M.

Definition 23. **Automorphism:** A mapping $f : M \to M$ is called an automorphism of M if

- f is a bijection that is one-to-one and onto.
- $f(x\,\alpha\,y) = f(x)\,\alpha\,f(y), \forall\, a, b \in M$

On the other hand, an isomorphic mapping of a Γ-ring M onto M itself is called an automorphism of M.

Definition 24. **Endomorphism:** Let M be a Γ-ring. An additive mapping $\phi : M \to M$ of the Γ-ring M into itself is said to be an endomorphism of M if $\phi(x+y) = \phi(x) + \phi(y)$ and $\phi(x\,\alpha\,y) = \phi(x)\,\alpha\,\phi(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 25. **Anti-Homomorphism:** Let M be a Γ-ring. An additive mapping $\psi : M \to M$ is said to be an anti-homomorphism if $\psi(x\,\alpha\,y) = \psi(y)\,\alpha\,\psi(x)$, for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 26. $(\phi,\, \psi)$**-Derivation:** Let ϕ and ψ are two endomorphisms of a Γ-ring M. An additive mapping $d : M \to M$ is said to be $(\phi,\, \psi)$-derivation if $d(x\,\alpha\,y) = d(x)\,\alpha\,\phi(y) + \psi(x)\,\alpha\,d(y)$, for all $x, y \in M$ and $\alpha \in \Gamma$. Every $(I,\, I)$-derivation on M is just a derivation on M, where I is the identity mapping.

Definition 27. **Generalized** $(\phi,\, \psi)$**-Derivation:** Let ϕ and ψ be two endomorphisms of a Γ-ring M. An additive mapping $G : M \to M$ is called generalized $(\phi,\, \psi)$-derivation associated with $(\phi,\, \psi)$-derivation g if

$$G(x\,\alpha\,y) = G(x)\,\alpha\,\phi(y) + \psi(x)\,\alpha\,g(y),\ \forall\, x, y \in M, \text{ and } \alpha \in \Gamma$$

And g is defined as

$$g(x\,\alpha\,y) = g(x)\,\alpha\,\phi(y) + \psi(x)\,\alpha\,g(y),\ \forall\, x, y \in M, \text{ and } \alpha \in \Gamma$$

If S is a nonempty subset of a Γ-ring M and if $(G,\, g)$ is a generalized $(\phi,\, \psi)$- derivation M such that

$$G(s\,\alpha\,t) = G(s)\,\alpha\,G(t) \quad \text{or} \quad G(s\,\alpha\,t) = G(t)\,\alpha\,G(s)$$

for all $s, t \in M$ and $\alpha \in \Gamma$, then G is called generalized $(\phi,\, \psi)$-derivation, which acts as a homomorphism or as an anti-homomorphism on S.

3. Main Results

We begin with the following known lemma, which is given below:

Lemma 3.1. ([9], *Theorem 2.2*) *For a σ-prime ring R, the following statements hold:*

- (i) *$I \cap S_{a_\sigma}(R) \neq 0$ for every non-zero σ-ideal I of R.*
- (ii) *If $0 \neq I$ is a σ-ideal and $aIb = 0 = aI\sigma(b)$, then $a = 0$ or $b = 0$.*

Lemma 3.2. *For a σ-prime Γ-ring M, the following statements hold:*

- (i) *$U \cap S_{a_\sigma}(M) \neq 0$, for every non-zero σ-ideal U of M.*
- (ii) *If U be a non-zero σ-ideal of M and if $a, b \in M$ such that $a\,\alpha\,U\,\beta\,b = 0 = a\,\alpha\,U\,\beta\,\sigma(b)$, then $a = 0$ or $b = 0$.*

Proof of (i): Let us suppose that $U \cap S_{a_\sigma}(M) = 0$, then for any $x \in U$ and $\alpha \in \Gamma$.
We have

$$x\,\alpha\,S_{a_\sigma}(M)\,\alpha\,\sigma(x) \subset U \cap S_{a_\sigma}(M) = 0$$

Hence for any $m \in M$, we write

$$x\,\alpha\,(\sigma(m) - m)\,\alpha\,\sigma(x) = 0$$

$$\text{or, } x\,\alpha\,\sigma(m)\,\alpha\,\sigma(x) - x\,\alpha\,m\,\alpha\,\sigma(x) = 0$$

$$\text{or, } x\,\alpha\,\sigma(m)\,\alpha\,\sigma(x) = x\,\alpha\,m\,\alpha\,\sigma(x)$$

This implies that

$$x\,\alpha\,m\,\alpha\,\sigma(x) \in U \cap S_{a_\sigma}(M)$$

Consequently,

$$x\,\alpha\,M\,\alpha\,\sigma(x) = 0$$

Since

$$\sigma(x) + x \in U \cap S_{a_\sigma}(M).$$

Then $\sigma(x) = -x$ and the σ-primeness of M yields $x = 0$. Therefore, if $U \neq 0$, we may conclude that $U \cap S_{a_\sigma}(M) \neq 0$. □

Proof of (ii): Let us consider that $a \neq 0$, there exists some $x \in U$ and $\alpha \in \Gamma$ such that

$$a\,\alpha\,x \neq 0$$

Indeed, otherwise, $\forall\, x \in U$

$$a\,\alpha\,M\,\alpha\,x = 0, \text{ and } a\,\alpha\,M\,\alpha\,\sigma(x) = 0, \text{ so } a = 0.$$

Since

$$a\,\alpha\,U\,\beta\,M\,\gamma\,b = 0, \text{ and } a\,\alpha\,U\,\beta\,M\,\gamma\,\sigma(b) = 0.$$

In view of the σ-primeness of M, yields $b = 0$. □

As we previously said, M. R. Khan and M. M. Hasnain [12] worked on sigma prime rings with differential identities on sigma ideals. In their article, they discussed the generalized (α, β)-derivation of a ring R that acts as a homomorphism or an anti-homomorphism on a nonempty subset A. They precisely proved the following theorem:

Theorem 3.1. ([12]): *Suppose that R is a 2-torsion-free σ-prime ring, $U \neq 0$ is a σ-ideal, and $(f \neq 0, d)$ is a generalized (α, β)-derivation with the additional condition that $\sigma D = D\sigma$, where β is an automorphism on R such that $\sigma\beta = \beta\sigma$. If f acts as a homomorphism or as an anti-homomorphism on U, then $d = 0$ or $f = \beta$.*

We extended the above theorem for gamma rings with generalized (ϕ, ψ)-derivations on σ-ideal of a σ-prime Γ-ring as follows:

Theorem 3.2. *Let U be a non-zero σ-ideal of a 2-torsion free σ-prime Γ-ring M and (G, g) is a generalized (ϕ, ψ)-derivation with the assumption that $\sigma g = g\sigma$, where ψ is an automorphism of M such that $\sigma\psi = \psi\sigma$. If G acts as a homomorphism or as an anti-homomorphism on U, then $g = 0$ or $G = \psi$.*

Proof: **Step I:** Let us consider that G acts as a homomorphism on U, then $\forall\, x, y \in U$ and $\alpha \in \Gamma$, we have

$$G(x\,\alpha\,y) = G(x)\,\alpha\,G(y) \tag{1}$$

Since, $G : M \to M$ is a generalized (ϕ, ψ)-derivation associated with (ϕ, ψ)-derivation g, then $\forall\, x, y \in U$ and $\alpha \in \Gamma$

$$G(x)\,\alpha\,G(y) = G(x)\,\alpha\,\phi(y) + \psi(x)\,\alpha\,g(y) \tag{2}$$

Using equation (1) and (2), we obtain

$$G(x)\,\alpha\,G(y) = G(x\,\alpha\,y) = G(x)\,\alpha\,\phi(y) + \psi(x)\,\alpha\,g(y) \tag{3}$$

If we replace x by $x\,\alpha\,z$ in equation (3), we get

$$\begin{aligned} G(x\,\alpha\,z)\,\alpha\,G(y) &= G(x\,\alpha\,z)\,\alpha\,\phi(y) + \psi(x\,\alpha\,z)\,\alpha\,g(y) \\ &= G(x)\,\alpha\,G(z)\,\alpha\,\phi(y) + \psi(x\,\alpha\,z)\,\alpha\,g(y) \end{aligned} \tag{4}$$

Since G is a homomorphism on U, then $\forall x, y \in U$ and $\alpha \in \Gamma$ we have

$$\begin{aligned} G(x\,\alpha\,z)\,\alpha\,G(y) &= G(x)\,\alpha\,G(z\,\alpha\,y) \\ &= G(x)\,\alpha\,[G(z)\,\alpha\,\phi(y) + \psi(z)\,\alpha\,g(y)] \\ &= G(x)\,\alpha\,G(z)\,\alpha\,\phi(y) + G(x)\,\alpha\,\psi(z)\,\alpha\,g(y) \end{aligned} \tag{5}$$

Using the equations (4) and (5), $\forall\, x, y, z \in U$ and $\alpha \in \Gamma$, we obtain

$$G(x)\,\alpha\,\psi(z)\,\alpha\,g(y) - \psi(x\,\alpha\,z)\,\alpha\,g(y) = 0$$

$$\text{or,}\;\; G(x)\,\alpha\,\psi(z)\,\alpha\,g(y) - \psi(x)\,\alpha\,\psi(z)\,\alpha\,g(y) = 0$$

$$\text{or,}\;\; ([G(x) - \psi(x)])\,\alpha\,\psi(z)\,\alpha\,g(y) = 0$$

Hence

$$(G(x) - \psi(x))\,\alpha\,\psi(U)\,\alpha\,g(y) = \{0\}$$

Say, $V = \psi(U)$. We see that V is a nonzero σ-ideal, that is, $\forall\, x, y \in U$ and $\alpha \in \Gamma$

$$(G(x) - \psi(x))\,\alpha\,V\,\alpha\,g(y) = \{0\} \tag{6}$$

Now, equation (6) yields

$$(G(x) - \psi(x))\,\alpha\,V\,\alpha\,g(y) = (G(x) - \psi(x))\,\alpha\,V\,\alpha\,\sigma(g(y)) = \{0\}$$

As g commutes with σ and V is a σ-ideal, then by Lemma 3.2 (ii), we have either $G(x) - \psi(x) = 0$ or $g(y) = 0$ for all $x, y \in U$; namely, $g = 0$ or $G = \psi$ on U.

Step II: Suppose, G acts as an anti-homomorphism on U. Then $\forall x, y \in U$ and $\alpha \in \Gamma$:

$$G(y)\,\alpha\,G(x) = G(x\,\alpha\,y) = G(x)\,\alpha\,\phi(y) + \psi(x)\,\alpha\,g(y) \tag{7}$$

Replacing x by $x\,\alpha\,y$ in equation (7), we get

$$\begin{aligned} G(y)\,\alpha\,G(x\,\alpha\,y) &= G(y)\,\alpha\,[G(x)\,\alpha\,\phi(y) + \psi(x)\,\alpha\,g(y)] \\ &= G(y)\,\alpha\,G(x)\,\alpha\,\phi(y) + G(y)\,\alpha\,\psi(x)\,\alpha\,g(y) \\ &= G(y)\,\alpha\,G(x)\,\alpha\,\phi(y) + \psi(x\,\alpha\,y)\,\alpha\,g(y). \end{aligned}$$

So that,

$$G(y)\,\alpha\,\psi(x)\,\alpha\,g(y) = \psi(x\,\alpha\,y)\,\alpha\,g(y),\ \forall\, x, y \in U \text{ and } \alpha \in \Gamma \tag{8}$$

Putting $x = z\,\alpha\,x$ in equation (8) and using it, we arrive at

$$\begin{aligned} &G(y)\,\alpha\,\psi(z\,\alpha\,x)\,\alpha\,g(y) = \psi(z\,\alpha\,x)\,\alpha\,\psi(y)\,\alpha\,g(y) \\ &\text{or, } G(y)\,\alpha\,\psi(z)\,\alpha\,\psi(x)\,\alpha\,g(y) = \psi(z)\,\alpha\,\psi(x)\,\alpha\,\psi(y)\,\alpha\,g(y) \\ &\text{or, } G(y)\,\alpha\,\psi(z)\,\alpha\,\psi(x)\,\alpha\,g(y) - \psi(z)\,\alpha\,\psi(x)\,\alpha\,\psi(y)\,\alpha\,g(y) = 0 \\ &\text{or, } ([G(y)\,\alpha\,\psi(z) - \psi(z)\,\alpha\,\psi(y)])\,\alpha\,\psi(x)\,\alpha\,g(y) = 0 \\ &\text{or, } ([G(y)\,\alpha\,\psi(z) - \psi(z)\,\alpha\,G(y)])\,\alpha\,\psi(x)\,\alpha\,g(y) = 0 \end{aligned}$$

$$[G(y), \psi(z)]_\alpha\,\alpha\,\psi(x)\,\alpha\,g(y) = 0,\ \ \forall\, x, y, z \in U, \text{ and } \alpha \in \Gamma \tag{9}$$

As ψ is an automorphism, that is, if we set $V = \psi(U)$, yields

$$[G(y), \psi(z)]_\alpha \alpha\,V\,\alpha\,g(y) = \{0\},\ \ \forall\, x, y, z \in U, \text{ and } \alpha \in \Gamma \tag{10}$$

For all $y \in U \cap S_{a_\sigma}(M)$ and from equation (10), we have

$$[G(y), \psi(z)]_\alpha \alpha\,V\,\alpha\,g(y) = [G(y), \psi(z)]_\alpha \alpha\,V\,\alpha\,\sigma(g(y)) = \{0\},\ \ \forall\, x, y, z \in U, \text{ and } \alpha \in \Gamma \tag{11}$$

In view of Lemma 3.2 (ii), the last equality yields

$$G(y) \in Z(M) \ \text{ or } \ g(y) = 0$$

Now assume that $g(x) = 0$. Then G is a left multiplier and equation (7) yields

$$\begin{aligned} G(y)\,\alpha\,G(x) &= G(y\,\alpha\,x) \\ &= G(y)\,\alpha\,\phi(x) + \psi(y)\,\alpha\,g(x) \\ &= G(y)\,\alpha\,\phi(x),\ \ \forall\, x, y \in U, \text{ and } \alpha \in \Gamma \end{aligned} \tag{12}$$

Taking $y\alpha x$ instead of y in equation (12) and using it, we reach the following

$$\begin{aligned} G(y\,\alpha\,x)\,\alpha\,G(x) &= [G(y)\,\alpha\,\phi(x) + \psi(y)\,\alpha\,g(x)]\,\alpha\,G(x) \\ &= G(y)\,\alpha\,\phi(x)\,\alpha\,G(x) + \psi(y)\,\alpha\,g(x)\,\alpha\,G(x) \\ &= G(y)\,\alpha\,\phi(x)\,\alpha\,G(x), \quad (\text{since } g(x) = 0) \end{aligned}$$

Therefore, $\forall\ x, y \in U\ and\ \alpha \in \Gamma$, we get

$$G(y\,\alpha\,x)\,\alpha\,G(x) = G(y)\,\alpha\,\phi(x)\,\alpha\,G(x)$$

$$\text{or, } G(y)\,\alpha\,G(x)\,\alpha\,G(x) = G(y)\,\alpha\,\phi(x)\,\alpha\,G(x)$$

$$\text{or, } G(y)\,\alpha\,\phi(x)\,\alpha\,G(x) = G(x)\,\alpha\,G(y)\,\alpha\,G(x)$$

$$\text{or, } G(y)\,\alpha\,\phi(x)\,\alpha\,G(x) - G(x)\,\alpha\,G(y)\,\alpha\,G(x) = 0$$

$$\text{or, } (G(y)\,\alpha\,\phi(x) - G(x)\,\alpha\,G(y))\,\alpha\,G(x) = 0 \tag{13}$$

As U is a σ-ideal and from Lemma 3.2 (ii), we get

$$G(x)\,\alpha\,G(y) = G(y)\,\alpha\,\phi(x), \quad \forall\ x, y \in U, \text{ and } \alpha \in \Gamma \tag{14}$$

Comparing equations (11) and (12), we have

$$G(y)\,\alpha\,G(x) = G(x)\,\alpha\,G(y), \quad \forall\ x, y \in U, \text{ and } \alpha \in \Gamma \tag{15}$$

Therefore, in both cases, we obtain

$$G(y)\,\alpha\,G(x) = G(x)\,\alpha\,G(y), \quad \forall\ x, y \in U, \text{ and } \alpha \in \Gamma$$

Thus, G acts as a homomorphism or as an anti-homomorphism on U, and hence, by the previous part of the theorem, we conclude that $g = 0$ or $G = \psi$. □

4. Conclusion

In classical ring theory, Γ-ring plays a very significant role in modern branches of algebra in mathematics. This concept of Γ-ring has a special place among generalizations of rings. In current years, many modern algebraists are engaged in analyzing and characterizing various properties of Γ-ring by different methods. In this paper, we tried to study generalized $(\phi,\ \psi)$-derivation, homomorphism, and anti-homomorphism in Γ-rings and also proved a theorem based on these properties. In this paper, the Γ-ring is supposed to be 2-torsion free, and we have established and analyzed the theorem in our work only for the σ-prime Γ-ring.

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