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NONCOERCIVE ELLIPTIC NEUMANN PROBLEM WITH L^1-DATA

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Abstract. In this study, we consider the equation $-\Delta u + \mathrm{div}(\mathbf{v}\, u) + bu = f$ in Ω paired with Neumann boundary conditions $\nabla u \cdot \eta - (\mathbf{v} \cdot \eta)u = 0$ on $\partial\Omega$. We introduce a notion of renormalized solution for this problem and establish both the existence and uniqueness of this solution for L^1-data.

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1. Introduction

Convection-diffusion-reaction equations are widely used in many physical, chemical, and biological phenomena. They model processes where a quantity (such as the concentration of a chemical, the temperature, or a polution) is not only diffused in a medium but also transported by a flow and subjected to internal reactions. The complexity of these interactions makes them an essential subject of study for understanding the dynamics at work in a variety of systems.

The following convection-diffusion-reaction problem is defined in a domain Ω bounded, polygonal, and connected open of $\mathbb{R}^d$ $(d \geq 2)$:

$$\begin{cases} -\Delta u + \mathrm{div}(\mathbf{v}\, u) + bu = f & \text{in } \Omega, \\ \nabla u \cdot \eta - (\mathbf{v} \cdot \eta)u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where

- u represents, for example, concentration, temperature, etc,
- $\mathbf{v}$ is a vector field modelling the flow velocity in Ω,

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- b is a coefficient relating to the intensity of the reaction,
- f is an outside source,
- η is the unit outward normal on $\partial\Omega$.

Similar problems have been studied by several authors. In 2024, the authors in [15] examined analogous non-coercive problems with Dirichlet boundary conditions, where existence and uniqueness were demonstrated for measure data. In 2023, the authors in [13] showed that the approximate solution obtained by the finite volumes method converges to the renormalized solution of (1.1). Additionally, in 2002, [8] studied the existence, uniqueness, and regularity of the solution to (1.1) under mixed boundary conditions using the duality method. In the context of Neumann boundary conditions, existence results were established in [5], with additional uniqueness results presented in [6]. Neumann boundary value problems with L^1-data and $\mathbf{v} = 0$ have been widely studied, with weak solutions proven to belong to an appropriate Sobolev space and to have a null mean value. In [1], the authors studied the case of irregular data and domains, obtaining solutions with zero median using an approximation method. Similarly in [5], authors used the median to establish the existence of a renormalised solution with zero median, recognising that the mean value of the solution is often not well defined.
In the present paper, we studied the problem (1.1) with L^1- data. The weak solution does not guarantee uniqueness, there are other types of solution which also offer this guarantee, notably the renormalized solution. The concept of renormalized solutions was introduced by DiPerna and Lions [7] in their work on the Boltzmann equations.
One way of dealing with the problem (1.1) is to reformulate it in a variational form. By multiplying the equation by a test function $\phi \in H^1(\Omega)$ and integrating over Ω, a bilinear form $a(u, \phi)$ is well-defined, due to Sobolev embedding. This form is continuous but not coercive on $H^1(\Omega)$ without assumptions on div($\mathbf{v}$). Even though the operator, it generally lacks coercivity unless $\mathbf{v}$ is sufficiently small. The study of elliptic equations with L^1-data is motivated by their application in physical models, such as the Thomas-Fermi models in atomic physics and reservoir models in porous media [9].

The rest of the paper is organized as follows: Section 2 recalls the assumptions, introduces relevant definitions (including the median of a measurable function), presents the concept of renormalized solutions for (1.1) and announces the main results. The section 3 is devoted to proving Theorem 2.1 below.

2. Assumptions and definitions

We consider the following non-linear elliptic problem with Neumann boundary conditions:

$$\begin{cases} -\Delta u + \operatorname{div}(\mathbf{v}\, u) + bu = f & \text{in } \Omega, \\ \nabla u \cdot \eta - (\mathbf{v} \cdot \eta) u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

We assume that

$$\mathbf{v} \in (L^p(\Omega))^d \text{ with } 2 < p < +\infty \text{ if } d = 2,\ p = d, \text{ if } d \geq 3, \tag{2.2}$$

$$b \in L^2(\Omega),\ b \geq 0 \text{ a.e. in } \Omega. \tag{2.3}$$

Moreover, we assume that

$$f \in L^1(\Omega), \tag{2.4}$$

and it satisfies the compatibility condition

$$\int_\Omega f dx = 0. \tag{2.5}$$

Note that in this paper the data f and the domain Ω are not regular. This situation in the Neumann case has been studied in [1] and solutions whose median is equal to zero are obtained. We recall that the median of a function $u \in H^1(\Omega)$ is defined as

$$med(u) := \sup\left\{\lambda \in \Omega : |\{x \in \Omega : u(x) > \lambda\}| \geq \frac{|\Omega|}{2}\right\}. \tag{2.6}$$

It is known that $med(u)$ is a non-empty compact interval (see [16]). Let us explicitly observe that if $0 \in med(u)$ then

$$|\{x \in \Omega : u(x) > 0\}| \leq \frac{|\Omega|}{2} \text{ and } |\{x \in \Omega : u(x) < 0\}| \leq \frac{|\Omega|}{2}. \tag{2.7}$$

In this case, we have the following Poincaré-Wirtinger inequality (see e.g. [16])

$$\|u - med(u)\|_{L^2(\Omega)} \leq C\|\nabla u\|_{(L^2(\Omega))^d}, \qquad \forall u \in H^1(\Omega), \tag{2.8}$$

where C is a constant depending on d, Ω.

We need a few notations. We denote by $meas(\Omega) = |\Omega|$ the d-dimensional Lebesgue measure of $\mathbb{R}^d$ and χ_Ω its characteristic function. Moreover, we adopt the notation $r^+ = \max(r, 0)$ and $r^- = \max(-r, 0)$ for the positive and the negative part of a function r, respectively. We also denote $\mathrm{sign}_0(\mathrm{r}) = 1, 0$ or -1 depending on whether $r > 0$, $r = 0$ or $r < 0$.

For any $\gamma > 0$ we denote by T_γ the truncation function at height γ, $T_\gamma(z) = \min(\gamma, \max(-\gamma, z))$ for any $z \in \mathbb{R}$ and we define for $\ell \geq 0$, the function S_ℓ by

$$S_\ell(r) = 1 - |T_{\ell+1}(r) - T_\ell(r)|, \forall r \in \mathbb{R}. \tag{2.9}$$

We now recall the gradient of functions whose truncates belong to $H^1(\Omega)$ (see [3]).

Lemma 2.1. *Let $u : \Omega \to \mathbb{R}$ be a measurable function, finite almost everywhere in Ω, such that $T_\gamma(u) \in H^1(\Omega)$ for any $\gamma > 0$. Then there exists a unique measurable vector field $w : \Omega \to \mathbb{R}^N$ such that*

$$\nabla T_\gamma(u) = \chi_{\{|u|<\gamma\}} w \quad a.e. \ in \ \Omega.$$

This function w is called the gradient of u and is denoted by ∇u.

We now recall the definition of a renormalized solution to (1.1).

Definition 2.1. *We say that a function $u : \Omega \to \overline{\mathbb{R}}$ is a renormalized solution to (1.1) if u is measurable and finite a.e. in Ω, satisfies the following conditions*

$$T_\gamma(u) \in H^1(\Omega), \text{ for any } \gamma > 0, \tag{2.10}$$

$$\lim_{\ell\to\infty} \int_{\{\ell<|T_\gamma(u)|<\ell+1\}} |\nabla T_\gamma(u)|^2 \, dx = 0, \tag{2.11}$$

and the following equation holds

$$\begin{aligned}\int_\Omega S(u)\,\nabla u \cdot \nabla\varphi\, dx + \int_\Omega S'(u)\,\varphi\,\nabla u\cdot\nabla u\, dx - \int_\Omega S(u)\, u\, \mathbf{v}\cdot\nabla\varphi\, dx \\ - \int_\Omega S'(u)\,\varphi\, u\,\mathbf{v}\cdot\nabla u\, dx + \int_\Omega b\, u\, S(u)\,\varphi\, dx = \int_\Omega f\,\varphi\, S(u)\, dx,\end{aligned} \tag{2.12}$$

for every $S \in C_c^1(\mathbb{R})$ having compact support and for every $\varphi \in H^1(\Omega)\cap L^\infty(\Omega)$.

Remark 2.1. *Condition (2.10) allows to define ∇u almost everywhere in Ω. Equality (2.12) is formally obtained by using in (1.1) the test function $S(u)\varphi$ and by taking into account Neumann boundary conditions. Since $S(u)\varphi \in H^1(\Omega)$ and since $Supp(S) \subset [-\gamma,\gamma]$for sufficiently large $\gamma > 0$, we can rewrite (2.12) as follows*

$$\begin{aligned}\int_\Omega S(u)\,\nabla T_\gamma(u)\cdot\nabla\varphi\, dx + \int_\Omega S'(u)\,\varphi\,\nabla T_\gamma(u)\cdot\nabla T_\gamma(u) dx - \int_\Omega S(u)\, u\,\mathbf{v}\cdot\nabla\varphi dx \\ - \int_\Omega S'(u)\,\varphi\, u\,\mathbf{v}\cdot\nabla T_\kappa(u)\, dx + \int_\Omega b\, uS(u)\,\varphi\, dx = \int_\Omega f\varphi\, S(u) dx.\end{aligned} \tag{2.13}$$

Let us observe that every integral in (2.13) is well defined thanks to the fact that $T_\gamma(u) \in H^1(\Omega)$ for every $\gamma > 0$, S has compact support and the assumptions (2.2)-(2.4). Condition (2.11) is classical in the framework of renormalized solutions.

The main results of the present work is the following.

Theorem 2.1. *Let us assume that (2.2)-(2.5) are fulfilled. Then, there exists a unique renormalized solution u to problem (1.1) with $med(u) = 0$.*

3. Proof of Theorem 2.1

3.1. Existence of renormalized solution.

The proof is split into several steps.

Step 1. Approximate problems

Let $f_\varepsilon = T_{1/\varepsilon}(f)$ be a sequence of bounded functions which strongly converges to f in $L^1(\Omega)$ such that

$\|f_\varepsilon\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}$ and $\displaystyle\int_\Omega f_\varepsilon\,dx = 0$ for all $\varepsilon > 0$.
We consider the following approximate problem

$$\begin{cases} -\Delta u_\varepsilon + \operatorname{div}(\mathbf{v}\,u_\varepsilon) + b\,u_\varepsilon = f_\varepsilon & \text{in } \Omega, \\ \nabla u_\varepsilon \cdot \eta - (\mathbf{v}\cdot\eta)u_\varepsilon = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

By known results there exists at least a weak solution $u_\varepsilon \in H^1(\Omega)$ to (3.1) ([11,12]) which $med(u_\varepsilon) = 0$ and for any $\varphi \in H^1(\Omega)$,

$$\int_\Omega \nabla u_\varepsilon \cdot \nabla\varphi\,dx - \int_\Omega u_\varepsilon\,\mathbf{v}\cdot\nabla\varphi\,dx + \int_\Omega b\,u_\varepsilon\,\varphi\,dx = \int_\Omega f_\varepsilon \varphi dx. \tag{3.2}$$

Step 2. A priori estimates

This step is devoted to finding a priori estimates of the solution of (3.1), which are crucial to pass to the limit.

Lemma 3.1.

(**i**) *For all $\gamma > 0$, the sequence $T_\gamma(u_\varepsilon)$ is bounded.*

(**ii**) *The sequence of renormalized solutions $(u_\varepsilon)_{\varepsilon>0}$ of the problems (3.1) verifies, for $\gamma > 0$ large enough, the following estimates:*

$$\|\ln(1+|u_\varepsilon|)\|^2_{L^2(\Omega)} \leq C, \tag{3.3}$$

$$meas\{|u_\varepsilon| \geq \gamma\} \leq \frac{C}{(\ln(1+\gamma))^2} \tag{3.4}$$

and

$$meas\{|\nabla u_\varepsilon| \geq \gamma\} \leq \frac{C}{\gamma} + \frac{C}{(\ln(1+\gamma))^2}, \tag{3.5}$$

where $C > 0$ is a constant.

(**iii**) *For all $\ell > 0$,*

$$\lim_{\ell\to\infty} \limsup_{\varepsilon\to 0} \int_{\{\ell<|T_\gamma(u_\varepsilon)|<\ell+1\}} |\nabla T_\gamma(u_\varepsilon)|^2\,dx = 0. \tag{3.6}$$

Proof. For (**i**), we use $\phi = T_\gamma(u_\varepsilon)$ as test function in (3.2) to obtain

$$\int_\Omega |\nabla T_\gamma(u_\varepsilon)|^2\,dx + \int_\Omega b\,u_\varepsilon\,T_\gamma(u_\varepsilon)\,dx = \int_\Omega T_\gamma(u_\varepsilon)\,\mathbf{v}\cdot\nabla T_\gamma(u_\varepsilon)\,dx + \int_\Omega f_\varepsilon T_\gamma(u_\varepsilon)\,dx. \tag{3.7}$$

Since b is non-negative and $T_\gamma(u_\varepsilon)$ has the same sign as u_ε then $bu_\varepsilon T_\gamma(u_\varepsilon) \geq 0$. Then, we deduce from (3.7) that

$$\int_\Omega |\nabla T_\gamma(u_\varepsilon)|^2\,dx \leq \gamma||f||_1 + \int_\Omega T_\gamma(u_\varepsilon)\,\mathbf{v}\cdot\nabla T_\gamma(u_\varepsilon)\,dx. \tag{3.8}$$

By Young's inequality and Sobolev embedding, we can control the second integral of the right-hand side of (3.8). Then, we can write

$$||\nabla T_\gamma(u_\varepsilon)||^2_{(L^2(\Omega))^d} \le C, \tag{3.9}$$

where $C = 2\gamma||f||_1 + S(\Omega,\gamma)\gamma^2\|\mathbf{v}\|^p_{(L^p(\Omega))^d}$.

To obtain (**ii**), we first prove (3.3). As stated in [15], we consider the test function $\phi(u_\varepsilon) = \int_0^{u_\varepsilon} \frac{1}{(1+|r|)^2} dr$ in (3.2). Observe that $med(\phi(u_\varepsilon)) = med(u_\varepsilon) = 0$. We have

$$\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(1+|u_\varepsilon|)^2}\,dx + \int_\Omega b u_\varepsilon \phi(u_\varepsilon)\,dx \le \|f\|_1 + \int_\Omega |u_\varepsilon|\mathbf{v}|\frac{|\nabla u_\varepsilon|}{(1+|u_\varepsilon|)^2}\,dx. \tag{3.10}$$

Since b is a non-negative function and moreover $\phi(u_\varepsilon)$ and u_ε have the same sign, then the second term in the left-hand side of (3.10) is non-negative. This leads to

$$\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(1+|u_\varepsilon|)^2}\,dx \le \|f\|_1 + \int_\Omega |u_\varepsilon||\mathbf{v}|\frac{|\nabla u_\varepsilon|}{(1+|u_\varepsilon|)^2}\,dx. \tag{3.11}$$

Furthermore, we have $|u_\varepsilon| \le 1 + |u_\varepsilon|$. Then, using Young's inequality and Sobolev embedding, the second term on the right-hand side of (3.11) can be estimated as follows:

$$\begin{aligned}\left|\int_\Omega u_\varepsilon \mathbf{v}\frac{\nabla u_\varepsilon}{(1+|u_\varepsilon|)^2}\,dx\right| &\le \int_\Omega \frac{|\mathbf{v}||\nabla u_\varepsilon|}{1+|u_\varepsilon|}dx \\ &\le \frac{S(\Omega,p)}{2}\|\mathbf{v}\|^p_{(L^p(\Omega))^d} + \frac{1}{2}\|\nabla \ln(1+|u_\varepsilon|)\|^2_{(L^2(\Omega))^d}.\end{aligned} \tag{3.12}$$

For the first term of (3.10), we have

$$\int_\Omega \left(\frac{|\nabla u_\varepsilon|}{1+|u_\varepsilon|}\right)^2 dx = \|\nabla \ln(1+|u_\varepsilon|)\|^2_{(L^2(\Omega))^d}. \tag{3.13}$$

Combining (3.11)-(3.13), we obtain

$$\|\nabla \ln(1+|u_\varepsilon|)\|^2_{(L^2(\Omega))^d} \le 2||f||_1 + S(\Omega,p)\|\mathbf{v}\|^2_{(L^p(\Omega))^d}.$$

Since $med(\ln(1+u_\varepsilon)) = 0$, Poincaré-Wirtinger inequality (2.8) implies that

$$\|\ln(1+|u_\varepsilon|)\|^2_{L^2(\Omega)} \le C, \tag{3.14}$$

where $C = C(f,\Omega,\mathbf{v})$ is a positive constant, and then (3.3) is proved.

Using again the inequality (3.14), we get

$$\int_{\{|u_\varepsilon|\ge\gamma\}} (\ln(1+\gamma))^2\,dx \le C(f,\Omega,\mathbf{v}),$$

which implies

$$meas\{|u_\varepsilon| \ge \gamma\} \le \frac{C(f,\Omega,\mathbf{v})}{(\ln(1+\gamma))^2}.$$

We set $\Phi(\gamma,\lambda) = \text{meas}\{|\nabla u_\varepsilon|^2 > \lambda,\ |u_\varepsilon| > \gamma\}$, for all$\gamma, \lambda > 0$. According to (3.4), we have

$$\Phi(\gamma, 0) \leq \frac{C}{(\ln(1+\gamma))^2}, \text{ for any } \gamma > 0 \text{ large enough.} \tag{3.15}$$

Thanks to (3.9), we obtain

$$\int_0^\infty \big(\Phi(0,s) - \Phi(\gamma,s)\big) ds \leq C. \tag{3.16}$$

We deduce from (3.15) and (3.16) that

$$\Phi(0,\lambda) \leq \frac{C}{\lambda} + \frac{C}{(\ln(1+\gamma))^2}, \quad \text{for all } \gamma, \lambda > 0. \tag{3.17}$$

Setting $\lambda = \gamma$ in (3.17) leads to (3.5).

For ((**iii**)), we take $\phi = T_1\big(T_\gamma(u_\epsilon) - T_\ell(T_\gamma(u_\epsilon))\big)$ as test function in (3.2). Using the same approach as in Lemma 4 of [15], we obtain (3.6). □

Step 3. Convergence results

We start by proving that the sequence $(u_\varepsilon)_\varepsilon$ of solutions of problem (3.1) converges in measure to a measurable function u.

Lemma 3.2. *Assume* (2.2)–(2.5) *hold and let* $u_\varepsilon \in H^1(\Omega)$ *be a solution of* (3.1).

(**i**) *For all* $\gamma > 0$, $T_\gamma(u_\varepsilon) \to T_\gamma(u)$ *strongly in* $L^2(\Omega)$ *and a.e. in* Ω, *as* $\varepsilon \to 0$.

(**ii**) *The sequence* $(u_\varepsilon)_\varepsilon$ *is Cauchy in measure. In particular, there exists a measurable function* u *such that* $u_\varepsilon \to u$ *in measure and a.e. in* Ω, *as* $\varepsilon \to 0$.

Proof. For any $\gamma > 0$, $\big(T_\gamma(u_\varepsilon)\big)_\varepsilon$ is bounded in $H^1(\Omega)$. Then, up to a subsequence we can assume that for any $\gamma > 0$, $T_\gamma(u_\varepsilon)$ converges weakly to σ_γ in $H^1(\Omega)$ and so $T_\gamma(u_\varepsilon)$ converges strongly to σ_γ in $L^2(\Omega)$. We now prove that $(u_\varepsilon)_\varepsilon$ is a Cauchy sequence in measure. Let $j > 0$. For all $\varepsilon > 0$, and all $\varepsilon', \gamma \geq 0$, we define

$$E_\varepsilon := \{|u_\varepsilon| > \gamma\}, \quad E_{\varepsilon'} := \{|u_{\varepsilon'}| > k\} \quad \text{and} \qquad E_{\varepsilon,\varepsilon'} := \{|T_\gamma(u_\varepsilon) - T_\gamma(u_{\varepsilon'})| > j\},$$

where $\gamma > 0$ is to be fixed. We note that $\{|u_\varepsilon - u_{\varepsilon'}| > j\} \subset E_\varepsilon \cup E_{\varepsilon'} \cup E_{\varepsilon,\varepsilon'}$, and therefore

$$\text{meas}\{|u_\varepsilon - u_{\varepsilon'}| > j\} \leq \ \text{meas}(E_\varepsilon) + \text{meas}(E_\varepsilon) + \text{meas}(E_{\varepsilon,\varepsilon'}). \tag{3.18}$$

We choose $\gamma = \gamma(\epsilon)$ such that

$$\text{meas}(E_\varepsilon) \leq \frac{\epsilon}{3} \text{ and } \text{meas}(E_{\varepsilon'}) \leq \frac{\epsilon}{3}. \tag{3.19}$$

Since $\big(T_\gamma(u_\varepsilon)\big)_\varepsilon$ converges strongly in $L^2(\Omega)$, then it is a Cauchy sequence in $L^2(\Omega)$. Thus,

$$\text{meas}(E_{\varepsilon,\varepsilon'}) \leq \frac{1}{s^2}\int_\Omega \big|T_\gamma(u_\varepsilon) - T_\gamma(u_{\varepsilon'})\big|^2 dx \leq \frac{\epsilon}{3}, \tag{3.20}$$

for all $\varepsilon, \varepsilon' \geq \varepsilon_0(j, \epsilon)$.

Finally, from (3.18)-(3.20), we obtain that

$$\text{meas}\{|u_\varepsilon - u_{\varepsilon'}| > j\} \leq \epsilon \text{ for all } \varepsilon, \varepsilon' \geq \varepsilon_0(j, \epsilon). \tag{3.21}$$

Hence, the sequence $(u_\varepsilon)_\varepsilon$ is a Cauchy sequence in measure and there exists a function u, which is finite almost everywhere on Ω, such that $u_\varepsilon \to u$ in measure. We can then extract a subsequence such that $u_\varepsilon \to u$ a.e. in Ω. Since T_γ is continuous, then $T_\gamma(u_\varepsilon) \to T_\gamma(u)$ a.e. in Ω, $\sigma_\gamma = T_\gamma(u)$ a.e. in Ω and $T_\gamma(u) \in H^1(\Omega)$.

It remains to prove that $med(u) = 0$. Since u_ε convergence to u the sequence $\chi_{\{u_\varepsilon > 0\}}$ converges to $\chi_{\{u>0\}}$ a.e. in Ω. Recalling that $med(u_\varepsilon) = 0$ then Fatou's lemma leads to

$$\begin{aligned} meas\{x \in \Omega : u(x) > 0\} &\leq \liminf_{\varepsilon \to 0} \int_\Omega \chi_{\{u_\varepsilon > 0\}} \chi_{\{u>0\}} dx \\ &\leq \liminf_{\varepsilon \to 0} meas\{u_\varepsilon > 0\} \\ &\leq \frac{meas(\Omega)}{2}. \end{aligned}$$

Similar to the convergence of $\chi_{\{u_\varepsilon < 0\}}$ to $\chi_{\{u_\varepsilon < 0\}}$ a.e. as $\varepsilon \to 0$.

$$meas\{x \in \Omega : u(x) < 0\} \leq \frac{meas(\Omega)}{2}.$$

It follows that $0 \in med(u)$. Since we have for γ large enough $med(T_\gamma(u)) = med(u)$ and $T_\gamma(u) \in H^1(\Omega)$, then $med(u) = 0$ is unique. □

Step 4. Passage to the limit

To establish that the function u is a renormalized solution of the problem (1.1), we prove the following lemma.

Lemma 3.3. *For all $S \in C_c^1(\mathbb{R})$ and $\psi \in H^1(\Omega) \cap L^\infty(\Omega)$,*

$$\nabla[S(u_\varepsilon)\psi] \to \nabla[S(u)\psi] \textit{ strongly in } (L^2(\Omega))^d, \textit{ as } \varepsilon \to 0.$$

Proof. The proof of this lemma follows the same lines as those proved in [15]. □

Next, taking $S(u_\varepsilon)\psi \in H^1(\Omega) \cap L^\infty(\Omega)$ as a test function in (3.1), we obtain

$$\begin{aligned} \int_\Omega \nabla u_\varepsilon \cdot \nabla[S(u_\varepsilon)\psi] dx - \int_\Omega u_\varepsilon \mathbf{v} \cdot \nabla[S(u_\varepsilon)\psi] dx + \int_\Omega b\, u_\varepsilon\, S(u_\varepsilon)\psi dx \\ = \int_\Omega f_\varepsilon S(u_\varepsilon)\psi dx. \end{aligned} \tag{3.22}$$

Now we pass to the limit when $\varepsilon \to 0$ in each term of (3.22). Since S has compact support, then there exists a positive real number γ such that $Supp(S) \subset [-\gamma, \gamma]$, so u_ε can be replaced by its truncation $T_\gamma(u_\varepsilon)$. We start with the first term of the left-hand side of (3.22). On the one hand, we have $\nabla[S(u_\varepsilon)\psi] \to$

$\nabla[S(u)\psi]$ strongly in $(L^2(\Omega))^d$ (see Lemma 3.3) and on the other hand, $\nabla T_\gamma(u_\varepsilon) \rightharpoonup \nabla T_\gamma(u)$ weakly in $(L^2(\Omega))^d$. Therefore, $\nabla T_\gamma(u_\varepsilon)\cdot\nabla[S(u_\varepsilon)\psi] \to \nabla T_\gamma(u_\varepsilon)\cdot\nabla[S(u_\varepsilon)\psi]$ strongly in $(L^1(\Omega))^d$. Hence, by Lebesgue's generalized convergence theorem, we deduce that

$$\lim_{\varepsilon\to 0}\int_\Omega \nabla T_\gamma(u_\varepsilon)\cdot\nabla[S(u_\varepsilon)\psi]dx = \int_\Omega \nabla T_\gamma(u)\cdot\nabla[S(u)\psi]dx, \text{ as } \varepsilon\to 0. \tag{3.23}$$

For the second integral of the left-hand side of (3.22), we have $\mathbf{v}\in(L^p(\Omega))^d$ and that $T_\gamma(u_\varepsilon)\in H^1(\Omega)\subset L^{2^*}(\Omega)$. Thus, $T_\gamma(u_\varepsilon)\mathbf{v}\in L^2(\Omega)$ given that $\dfrac{1}{p}+\dfrac{1}{2^*}=\dfrac{1}{2}$ where $2^*=\dfrac{2p}{p-2}$, $p=d$, $d\geq 3$. Consequently, the sequence $(T_\gamma(u_\varepsilon)\mathbf{v})_{\varepsilon>0}$ converges weakly to $T_\gamma(u)\mathbf{v}$ in $(L^2(\Omega))^d$. Moreover, using Lemma 3.3 and Lebesgue's generalized convergence theorem, we deduce that

$$\lim_{\varepsilon\to 0}\int_\Omega T_\gamma(u_\varepsilon)\mathbf{v}\cdot\nabla[S(u_\varepsilon)\psi]dx = \int_\Omega T_\gamma(u)\mathbf{v}\cdot\nabla[S(u)\psi]dx, \text{ as } \varepsilon\to 0. \tag{3.24}$$

Since $S(u_\varepsilon)\psi$ converges weakly-$\star$ to $S(u)\psi$ in $L^\infty(\Omega)$, using the fact that $b\in L^2(\Omega)$ and $T_\gamma(u_\varepsilon)$ converges strongly to $T_\gamma(u)$ in $L^2(\Omega)$ and a.e. in Ω when $\varepsilon\to 0$, we deduce that $bT_\gamma(u_\varepsilon)\longrightarrow bT_\gamma(u)$ strongly in $L^1(\Omega)$. We also have $f_\varepsilon\to f$ strongly in $L^1(\Omega)$. Thus, when ε tends to zero, we can pass to the limit in the third integral of the left-hand side of (3.22) and in the first integral of the right-hand side of (3.22), respectively, to obtain

$$\lim_{\varepsilon\to 0}\int_\Omega bT_\gamma(u_\varepsilon)S(u_\varepsilon)\psi dx = \int_\Omega bT_\gamma(u)S(u)\psi dx \tag{3.25}$$

and

$$\lim_{\varepsilon\to 0}\int_\Omega f_\varepsilon S(u_\varepsilon)\psi dx = \int_\Omega fS(u)\psi dx. \tag{3.26}$$

Combining (3.23)-(3.26) yields (2.12).

3.2. **Uniqueness of renormalized solution.**

Here we prove the uniqueness follows the same ideas as in [6]. Let u_1 and u_2 be two renormalized solutions of (1.1) having $med(u_1)=med(u_2)=0$. Our goal is to prove that $u_1=u_2$. Let $\gamma>0, k>0$ and $\ell>0$. Using $T_\gamma\big(T_k(u_1)-T_k(u_2)\big)S_\ell(u)\in H^1(\Omega)\cap L^\infty(\Omega)$ as test function in (1.1). Since $T_\gamma\big(T_k(u_1)-T_k(u_2)\big)S_\ell(u)$ converges to $T_\gamma(u_1-u_2)S_\ell(u)$ almost everywhere in Ω as $k\to\infty$. Then $T_\gamma(u_1-u_2)S_\ell(u)\in H^1(\Omega)\cap L^\infty(\Omega)$. So, taking (2.12) on the one hand $S=S_\ell(u_1)$ and $\varphi=T_\gamma(u_1-u_2)$ written in u_1, $S=S_\ell(u_2)$ and $\varphi=T_\gamma(u_1-u_2)$ written in u_2 on the other hand, we obtain by subtracting the two equalities

$$A_{\gamma,\ell}+B_{\gamma,\ell}+C_{\gamma,\ell}=D_{\gamma,\ell}+E_{\gamma,\ell}+F_{\gamma,\ell}, \tag{3.27}$$

where

$$A_{\gamma,\ell}=\int_\Omega\big[\nabla u_1\,S_\ell(u_1)-\nabla u_2\,S_\ell(u_2)\big]\nabla T_\gamma(u_1-u_2)dx,$$

$$B_{\gamma,\ell}=\int_\Omega\left[\nabla u_1.\nabla u_1\,S'_\ell(u_1)-\nabla u_2.\nabla u_2\,S'_\ell(u_2)\right]T_\gamma(u_1-u_2)dx,$$

$$C_{\gamma,\ell} = \int_\Omega \big[u_1 S_\ell(u_1) - u_2 S_\ell(u_2)\big] b\, T_\gamma(u_1-u_2)dx,$$
$$D_{\gamma,\ell} = \int_\Omega \big[u_1\mathbf{v}\ S_\ell(u_1) - u_2\mathbf{v}\ S_\ell(u_2)\big].\nabla T_\gamma(u_1-u_2)dx,$$
$$E_{\gamma,\ell} = \int_\Omega \big[u_1\mathbf{v}S'_\ell(u_1)\nabla u_1 - u_2\mathbf{v}S'_\ell(u_2)\nabla u_2\big]\ T_\gamma(u_1-u_2)dx,$$
$$F_{\gamma,\ell} = \int_\Omega \big[S_\ell(u_1) - S_\ell(u_2)\big] f\ T_\gamma(u_1-u_2)dx.$$

We now pass to the limit in (3.27) as $\ell \to \infty$ first and then as $\gamma \to 0$. We now study all the terms in (3.27).

Step 1. The idea behind this step is to prove that

$$\lim_{\gamma\to 0}\frac{1}{\gamma^2}\int_\Omega |\nabla T_\gamma(u_1-u_2)|^2 dx = 0. \tag{3.28}$$

For $\gamma > 0$ fixed, we examine the limit of each term in (3.27) as $\ell \to \infty$. Given that $S_\ell \to 1$ as $\ell \to \infty$, we have $\big(S_\ell(u_1) - S_\ell(u_2)\big) f\ T_\gamma(u_1-u_2) \to 0$ a.e. in Ω as $\ell \to \infty$ and

$$\big|\big(S_\ell(u_1) - S_\ell(u_2)\big) f\ T_\gamma(u_1-u_2)\big| \le 2\gamma|f| \in L^1(\Omega).$$

By using Lebesgue's dominated convergence theorem, we deduce that

$$\lim_{\ell\to\infty} F_{\gamma,\ell} = \lim_{\ell\to\infty}\int_\Omega \big(S_\ell(u_1) - S_\ell(u_2)\big) f\ T_\gamma(u_1-u_2)dx = 0. \tag{3.29}$$

On the other hand, $\big[u_1\, S_\ell(u_1) - u_2\, S_\ell(u_2)\big] b\, T_\gamma(u_1-u_2) \to b\,(u_1-u_2)T_\gamma(u_1-u_2)$ as $\ell\to\infty$. Moreover,

$$\big|\big(u_1 S_\ell(u_1) - u_2 S_\ell(u_2)\big) b\, T_\gamma(u_1-u_2)\big| \le \gamma|b\,u_1| + \gamma|b\,u_2| \in L^1(\Omega).$$

Again, we derive from Lebesgue's dominated convergence theorem that

$$\begin{aligned}\lim_{\ell\to\infty} C_{\gamma,\ell} &= \lim_{\ell\to\infty}\int_\Omega \big[u_1\, S_\ell(u_1) - u_2\, S_\ell(u_2)\big] b\, T_\gamma(u_1-u_2)\, dx \\ &= \int_\Omega b\,(u_1-u_2)T_\gamma(u_1-u_2)\, dx.\end{aligned} \tag{3.30}$$

Since $b \ge 0$ and (u_1-u_2) has the same sign as $T_\gamma(u_1-u_2)$, then

$$\lim_{\ell\to\infty} C_{\gamma,\ell} \ge 0. \tag{3.31}$$

We now turn to $D_{\gamma,\ell}$ and write it as follows

$$\begin{aligned}D_{\gamma,\ell} &= \int_\Omega \big[u_1\, S_\ell(u_1) - u_2\, S_\ell(u_2)\big]\mathbf{v}.\nabla T_\gamma(u_1-u_2)dx \\ &= \frac{1}{2}\int_\Omega (u_1-u_2)\,(S_\ell(u_1) + S_\ell(u_2))\mathbf{v}.\nabla T_\gamma(u_1-u_2)dx \\ &+ \frac{1}{2}\int_\Omega (u_1+u_2)\,(S_\ell(u_1) - S_\ell(u_2))\mathbf{v}.\nabla T_\gamma(u_1-u_2)dx.\end{aligned} \tag{3.32}$$

Since $S_\ell \to 1$ as $\ell \to \infty$, we get $|S_\ell(u_1) - S_\ell(u_2)| \to 0$ as $\ell \to \infty$. Moreover, $|(u_1 + u_2)(S_\ell(u_1) - S_\ell(u_2))\mathbf{v}.\nabla T_\gamma(u_1 - u_2)| \leq |(u_1 + u_2)\mathbf{v}.\nabla T_\gamma(u_1 - u_2)| \in L^1(\Omega)$. Thus, by the Lebesgue dominated convergence theorem, it follows that

$$\lim_{\ell\to\infty} \frac{1}{2} \int_\Omega (u_1 + u_2)(S_\ell(u_1) - S_\ell(u_2))\mathbf{v}.\nabla T_\gamma(u_1 - u_2)dx = 0. \tag{3.33}$$

As in the previous case, we deduce that

$$\frac{1}{2} \int_\Omega (u_1 - u_2)(S_\ell(u_1) + S_\ell(u_2))\mathbf{v}.\nabla T_\gamma(u_1 - u_2)dx \to \int_\Omega (u_1 - u_2)\mathbf{v}.\nabla T_\gamma(u_1 - u_2)dx,$$

as $\ell \to \infty$. Then, just remember that $\nabla T_\gamma(u_1 - u_2) = \chi_{\{0<|u_1-u_2|<\gamma\}}\nabla(u_1 - u_2)$,

$$\int_\Omega (u_1 - u_2)\mathbf{v}.\chi_{\{0<|u_1-u_2|<\gamma\}}\nabla(u_1 - u_2)dx = \int_{\omega_\gamma} (u_1 - u_2)\mathbf{v}.\nabla(u_1 - u_2)dx.$$

where we set $\omega_\gamma = \{x \in \Omega : 0 < |u_1 - u_2| < \gamma\}$. On the other hand, $|u_1 - u_2| < \gamma$, then $|u_1 - u_2|\,|\mathbf{v}|.|\nabla(u_1 - u_2)| < \gamma|\mathbf{v}| \cdot |\nabla(u_1 - u_2)|$ and we have

$$\int_{\omega_\gamma} |u_1 - u_2|\,|\mathbf{v}| \cdot |\nabla(u_1 - u_2)|dx \leq \gamma \int_{\omega_\gamma} |\mathbf{v}| \cdot |\nabla(u_1 - u_2)|dx. \tag{3.34}$$

By using the Young's inequality, the right hand side can be write

$$\gamma \int_{\omega_\gamma} |\mathbf{v}|.|\nabla(u_1 - u_2)|dx \leq \frac{\gamma^2}{2} \int_{\omega_\gamma} |\mathbf{v}|^2 dx + \frac{1}{2} \int_{\omega_\gamma} |\nabla(u_1 - u_2)|^2 dx. \tag{3.35}$$

By combining (3.33)-(3.35), we deduce that

$$\lim_{\ell\to\infty} |D_{\gamma,\ell}| \leq \frac{\gamma^2}{2} \int_{\omega_\gamma} |\mathbf{v}|^2 dx + \frac{1}{2} \int_\Omega |\nabla T_\gamma(u_1 - u_2)|^2 dx. \tag{3.36}$$

Next, due to the definition of S_ℓ, it follows that $|S'_\ell(s)| = 1\chi_{\{\ell<|s|<\ell+1\}}$ and zero for $|s| \geq \ell + 1$ or $|s| \leq \ell$. For $B_{\gamma,\ell}$, we have

$$\left| \int_\Omega \Big(\nabla u_1.\nabla u_1\ S'_\ell(u_1) - \nabla u_2.\nabla u_2\ S'_\ell(u_2) \Big) T_\gamma(u_1 - u_2)dx \right|$$
$$\leq \gamma \Big(\int_{\{\ell<|u_1|<\ell+1\}} |\nabla u_1|^2 + \int_{\{\ell<|u_2|<\ell+1\}} |\nabla u_2|^2 \Big),$$

which converges to zero, as $\ell \to \infty$, according to (2.11). Thus,

$$\lim_{l\to\infty} |B_{\gamma,\ell}| = 0. \tag{3.37}$$

For $E_{\gamma,\ell}$, we note that $T_\gamma(u_1 - u_2) \in L^\infty(\Omega)$ with $\|T_\gamma(u_1 - u_2)\|_{L^\infty(\Omega)} \leq \gamma$, we have

$$|E_{\gamma,\ell}| \leq \gamma \int_{\{\ell<|u_1|<\ell+1\}} |u_1\mathbf{v} \cdot \nabla u_1|dx + \gamma \int_{\{\ell<|u_1|<\ell+1\}} |u_2\mathbf{v} \cdot \nabla u_2|dx, \tag{3.38}$$

hence, according to [15], Lemma 4 we deduce that

$$\lim_{l\to\infty} |E_{\gamma,l}| = 0. \tag{3.39}$$

We write $A_{\gamma,\ell}$ as

$$
\begin{aligned}
A_{\gamma,\ell} &= \int_\Omega \big[\nabla u_1\ S_\ell(u_1) - \nabla u_2\ S_\ell(u_2)\big]\nabla T_\gamma(u_1-u_2)dx \\
&= \int_\Omega \big(\nabla u_1 - \nabla u_2\big)\ S_\ell(u_1)\,\nabla T_\gamma(u_1-u_2)dx + \int_\Omega (S_\ell(u_1)-S_\ell(u_2))\nabla u_2\,\nabla T_\gamma(u_1-u_2)dx. \quad (3.40)
\end{aligned}
$$

We use the previous results to pass to the limit in the two terms in the right hand-side of (3.40) to obtain

$$
\lim_{\ell\to\infty}\int_\Omega (S_\ell(u_1)-S_\ell(u_2))\nabla u_2\,\nabla T_\gamma(u_1-u_2) = 0 \quad (3.41)
$$

and

$$
\lim_{\ell\to\infty}\int_\Omega \big(\nabla u_1 - \nabla u_2\big)\ S_\ell(u_1)\,\nabla T_\gamma(u_1-u_2)dx = \int_\Omega \big(\nabla u_1 - \nabla u_2\big)\ \nabla T_\gamma(u_1-u_2)dx. \quad (3.42)
$$

From (3.41) and (3.42) and using Lemma 2.12 of [4], we have

$$
(\nabla u_1 - \nabla u_2)\chi_{\omega_\gamma} = \nabla(u_1-u_2)\chi_{\omega_\gamma} = \nabla T_\gamma(u_1-u_2),
$$

next, we deduce that

$$
\lim_{\ell\to\infty} A_{\gamma,\ell} = \int_\Omega |\nabla T_\gamma(u_1-u_2)|^2 dx. \quad (3.43)
$$

Bz combining (3.29)-(3.43), we can deduce from (3.27) that

$$
\int_\Omega |\nabla T_\gamma(u_1-u_2)|^2 dx \le \frac{\gamma^2}{2}\int_{\omega_\gamma} |\mathbf{v}|^2 dx. \quad (3.44)
$$

Note that the function $\chi_{\omega_\gamma} \to 0$ a.e. in Ω as $\gamma \to 0$. Thus dividing the above inequality by γ^2, the Lebesgue dominated convergence theorem and (3.44) allow us to conclude that

$$
\lim_{\gamma\to 0}\frac{1}{\gamma^2}\int_\Omega |\nabla T_\gamma(u_1-u_2)|^2 dx = 0. \quad (3.45)
$$

Hence the result.

Step 2. We prove that either

$$
\begin{cases}
u_1 = u_2 & \text{a.e. in } \Omega, \text{ or} \\
u_1 > u_2 & \text{a.e. in } \Omega, \text{ or} \\
u_1 < u_2 & \text{a.e. in } \Omega.
\end{cases}
$$

For $\gamma < \ell$, we have $S_\ell(u_1)\frac{1}{\gamma}T_\gamma(u_1-u_2) \in H^1(\Omega)$ and $\frac{1}{\gamma}T_\gamma(u_1-u_2)$ converges to $sign_0(u_1-u_2)$ a.e. in Ω. Next, Poincaré-Wirtinger inequality leads to

$$
\begin{aligned}
&\int_\Omega \left| S_\ell(u_1)\frac{1}{\gamma}T_\gamma(u_1-u_2) - med\Big(S_\ell(u_1)\frac{1}{\gamma}T_\gamma(u_1-u_2)\Big)\right|^2 dx \\
&\qquad\qquad \le C\int_\Omega \left|\nabla\Big(S_\ell(u_1)\frac{1}{\gamma}T_\gamma(u_1-u_2)\Big)\right|^2 dx. \quad (3.46)
\end{aligned}
$$

Let us show that the right-hand side goes to zero as $\gamma \to 0$ and $\ell \to \infty$.

Since

$$\nabla\Big(S_\ell(u_1)\frac{1}{\gamma}T_\gamma(u_1-u_2)\Big) = S_\ell(u_1))\frac{1}{\gamma}\nabla T_\gamma(u_1-u_2) + S'_\ell(u_1)\nabla u_1)\frac{1}{\gamma}T_\gamma(u_1-u_2)$$

almost everywhere in Ω. Note that $\left|\frac{1}{\gamma}T_\gamma(u_1-u_2)\right| \le 1$ and $|S'_\ell(u_1)| = 1\chi_{\{\ell<|u_1|<\ell+1\}}$, then the term of the right hand side of (3.46) leads to

$$\int_\Omega \left|\nabla\Big(S_\ell(u_1)\frac{1}{\gamma}T_\gamma(u_1-u_2)\Big)\right|^2 dx \le \frac{1}{\gamma^2}\int_\Omega S_\ell^2(u_1)|\nabla T_\gamma(u_1-u_2)|^2 dx + \int_{\{\ell<|u_1|<\ell+1\}} |\nabla u_1|^2 dx. \quad (3.47)$$

Passing to the limit in (3.47) first as $\gamma \to 0$ and then as $\ell \to \infty$, we have on the one hand

$$\lim_{\ell\to\infty}\lim_{\gamma\to 0}\frac{1}{\gamma^2}\int_\Omega S_\ell^2(u_1)|\nabla T_\gamma(u_1-u_2)|^2 dx = 0 \quad (3.48)$$

and on the other hand

$$\lim_{\ell\to\infty}\int_{\{\ell<|u_1|<\ell+1\}} |\nabla u_1|^2 dx = 0. \quad (3.49)$$

Gathering (3.48) and (3.49), it follows that

$$\lim_{\ell\to\infty}\lim_{\gamma\to 0}\int_\Omega \left|\nabla\Big(S_\ell(u_1)\frac{1}{\gamma}T_\gamma(u_1-u_2)\Big)\right|^2 dx = 0. \quad (3.50)$$

Passing to the limit in (3.46), we obtain

$$\lim_{\ell\to\infty}\lim_{\gamma\to 0}\int_\Omega \left|S_\ell(u_1)\frac{1}{\gamma}T_\gamma(u_1-u_2) - med\Big(S_\ell(u_1)\frac{1}{\gamma}T_\gamma(u_1-u_2)\Big)\right|^2 dx = 0. \quad (3.51)$$

For all $\gamma > 0$, $\left|S_\ell(u_1)\frac{1}{\gamma}T_\gamma(u_1-u_2)\right| \le 1$ and $\left|med\big(S_\ell(u_1)\frac{1}{\gamma}T_\gamma(u_1-u_2)\big)\right| \le 1$.

Using (3.51) on the one hand, up to a subsequence we obtain

$$\lim_{\ell\to+\infty}\lim_{\gamma\to 0} med\big(S_\ell(u_1)\frac{1}{\gamma}T_\gamma(u_1-u_2)\big) \le \rho, \quad (3.52)$$

for a given constant $\rho \in \mathbb{R}$, $|\rho| \le 1$.

On the other hand, since u is finite almost everywhere in Ω, so

$$\lim_{\ell\to\infty}\lim_{\gamma\to 0} S_\ell(u_1)\frac{T_\gamma(u_1-u_2)}{\gamma} = sign_0(u_1-u_2), \text{ a.e. in } \Omega \text{ and } L^\infty(\Omega) \text{ weak-}*.$$

Subsequently, we infer from (3.51) that

$$\int_\Omega |sign_0(u_1-u_2) - \rho|^2 dx = 0. \quad (3.53)$$

This suggests that $\rho = 0$ or $\rho = -1$ or $\rho = 1$. It either leads to

$$u_1 = u_2, \text{ a.e. in } \Omega \;\; or \;\; u_1 > u_2, \text{ a.e. in } \Omega \;\; or \;\; u_1 < u_2, \text{ a.e. in } \Omega.$$

Step 3. We prove that $u_1 < u_2$, a.e. in Ω or $u_1 > u_2$, a.e. in Ω can not occur.

To do this, we first prove that u_1 and u_2 have the same sign. Assume that

$$u_1 < u_2, \text{ a.e. in } \Omega. \tag{3.54}$$

Fix any $\ell > 0$, $\gamma > 0$, $h > 0$ and let us consider the test function

$$W_{\ell,\gamma,h} = S_\ell(u_1)\frac{T_\gamma(u_1-u_2)}{\gamma}\Big(\frac{T_h(u_2^+)}{h} - \frac{T_h(u_1^-)}{h}\Big).$$

Due to the fact that $S_\ell(u_1)\dfrac{T_\gamma(u_1-u_2)}{\gamma} \in H^1(\Omega)\cap L^\infty(\Omega)$, for any $\gamma < \ell$, then $W_{\ell,\gamma,h} \in H^1(\Omega)\cap L^\infty(\Omega)$. Moreover we have

$$\begin{aligned}\nabla W_{\ell,\gamma,h} &= \Big(\frac{T_h(u_2^+)}{h} - \frac{T_h(u_1^-)}{h}\Big).\nabla\Big(S_\ell(u_1)\frac{T_\gamma(u_1-u_2)}{\gamma}\Big)\\ &\quad + \Big(S_\ell(u_1)\frac{T_\gamma(u_1-u_2)}{\gamma}\Big)\Big(\frac{\nabla u_2\chi_{\{0<u_2<h\}}}{h} - \frac{\nabla u_1\chi_{\{-h<u_1<0\}}}{h}\Big) \text{ a.e. in } \Omega. \end{aligned}\tag{3.55}$$

We evaluate the limit when $h \to 0$, $\gamma \to 0$ and then $\ell \to \infty$. Let us first prove that $med(W_{\ell,\gamma,h}) = 0$. Let σ such that $0 < \sigma < \dfrac{1}{2}$.

$$\begin{aligned}&\{x \in \Omega : W_{\ell,\gamma,h} > \sigma\}\\ &\qquad = \{x \in \Omega : W_{\ell,\gamma,h} > \sigma, 0 < u_2 < 2\ell\} \subset \{x \in \Omega : \frac{T_h(u_2^+)}{h} > \sigma\}\\ &\qquad = \{x \in \Omega : u_2 > h\sigma\}.\end{aligned}$$

Since $med(u_2) = 0$, we have

$$meas\{x \in \Omega : u_2 > h\sigma\} \le \frac{|\Omega|}{2}.$$

It follows that $\forall\, \sigma < \dfrac{1}{2}$,

$$meas\{x \in \Omega : W_{\ell,\gamma,h} > \sigma\} \le \frac{|\Omega|}{2},$$

which implies that $med(W_{\ell,\gamma,h}) \le 0$.

In addition, since $\forall \sigma > 0$,

$$\{x \in \Omega : u_2 \ge 0\} \subset \{x \in \Omega : W_{\ell,\gamma,h} > -\sigma\}$$

and

$$meas\{x \in \Omega : u_2 \ge 0\} \le \frac{|\Omega|}{2},$$

we deduce that

$$meas\{x \in \Omega : W_{\ell,\gamma,h} > -\sigma\} \le \frac{|\Omega|}{2},$$

which implies that $med(W_{\ell,\gamma,h}) \geq 0$. Consequently, we have $med(W_{\ell,\gamma,h}) = 0$.

Thus, the Poincaré-Wirtinger inequality becomes

$$\int_\Omega |W_{\ell,\gamma,h}|^2 dx \leq C \int_\Omega |\nabla W_{\ell,\gamma,h}|^2 dx. \tag{3.56}$$

We now evaluate the gradient of $W_{\ell,\gamma,h}$. By using (3.47) and (3.55),

$$\int_\Omega |\nabla W_{\ell,\gamma,h}|^2 dx \leq \frac{1}{l^2}\int_\Omega |\nabla T_{2\ell}(u_1)|^2 dx + \frac{1}{\gamma^2}\int_\Omega S_\ell^2(u_1)|\nabla T_\gamma(u_1-u_2)|^2 dx$$
$$+ \frac{1}{h^2\,\gamma^2}\int_\Omega S_\ell^2(u_1)|\nabla T_\gamma(u_1-u_2)|^2\Big(|\nabla T_h(u_2^+)| + |\nabla T_h(u_1^-)|\Big)^2 dx. \tag{3.57}$$

We claim that

$$\lim_{\ell\to\infty} \frac{1}{\ell^2}\int_\Omega |\nabla T_{2\ell}(u_1)|^2 dx = 0, \tag{3.58}$$

$$\lim_{\gamma\to 0} \frac{1}{\gamma^2}\int_\Omega S_\ell^2(u_1)|\nabla T_\gamma(u_1-u_2)|^2 dx = 0, \tag{3.59}$$

$$\lim_{h\to 0} \frac{1}{h^2\,\gamma^2}\int_\Omega S_\ell^2(u_1)|\nabla T_\gamma(u_1-u_2)|^2\,|\nabla T_h(u_2^+)|^2 dx = 0, \tag{3.60}$$

$$\lim_{h\to 0} \frac{1}{h^2\,\gamma^2}\int_\Omega S_\ell^2(u_1)|\nabla T_\gamma(u_1-u_2)|^2\,|\nabla T_h(u_1^-)|^2 dx = 0. \tag{3.61}$$

Indeed, (3.58) is deduced from (3.49). Moreover, thanks to (3.45) we also deduce (3.59), since $S_\ell(u_1)$ converges to 1 when l tends to ∞. We now consider (3.60) and (3.61). Noting that

$$\frac{1}{h^2\,\gamma^2}\int_\Omega |\nabla T_\gamma(u_1-u_2)|^2\,|\nabla T_h(u_1^-)|^2 = \frac{1}{h^2\,\gamma^2}\int_\Omega |\nabla T_\gamma(u_1-u_2)|^2\,|\nabla u_1)|^2 \chi_{\{-h<u_1<0\}}$$

a.e. in Ω. Since $u_1 < u_2$ a.e. in Ω it results

$$0 < \frac{T_\gamma(u_1-u_2)}{h\gamma}\chi_{\{-h<u_1<0\}} \leq \frac{1}{\gamma}\chi_{\{-h<u_1<0\}},$$

and then

$$\frac{1}{h^2\,\gamma^2}|\nabla T_\gamma(u_1-u_2)|^2\,|\nabla T_h(u_1^-)|^2 = \frac{1}{\gamma^2}|\nabla u_1)|^2\chi_{\{-h<u_1<0\}}.$$

For fixed $h > 0$, since $|\nabla u_1| \in L^2(\Omega)$ and $\frac{1}{\gamma^2}|\nabla u_1)|^2\chi_{\{-h<u_1<0\}} \to 0$ strongly as $h \to 0$, by the Lebesgue dominated convergence Theorem, we deduce (3.60). By analogy, we get (3.61) By gathering (3.58), (3.59), (3.60), (3.61), we get

$$\lim_{h\to 0}\lim_{\gamma\to 0}\lim_{\ell\to\infty}\int_\Omega |\nabla W_{\ell,\gamma,h}|^2 dx = 0. \tag{3.62}$$

From (3.62) and (3.56), it follows that

$$\lim_{h\to 0}\lim_{\gamma\to 0}\lim_{\ell\to\infty}\int_\Omega |W_{\ell,\gamma,h}|^2 dx = 0, \tag{3.63}$$

which gives, via Lebesgue dominated Theorem,

$$|sign_0(u_1-u_2)\big(\chi_{\{u_2>0\}} - \chi_{\{u_1<0\}}\big)| = 0. \tag{3.64}$$

This means that $\chi_{\{u_2>0\}} = \chi_{\{u_1<0\}}$ a.e. in Ω since u_1 and u_2 have the same sign, which leads to a contradiction.

Using the same arguments, we can prove that $u_1 > u_2$, a.e. in Ω, cannot occur. From steps 2 and 3 it follows that $u_1 = u_2$.

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