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APPROXIMATE ANALYTICAL SOLUTIONS OF FRACTIONAL SCHRODINGER EQUATIONS

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Abstract. The fractional Natural homotopy perturbation method (FNHPM) is employed in the present investigation to find the solution for linear and nonlinear fractional Schrodinger Equation. This method is coupled by the Natural transform (NT) and homotopy perturbation method (HPM). The method in general is easy to implement and yields good results. Illustrative examples are included to demonstrate the validity and applicability of the presented method.

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1. Introduction

Fractional order partial differential equations are generalizations of classical partial differential equations. These have been of considerable interest in the recent literatures. These topics have received a great deal of attention especially in the fields of viscoelasticity materials, electrochemical processes, dielectric polarization, colored noise, anomalous diffusion, signal processing, control theory and others. Increasingly, these models are used in applications such as fluid flow, finance and others. Most nonlinear fractional differential equations do not have analytic solutions, so approximation and numerical techniques must be used. In recent years, many researchers have paid attention to study the behavior of physical problems by using various analytical and numerical techniques which are not described by the common observations, such as the FVIM [1–5], FDTM [6–8], FSEM [9], FSTM [10], FLTM [11,12], FHPM [13], FLDM [14,15], FFSM [16], FLVIM [17–21], FRDTM [22–25], FFDM [26,27] FSDM [28,29] and other methods [30–70].

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This paper considers the efficiency of fractional Natural homotopy perturbation method (FNHPM) to solve linear and nonlinear time-fractional fractional Schrodinger Equation. The FNHPM is a graceful coupling of two powerful techniques namely HPM and Natural transform methods and gives more refined convergent series solution. The remaining sections of this work are organized as follows. In Section 2, some background notations of fractional calculus are presented. In Section 3, the analysis of fractional Natural homotopy perturbation method are discussed. Applications of FNHPM are shown in Section 4. The conclusion of this paper is given in Section 5.

2. Preliminaries of Fractional Calculus

Some fractional calculus definitions and notation needed [29, 30] in the course of this work are discussed in this section

Definition 2.1. *A real function $x(\tau), \tau > 0$, is said to be in the space $C_\vartheta, \vartheta \in R$ if there exists a real number $q, (q > \vartheta)$, such that $x(\tau) = \tau^q x_1(\tau)$, where $x_1(\tau) \in C[0, \infty)$, and it is said to be in the space C_ϑ^m if $x^{(m)} \in C_\vartheta, m \in N$.*

Definition 2.2. *The Riemann Liouville fractional integral operator of order ν of a function $x(\tau) \in C_\vartheta, \vartheta \geq -1$ is defined as*

$$I^\nu x(\tau) = \begin{cases} \frac{1}{\Gamma(\nu)} \int_0^\tau (\tau-\kappa)^{\nu-1} x(\kappa) d\kappa, & \nu > 0, \tau > 0, \\ x(\tau), & \nu = 0, \end{cases}$$

where $\Gamma(\cdot)$ is the well-known Gamma function.

Properties of the operator I^ν, which we will use here, are as follows:
For $x \in C_\vartheta, \vartheta \geq -1, \nu, \sigma \geq -1$, then

(1) $I^\nu I^\sigma \varphi(\mu) = I^{\gamma+\sigma} x(\tau)$.
(2) $I^\nu I^\sigma \varphi(\mu) = I^\sigma I^\nu x(\tau)$.
(3) $I^\nu \tau^m = \frac{\Gamma(m+1)}{\Gamma(\nu+m+1}\tau^{\nu+m}$.

Definition 2.3. *The fractional derivative of $x(\tau)$ in the Caputo sense is defined as*

$$\begin{aligned} D^\nu x(\tau) &= I^{m-\nu} D^m x(\tau) \\ &= \frac{1}{\Gamma(m-\nu)} \int_0^\tau (\tau-\xi)^{m-\nu-1} x^{(m)}(\xi) d\xi, \end{aligned} \tag{2.1}$$

for $m-1 < \nu \leq m, m \in N, \mu > 0, x \in C_{-1}^m$.

The following are the basic properties of the operator D^ν:

(1) $D^\nu I^\nu x(\tau) = x(\tau)$.
(2) $D^\nu I^\nu x(\tau) = x(\tau) - \sum_{k=0}^{m-1} x^{(k)}(0)\frac{\tau^k}{k!}$.

Definition 2.4. *The Mittag-Leffler function E_ν with $\nu > 0$ is defined as*

$$E_\nu(z) = \sum_{m=0}^{\infty} \frac{z^\nu}{\Gamma(m\nu+1)}. \tag{2.2}$$

Definition 2.5. *The natural transform (NT) of $x(\tau)$ is symbolized by $N\{x(\tau)\}$ presented by*

$$N\{x(\tau)\} = \int_0^\infty e^{-s\tau} x(w\tau) d\tau, \tag{2.3}$$

where s and w are the NT variables.

Definition 2.6. *The Natural transform of the Caputo fractional derivative is defined as*

$$N\left\{D_\tau^{(\nu)}[x(\rho,\tau)]\right\} = \frac{s^\nu}{w^\nu} N\{x(\rho,\tau)\} - \sum_{j=0}^{n-1} s^{\nu-k-1} w^{-(\nu-k)} x^{(k)}(\rho,0). \tag{2.4}$$

Some basic properties of the natural transform are defined as below:

(1) $N\{1\} = \frac{1}{s^\nu}$.

(2) $N\{\tau^\nu\} = \frac{\Gamma(\nu+1)w^\nu}{s^{\nu+1}}$.

3. Fractional Natural Homotopy Perturbation Method

Let us consider the following fractional partial differential equation:

$$D_\tau^{(\nu)}[x(\rho,\tau)] + R[x(\rho,\tau)] + F[x(\rho,\tau)] = g(\rho,\tau), \tag{3.1}$$

with $n-1 < \nu \leq n$ and subject to initial conditions

$$\frac{\partial^r x(\rho,0)}{\partial \tau^r} = x^{(r)}(\rho,0), \quad r = 0,1,... \tag{3.2}$$

where $D_\tau^{(\nu)} = \frac{\partial^\nu}{\partial \tau^\nu}$ denote Caputo fractional derivative, R is linear LFDOs, F is nonlinear LFDOs and g is an inhomogenuous term.

Taking the NT on both side of (3.1), we construct

$$N\left\{D_\tau^{(\nu)}[x(\rho,\tau)]\right\} + N\{R[x(\rho,\tau)]\} + N\{F[x(\rho,\tau)]\} = N\{g(\rho,\tau)\}, \tag{3.3}$$

or equivalent

$$\begin{aligned} N\{x(\rho,\tau)\} &= \frac{w^\nu}{s^\nu} \sum_{j=0}^{n-1} s^{\nu-k-1} w^{-(\nu-k)} x^{(k)}(\rho,0) + \\ &\quad \frac{w^\nu}{s^\nu} \left(N\{g(\rho,\tau)\} - N\{R[x(\rho,\tau)]\} - N\{F[x(\rho,\tau)]\}\right). \end{aligned} \tag{3.4}$$

Now applying inverse NT on both side of (3.4), we get

$$x(\rho,\tau) = N^{-1}\left[\frac{w^\nu}{s^\nu} \sum_{j=0}^{n-1} s^{\nu-k-1} w^{-(\nu-k)} x^{(k)}(\rho,0)\right] +$$

$$N^{-1}\left[\frac{w^{\nu}}{s^{\nu}}\left(N\left\{g(\rho,\tau)\right\}-N\left\{R\left[x(\rho,\tau)\right]\right\}-N\left\{F\left[x(\rho,\tau)\right]\right\}\right)\right]. \tag{3.5}$$

Now we apply the HPM.

$$x(\rho,\tau)=\sum_{n=0}^{\infty}p^{n}x_{n}(\rho,\tau), \tag{3.6}$$

and the nonlinear terms $F\left[x(\rho,\tau)\right]$ is decomposed as:

$$F\left[x(\rho,\tau)\right]=\sum_{n=0}^{\infty}p^{n}H_{n}(x), \tag{3.7}$$

where $H_n(x)$ is the Heâ€™s polynomial and be computed using the following formula:

$$H_{n}(x_1,x_2,...,x_n)=\frac{1}{n!}\frac{\partial^{n}}{\partial p^{n}}\left[F\left(\sum_{i=0}^{n}p^{i}x_{i}(\rho,\tau)\right)\right]_{p=0},\quad n=0,1,2,.... \tag{3.8}$$

Substituting (3.6) and (3.7) into (3.5), we have:

$$\sum_{n=0}^{\infty}p^{n}x_{n}(\rho,\tau)=N^{-1}\left[\frac{w^{\nu}}{s^{\nu}}\sum_{j=0}^{n-1}s^{\nu-k-1}w^{-(\nu-k)}x^{(k)}(\rho,0)\right]+$$
$$N^{-1}\left[\frac{w^{\nu}}{s^{\nu}}\left(N\left\{g(\rho,\tau)\right\}-N\left\{R\left[\sum_{n=0}^{\infty}p^{n}x_{n}\right]\right\}-N\left\{\sum_{n=0}^{\infty}p^{n}H_{n}\right\}\right)\right]. \tag{3.9}$$

Using the coefficient of the likes powers of p in (3.9), the following approximations are obtained:

$$p^{0}:x_{0}(\rho,\tau)=N^{-1}\left[\frac{w^{\nu}}{s^{\nu}}\sum_{j=0}^{n-1}s^{\nu-k-1}w^{-(\nu-k)}x^{(k)}(\rho,0)\right]+N^{-1}\left[\frac{w^{\nu}}{s^{\nu}}\left(N\left\{g(\rho,\tau)\right\}\right)\right],$$
$$p^{n+1}:x_{n+1}(\rho,\tau)=N^{-1}\left[\frac{w^{\nu}}{s^{\nu}}\left(N\left\{R\left[x_{n}\right]\right\}-N\left\{H_{n}\right\}\right)\right],\ n\geq 0. \tag{3.10}$$

Hence, the series solution of (3.1) is given by:

$$x(\rho,\tau)=lim_{M\longrightarrow\infty}\sum_{n=0}^{M}x_{n}(\rho,\tau), \tag{3.11}$$

4. Applications

Example 4.1. Let us consider the linear fractional Schrodinger equation of the form

$$iD_{\tau}^{\nu}x(\rho,\tau)+x_{\rho\rho}(\rho,\tau)=0, \tag{4.1}$$

subject to initial condition

$$x(\rho,0)=sin(\rho). \tag{4.2}$$

Applying the Natural transform on both sides of (3.5), we get:

$$N\left\{x(\rho,\tau)\right\}=\frac{1}{s}\sin(\rho)-i\frac{w^{\nu}}{s^{\nu}}\left[N\left\{x_{\rho\rho}(\rho,\tau)\right\}\right]. \tag{4.3}$$

Taking the inverse Natural transform of (4.3), we obtain

$$x(\rho,\tau)=\sin(\rho)-iN^{-1}\left(\frac{w^{\nu}}{s^{\nu}}\left[N\left\{x_{\rho\rho}(\rho,\tau)\right\}\right]\right). \tag{4.4}$$

We now assume that

$$x(\rho,\tau)=\sum_{n=0}^{\infty}p^{n}x_{n}(\rho,\tau). \tag{4.5}$$

Then by using (4.5), we can re-write (4.4) in the form:

$$\sum_{n=0}^{\infty}p^{n}x_{n}=\sin(\rho)-iN^{-1}\left(\frac{w^{\nu}}{s^{\nu}}\left[N\left\{\sum_{n=0}^{\infty}p^{n}(x_{n})_{\rho\rho}\right\}\right]\right). \tag{4.6}$$

By comparing both sides of (4.6), we can easily generate the recursive relation as follows:

$$\begin{aligned}
p^{0}:x_{0}(\rho,\tau) &= \sin(\rho).\\
p^{1}:x_{1}(\rho,\tau) &= -iN^{-1}\left(\frac{w^{\nu}}{s^{\nu}}\left[N\left\{(x_{0})_{\rho\rho}\right\}\right]\right)\\
&= -iN^{-1}\left(\frac{w^{\nu}}{s^{\nu}}\left[N\left\{-\sin(\rho)\right\}\right]\right)\\
&= i\sin(\rho)N^{-1}\left(\frac{w^{\nu}}{s^{2\nu}}\right)\\
&= \sin(\rho)\frac{i\tau^{\nu}}{\Gamma(1+\nu)}.
\end{aligned}$$

$$\begin{aligned}
p^{2}:x_{2}(\rho,\tau) &= -iN^{-1}\left(\frac{w^{\nu}}{s^{\nu}}\left[N\left\{(x_{1})_{\rho\rho}\right\}\right]\right)\\
&= -iN^{-1}\left(\frac{w^{\nu}}{s^{\nu}}\left[N\left\{-\sin(\rho)\frac{i\tau^{\nu}}{\Gamma(1+\nu)}\right\}\right]\right)\\
&= i^{2}\sin(\rho)N^{-1}\left(\frac{w^{2\nu}}{s^{3\nu}}\right)\\
&= \sin(\rho)\frac{(i\tau^{\nu})^{2}}{\Gamma(1+2\nu)}.
\end{aligned}$$

$$\begin{aligned}
p^{3}:x_{3}(\rho,\tau) &= -iN^{-1}\left(\frac{w^{\nu}}{s^{\nu}}\left[N\left\{(x_{2})_{\rho\rho}\right\}\right]\right)\\
&= -iN^{-1}\left(\frac{w^{\nu}}{s^{\nu}}\left[N\left\{-\sin(\rho)\frac{(i\tau^{\nu})^{2}}{\Gamma(1+2\nu)}\right\}\right]\right)\\
&= i^{3}\sin(\rho)N^{-1}\left(\frac{w^{3\nu}}{s^{4\nu}}\right)\\
&= \sin(\rho)\frac{(i\tau^{\nu})^{3}}{\Gamma(1+3\nu)}.\\
&\vdots
\end{aligned}$$

Hence, the approximate series solution is given by

$$\begin{aligned} x(\rho,\tau) &= \sin(\rho)\left[1+\frac{i\tau^{\nu}}{\Gamma(1+\nu)}+\frac{(i\tau^{\nu})^2}{\Gamma(1+2\nu)}+\frac{(i\tau^{\nu})^3}{\Gamma(1+3\nu)}+\cdots\right] \\ &= \sin(\rho)E_{\nu}(i\tau). \end{aligned} \tag{4.7}$$

For $\nu=1$ the above solution reduces to exact solution $x(\rho,\tau)=e^{i\tau}\sin(\rho)$.

From Eqs. (4.7) , the approximate solution of the given problem (4.1) by using FNHPM is the same results as that obtained by FDTM and it clearly appears that the approximate solution remains closed form to exact solution.

Example 4.2. Consider the following nonlinear Schrödinger equation with Caputo fractional operator:

$$iD_{\tau}^{\nu}x(\rho,\tau)+x_{\rho\rho}(\rho,\tau)-2|x(\rho,\tau)|^2x(\rho,\tau)=0, \tag{4.8}$$

subject to initial condition

$$x(\rho,0)=e^{i\rho}. \tag{4.9}$$

Taking the Natural transform on both sides of (4.8), we get:

$$\begin{aligned} N\{x(\rho,\tau)\}=\frac{1}{s^{\nu}}x(\rho,0) \quad &+ \quad i\frac{w^{\nu}}{s^{\nu}}\left[N\{x_{\rho\rho}(\rho,\tau)\}\right] \\ &-2i\frac{w^{\nu}}{s^{\nu}}\left[N\{|x(\rho,\tau)|^2x(\rho,\tau)\}\right]. \end{aligned} \tag{4.10}$$

Applying the inverse Natural transform of (4.10), we obtain

$$x(\rho,\tau)=e^{i\rho}+iN^{-1}\left(\frac{w^{\nu}}{s^{\nu}}\left[N\{x_{\rho\rho}(\rho,\tau)\}\right]\right)-2iN^{-1}\left(\frac{w^{\nu}}{s^{\nu}}\left[N\{|x|^2x\}\right]\right). \tag{4.11}$$

We now suppose that

$$\begin{aligned} x(\rho,\tau) &= \sum_{n=0}^{\infty}p^n x_n(\rho,\tau), \\ |x|^2x &= \sum_{n=0}^{\infty}p^n\Omega_n. \end{aligned} \tag{4.12}$$

Then by using (4.12), we can re-write (4.11) in the form:

$$\begin{aligned} \sum_{n=0}^{\infty}p^n x_n=e^{i\rho} \quad &+ \quad iN^{-1}\left(\frac{w^{\nu}}{s^{\nu}}\left[N\left\{\sum_{n=0}^{\infty}p^n(x_n)_{\rho\rho}\right\}\right]\right) \\ &-2iN^{-1}\left(\frac{w^{\nu}}{s^{\nu}}\left[N\left\{\sum_{n=0}^{\infty}p^n\Omega_n\right\}\right]\right), \end{aligned} \tag{4.13}$$

where

$$\Omega_0=x_0^2\overline{x_0},$$

$$\Omega_1=2x_0\overline{x_0}x_1+x_0^2\overline{x_1},$$

$$\Omega_2 = 2x_0\overline{x_0}x_2 + x_1^2\overline{x_0} + 2x_0x_1\overline{x_1} + x_0^2\overline{x_2},$$
$$\vdots$$

By comparing both sides of (4.13), we can easily generate the recursive relation as follows:

$$\begin{aligned}
p^0 : x_0(\rho,\tau) &= e^{i\rho}.\\
p^1 : x_1(\rho,\tau) &= iN^{-1}\left(\frac{w^\nu}{s^\nu}\left[N\left\{(x_0)_{\rho\rho}\right\}\right]\right) - 2iN^{-1}\left(\frac{w^\nu}{s^\nu}\left[N\left\{\Omega_0\right\}\right]\right)\\
&= -e^{i\rho}\frac{3i\tau^\nu}{\Gamma(1+\nu)}.\\
p^2 : x_2(\rho,\tau) &= iN^{-1}\left(\frac{w^\nu}{s^\nu}\left[N\left\{(x_1)_{\rho\rho}\right\}\right]\right) - 2iN^{-1}\left(\frac{w^\nu}{s^\nu}\left[N\left\{\Omega_1\right\}\right]\right)\\
&= e^{i\rho}\frac{(3i\tau^\nu)^2}{\Gamma(1+2\nu)}.\\
p^3 : x_3(\rho,\tau) &= iN^{-1}\left(\frac{w^\nu}{s^\nu}\left[N\left\{(x_2)_{\rho\rho}\right\}\right]\right) - 2iN^{-1}\left(\frac{w^\nu}{s^\nu}\left[N\left\{\Omega_2\right\}\right]\right)\\
&= -e^{i\rho}\frac{(3i\tau^\nu)^3}{\Gamma(1+3\nu)}.\\
&\vdots
\end{aligned}$$

Hence, the approximate series solution is given by

$$\begin{aligned}
x(\rho,\tau) &= e^{i\rho}\left(1 - \frac{3i\tau^\nu}{\Gamma(1+\nu)} + \frac{(3i\tau^\nu)^2}{\Gamma(1+2\nu)} - \frac{(3i\tau^\nu)^3}{\Gamma(1+3\nu)} + \cdots\right)\\
&= e^{i\rho}E_\nu(-3i\tau). \qquad (4.14)
\end{aligned}$$

For $\nu = 1$ the above solution reduces to exact solution $x(\rho,\tau) = e^{i(\rho-3\tau)}$.

From Eqs. (4.14) , the approximate solution of the given problem (4.8) by using FNHPM is the same results as that obtained by FDTM and it clearly appears that the approximate solution remains closed form to exact solution.

Conclusions

In this work, we utilized new technique called the fractional Natural homotopy perturbation method for solving fractional Schrödinger equation. The new technique provides an elegant series solution which converge very rapidly with reduced computational size. The results obtained by the FNHPM are in excellent agreement with the results of the existing methods. Thus, the proposed technique is a powerful, reliable and efficient mathematical tool for solving nonlinear PDEs.

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